Using a Modeling First Approach and Maple in a Traditional Differential Equations Class

SIMIODE Expo 2021

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1853
MANHATTAN
COLLEGE

Manhattan College Facts

Location: Bronx, N.Y.   Founded: 1853

Full-time Enrollment: about 3600

Schools:

Business

Education and Health

Engineering,

Liberal Arts

Science

Continuing and Professional Studies

The differential equations course at Manhattan College is a three credit, 200-level course. It is required of every student in the School of Engineering. It is an elective in the Mathematics Dept.

There is a syllabus with topics that have to be covered in preparation for a common cumulative final. **

All the traditional methods of solving differential equations by hand must be covered.

Systems of first order differential equations are in the curriculum and linear algebra is not a prerequisite for the course.

Decisions:

1/3 of the in-class time is allotted to modeling first activities adapted from modeling scenarios on the SIMIODE website.

Lectures are supplemented with videos on the material covered. All the material in the videos is covered in class, but more examples and explanations were provided on the videos.

More decisions:

We make extensive use the computer algebra system Maple. We have found that the modeling first approach works best when students actually solve problems and get answers that make sense. Any computer algebra system will work.

We ask questions on the tests that covered the ideas from lab.

Modeling in Differential Equations

Example 1:

Common Cold Spread (Scenario 1-37-S)

by Corban Harwood, George Fox University

Adapted by:

Catherine Bonan-Hamada,
Lisa Driskell & Tracii Friedman

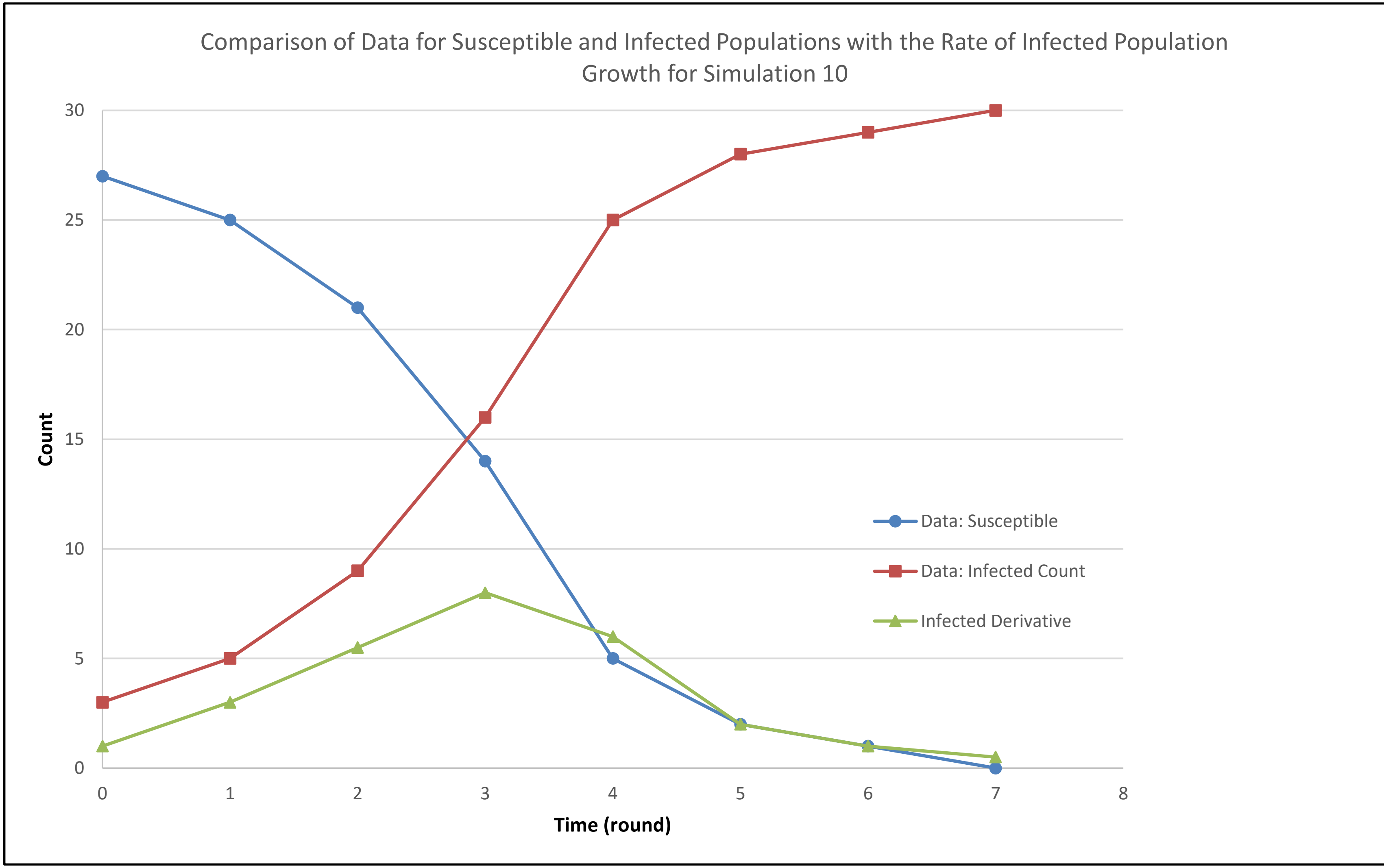
Common Cold Spread

Simulation

- N = 30 residents (beans) **
- Initially y_0 = 3 residents are infected with the common cold.
- After each round, remove the beans that become infected.
- Continue the simulation until no beans remain.

Common Cold Spread —Sample Data

Simulation Table with $y_0 = 3$ and $N = 30$								
Time (t) (rounds)	0	1	2	3	4	5	6	7
Susceptible Count	27	25	21	14	5	2	1	0
Infected Count ($y(t)$)	3	5	9	16	25	28	29	30
Change in Infected	2	4	7	9	3	1	1	0
Infected Derivative (dy/dt)	1	3	5.5	8	6	2	1	0.5
Infected Count (given by model)	3.00	5.84	10.34	16.00	21.39	25.32	27.65	28.87


Comparison of Data for Susceptible and Infected Populations with the Rate of Infected Population Growth for Simulation 10
Count
Time (round)
Data: Susceptible
Data: Infected Count
Infected Derivative
0
5
10
15
20
25
30
0
1
2
3
4
5
6
7
8

Common Cold Spread

Model Development

– Choose the most appropriate model

$$\frac{dy}{dt} = a\sin(bt) \qquad \frac{dy}{dt} = a(t-b)(t-c)$$

$$\frac{dy}{dt} = a(30-y) \qquad \frac{dy}{dt} = ay(30-y)$$

– Solve the differential equation

Common Cold Spread

In a simulation we let $y(t)$ = the number of infected students at time t. You were given that 3 students come back to school infected. This is an initial condition. Write an equation representing this initial condition.

In words, what does dy/dt represent?

On what does the value of dy/dt depend at any given time?

Write a differential equation that models this situation.

The Boarding School Problem
(DISEASE SPREAD—SIMIODE)

A student carrying a flu virus returns to an isolated college campus of 1000 students.

After 4 days, 50 students are infected.

How many students are infected after 6 days?

Find a function describing the number of infected students.

Plot the function and analyze the plot.

Create the differential equation.

$$eq := x'(t) = k \cdot x(t) \cdot (1000 - x(t))$$

$$D(x)(t) = k\,x(t)\,(1000 - x(t))$$

Solve the initial value problem.

$$dsolve([eq, x(0) = 1], x(t))$$

$$x(t) = \frac{1000}{1 + 999\,\mathrm{e}^{-1000\,k\,t}}$$

$$x := t \rightarrow \frac{1000}{1 + 999\,\mathrm{e}^{-1000\,k\,t}}$$

$$t \rightarrow \frac{1000}{1 + 999\,\mathrm{e}^{-1000\,k\,t}}$$

Use the fact that after 4 days, 50 students are infected.

$solve(x(4) = 50, k)$

$$-\frac{1}{4000}\ln\left(\frac{19}{999}\right)$$

$k := evalf(\%)$

$$0.0009905789498$$

How many students are infected after 6 days?

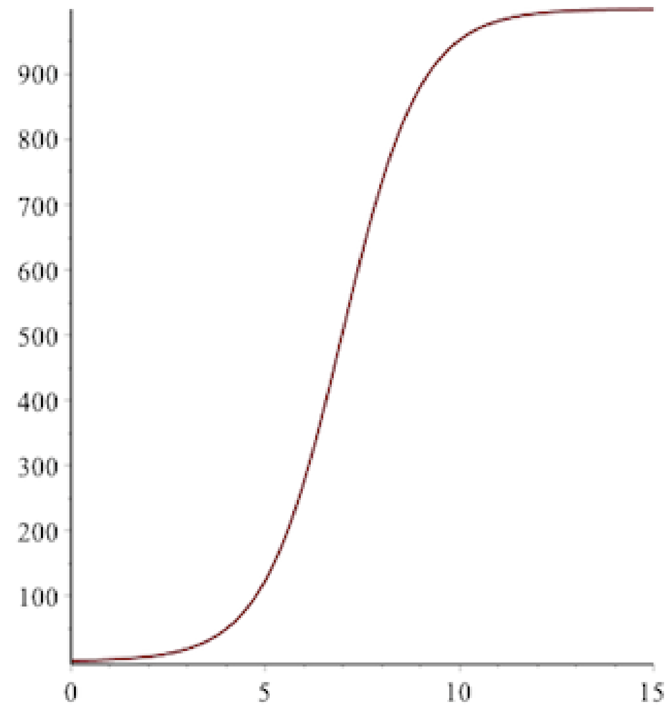
$x(6)$

$$276.2217492$$

There are about 276 students infected after 6 days.

The plot:

$plot(x, 0..15)$



Test question: From calculus you know that an inflection point is the place where the function changes concavity. Describe in words what happens at the inflection point in the above graph in terms of the change in the number of infected with respect to time.

Ant Tunnel Building

How long does it take an ant to build a tunnel?

To answer the question we might need some narrowing of scope, some simplification, and certainly some identification of terms and variables before we can get a nice answer. Let us identify some variables and then together make some assumptions which will lead to a mathematical model.

Assumptions:

The ant will build a level straight tunnel.

The ant will build in uniform, moist, fine sand.

The ant has to carry sand back to the opening of the tunnel.

The ant is digging into the side of a sand wall.

The ant will walk as fast in each direction.

The tunnel’s cross-sectional area is constant.

Here are some variables that students assign:

Let x be the length of the tunnel in feet that an ant builds.

Let $T(x)$ be the time in hours it takes the ant to build the tunnel of length x.

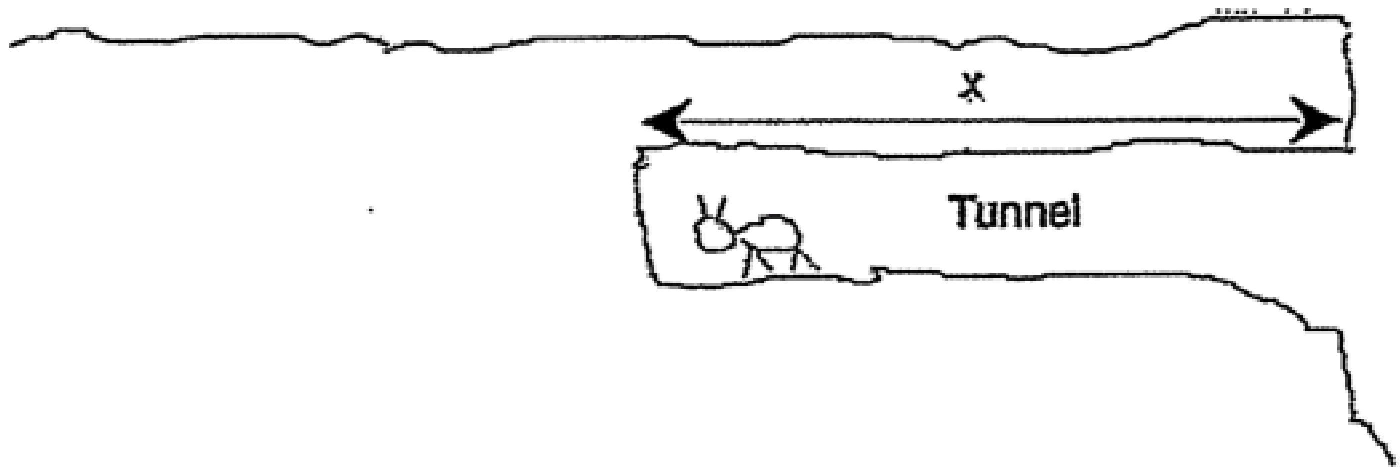


Figure 1. Crude drawing for ant tunnel building model. x is the length of the tunnel and $T(x)$ is the time it takes an ant to build a tunnel of length x.

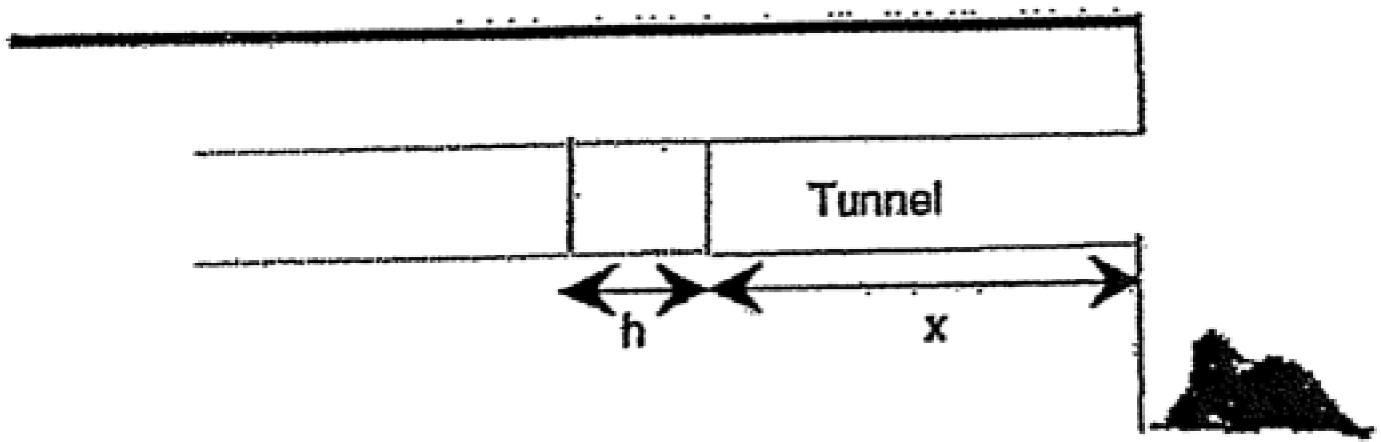


Figure 2. Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x + h$.

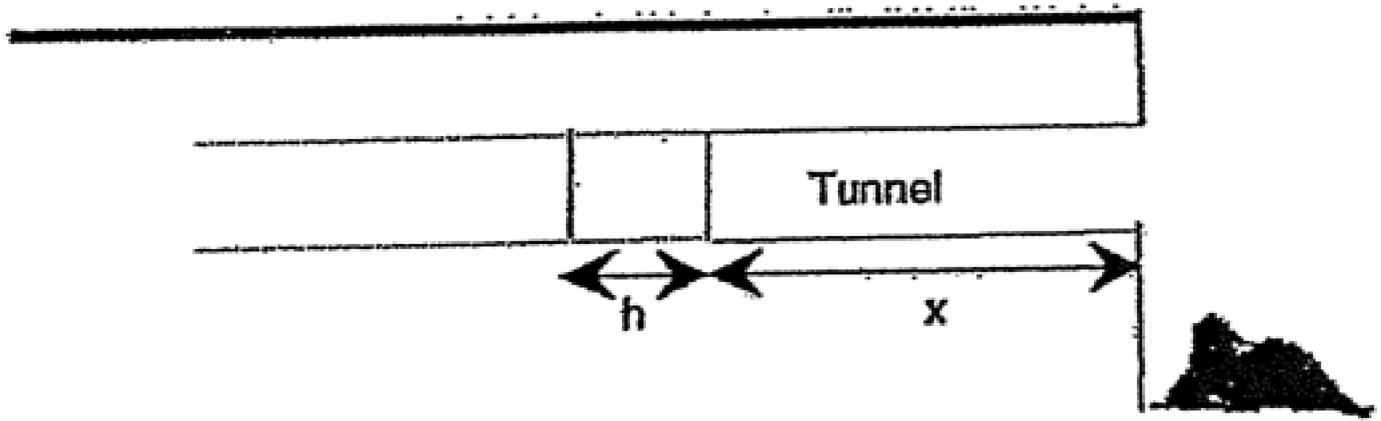


Figure 2. Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x + h$.

Let T(x) be the time in hours it takes the ant to build the tunnel of length x starting from the beginning of the tunnel.

The time it takes to build the small section of tunnel:

$$T(x + h) - T(x)$$

$T(x + h) - T(x)$ is proportional to both x and h.

$T(x + h) - T(x) = k\,x\,h$ so $T'(x) = k\,x$.

Using Maple with the snowplow problem:

It starts snowing in the morning and continues steadily throughout the day. A snowplow that removes snow at a constant rate starts plowing at noon. It plows 2 miles the first hour, and 1 mile in the second hour. Assume that the rate the snowplow travels is inversely proportional to the height of the snow. What time did it start snowing?

Students typically decide that to answer the question they need some narrowing of scope, some simplification, and certainly some identification of terms and variables.

They identify some variables and make some assumptions which leads to a mathematical model.

Let t be the time measured in hours after noon.

Let $h(t)$ be the height of the snow at time t.

Let $b > 0$ be the (unknown) number of hours before noon that it started snowing.

Let $x(t)$ be the distance the snowplow has traveled.

Let k be the constant rate at which snow falls.

The snow is falling at a constant rate so $h(t) = k(b+t)$.

Now the rate at which the snowplow travels is inversely proportional to the height of the snow.

$$eq1 := x'(t) = \frac{c}{k \cdot t + k \cdot b}$$

$$eq1 := \mathrm{D}(x)(t) = \frac{c}{k\,b + k\,t}$$

Also we have an initial condition, so we will solve an IVP.

$$dsolve([eq1, x(0) = 0], x(t))$$

$$x(t) = \frac{c \ln(b+t)}{k} - \frac{c \ln(b)}{k}$$

Since it plows 2 miles the first hour and 1 mile in the second hour.

$evalf(solve(\{x(1) = 2, x(2) = 3\}))$

$$\{b = 0.6180339888, c = c, k = 0.4812118252\, c\}$$

Remember that b is the number of hours before noon that it started snowing.

So it started snowing at about 0.618 hours before noon.

It started to snow at approximately 37 minutes before noon.

That is, it started to snow at approximately 11:23 AM.

<u>START</u> with the problem:

Suppose we have two tanks, A and B, that are connected.

Tank A contains 50 gallons of water in which 25 pounds of salt is dissolved. Tank B contains 50 gallons of pure water. Liquid is pumped into and out of each tank. Pure water is pumped into tank A at a rate of 3 gal/min. A well mixed solution flows from tank A into tank B at a rate of 4 gal/min. A well mixed solution flows from tank B to tank A at a rate of 1 gal/min. A well mixed solution flows out of tank B and is discarded at a rate of
3 gal/min.

It was simply fascinating, and a bit unnerving, to watch as students tried to cram every piece of information into one differential equation.
The reaction was one of relief as they realized they needed two differential equations! This lab was great.

Students saw systems of differential equations from a modeling first perspective. They analyzed their solutions and answered all the subsequent questions from several different perspectives.

For Tank A:

$$\frac{dA}{dt}\ \frac{lb}{\min} = 3\frac{gal}{\min}\cdot 0\frac{lb}{gal} + 1\frac{gal}{\min}\cdot\frac{B(t)}{50}\frac{lb}{gal} - 4\frac{gal}{\min}\cdot\frac{A(t)}{50}\frac{lb}{gal}$$

For Tank B:

$$\frac{dB}{dt}\ \frac{lb}{\min} = 4\frac{gal}{\min}\cdot\frac{\mathrm{A(t)}}{50}\frac{lb}{gal} - 3\frac{gal}{\min}\cdot\frac{B(t)}{50}\frac{lb}{gal} - 1\frac{gal}{\min}\cdot\frac{B(t)}{50}\frac{lb}{gal}$$

with initial conditions:

$$\mathrm{A}(0) = 25 \text{ and } \mathrm{B}(0) = 0$$

$$eq1 := \frac{d}{dt} \mathrm{A}(\mathrm{t}) = -2/25\, A(\mathrm{t}) + 1/50\, B(\mathrm{t})$$

$$\frac{d}{dt} A(t) = -\frac{2}{25} A(t) + \frac{1}{50} B(t)$$

$$eq2 := \frac{d}{dt} B(t) = 2/25\, A(t) - 2/25\, B(t)$$

$$\frac{d}{dt} B(t) = \frac{2}{25} A(t) - \frac{2}{25} B(t)$$

$$sol := dsolve([eq1, eq2, A(0) = 25, B(0) = 0])$$

$$\left\{ A(t) = \frac{25}{2} e^{-\frac{1}{25} t} + \frac{25}{2} e^{-\frac{3}{25} t}, B(t) = 25 e^{-\frac{1}{25} t} - 25 e^{-\frac{3}{25} t} \right\}$$

Using Technology in the Lab:

It is imperative in every lab setting that mathematics takes center stage. Code is de-emphasized. We learned this while using technology in calculus courses.

Students will say, "I don't know what to type." Typically what they really mean is "I don't know what to do."

It takes about two weeks, but finally all students realize that asking for code was not going to help them actually create a differential equation.

The use of technology enhances this modeling first approach

Students are no longer limited to exploring modeling problems that are solved by hand.

The appropriate technological tool can encourage students to hypothesize and experiment.

A computer algebra system allows students to analyze their solutions, and easily readjust their hypothesis if a particular graph or solution makes no sense.

For the labs involving first order differential equations, we drew on portions of the SIMIODE scenarios *1-13 Sleuthing*, *1-17 Disease Spread*, and *1-25 Mixing It Up*.

For the labs involving systems of first order differential equations, we used SIMIODE scenario *6-28 Salt Compartments*.

For the labs involving second order differential equations, we used SIMIODE scenarios *3-1 Spring Mass Data Analysis* and *3-30 Second Order Intro.*

Parting Words:

Getting started with the modeling first approach is much easier with the support of the SIMIODE community.

Check the SIMIODE site often. Scenarios are added all the time. Once you get started you might add your own scenario to the site.

The use of technology encourages students to hypothesize, experiment, analyze their solutions, and if necessary readjust their hypothesis and try again--without frustration! And students really like answers.

- THANK YOU!