THE FINITE ELEMENT METHOD IN SCHOOL, IN INDUSTRY, AND ONE EXAMPLE OF A CAREER IN IT.

SORRY FOR THE LONG TITLE

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ABSTRACT

FEW MATHEMATICAL TECHNIQUES HAVE HAD GREATER IMPACT ON OUR DAILY LIVES THAN THE FINITE ELEMENT METHOD (FEM). IT HAS BEEN ESSENTIAL TO THE PROGRESS OF OUR ENTIRE ENGINEERED WORLD, FROM THE DRAMATICALLY POWERFUL ROCKETS THAT PUT MAN AND MACHINE INTO SPACE TO THE MUNDANE PACKAGING THAT PROTECTS WHAT WE BUY WHILE IN TRANSIT. AT ITS CORE, THE METHOD IS SIMPLY A WAY TO APPROXIMATE SOLUTIONS TO DIFFERENTIAL EQUATIONS. AND, AS WITH MOST SUCCESSFUL NUMERICAL METHODS ITS STRENGTH IS THE SIMPLICITY OF ITS FOUNDATION, ALLOWING IT TO BE ADAPTED TO A LARGE VARIETY OF PROBLEMS.

IN UNIVERSITIES, THE FINITE ELEMENT METHOD IS TAUGHT REGULARLY IN MATHEMATICS AND ENGINEERING DEPARTMENTS, WITH PREDICTABLE DIFFERENCES IN EMPHASIS, BUT STILL TOUCHING ON THE SAME ESSENTIAL TENETS. ITS USE IN INDUSTRY IS THROUGH LARGE COMMERCIAL SOFTWARE PACKAGES THAT EFFICIENTLY CREATE, SOLVE, AND POST-PROCESS MATHEMATICAL MODELS. IN THIS TALK I WANT TO HIGHLIGHT MY EXPERIENCE WITH THE FEM, AS A STUDENT AND TEACHER OF THE FIRST, AND ONLY, CLASS MOST PROFESSIONAL USERS OF THE FEM WILL TAKE. AND CONTRAST THAT WITH THE REQUIREMENTS OF USING THE METHOD PROFESSIONALLY WITH COMMERCIAL SOFTWARE. FINALLY, I WILL BRING IN MY PERSONAL EXPERIENCE IN A RATHER NARROW, BUT EXCITING CORNER OF THE FEM COMMUNITY.

A UNIVERSITY CLASS IN THE FINITE ELEMENT METHOD

- TYPICAL CLASS: 3 X 50 MINUTES OR 2 X 75 MINUTES PER WEEK, 15 WEEK TERM.
  - 2250 MIN OF CLASS = **37 HOURS, 30 MINUTES**
  - LECTURES, DISCUSSIONS (CLASS LOGISTICS, ANSWERING BACKGROUND/HW/SOFTWARE QUESTIONS), REVIEWS FOR EXAMS, EXAMS, LOST TIME (TECHNOLOGY). – NOT A LOT OF TIME.
- THE STARTING LINE IS SOME VARIATION OF THIS EXAMPLE:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=q, 0<x<L$$

$$u(0)=u_0, \left.\left(a\frac{du}{dx}\right)\right|_{x=L}=Q_L$$

- THE FINITE ELEMENT METHOD:
  - PARTITION THE INTERVAL $(0, L)$ INTO NON-OVERLAPPING SUB-INTERVALS (x_A, x_B) CALLED ELEMENTS.
  - ON EACH ELEMENT TRANSLATE DIFFERENTIAL EQUATION TO AN EQUIVALENT VARIATIONAL PROBLEM (LESS SMOOTHNESS REQUIRED)
    - FIND $u \in C^1$, SUCH THAT $\int_{x_A}^{x_B}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx=\int_{x_A}^{x_B}qwdx+w(x_B)\left.\left(a\frac{du}{dx}\right)\right|_{x=x_B}-w(x_A)\left.\left(a\frac{du}{dx}\right)\right|_{x=x_A}$ FOR ALL $w \in C^1$ WITH $w(0)=0$. USE REALLY SIMPLE FUNCTIONS (LINEAR) TO MAKE THE APPROXIMATION.
  - FORCE CONTINUITY OF THE SOLUTION OVER ELEMENT BOUNDARIES.
  - THE REST IS BOOKKEEPING!

WHERE MIGHT A CLASS GO FROM THAT STARTING EXAMPLE?

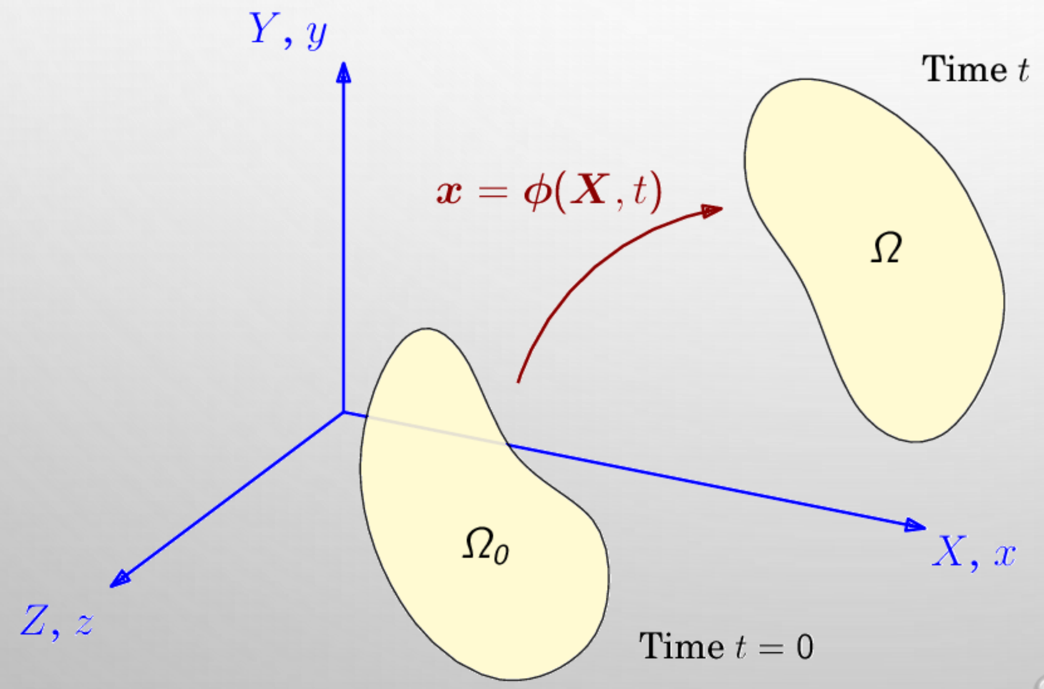
- OTHER APPLICATIONS OF THE FIRST EXAMPLE EQUATION:
  - ROD/BAR, CABLE, STEADY STATE HEAT IN ROD, OTHER BOUNDARY CONDITIONS
- OTHER TYPICAL EQUATIONS
  - BEAMS
    - EULER-BERNOULLI (4^{TH} ORDER ODE)
    - TIMOSHENKO (SYSTEM OF 2^{ND} ORDER ODES)
  - MEMBRANE OR STEADY IRROTATIONAL FLOW (2^{ND} ORDER PDE – POISSON'S EQUATION)
  - KIRCHOFF PLATE (4^{TH} ORDER PDE – PLATE EQUATION)
  - PLANE STRESS/STRAIN (SYSTEM OF 2^{ND} ORDER PDES - NAVIER'S EQUATIONS)
  - EVOLUTION EQUATIONS
    - VIBRATION (EIGENVALUE) PROBLEMS FOR ANY OF THE ABOVE
    - DYNAMIC PROBLEMS – COULD BE ANY OF THE ABOVE, BUT TYPICALLY LIMITED TO 1D
- OTHER TOPICS THAT MIGHT BE COVERED
  - NUMERICAL LINEAR ALGEBRA
  - NUMERICAL INTEGRATION
  - COMMERCIAL SOFTWARE

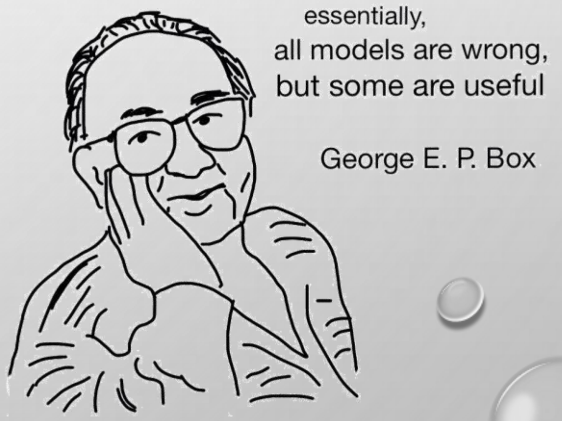
These don't look like the equations of Continuum Mechanics (although, they can all be derived by simplifying the CM equations).

COMMERCIAL SOFTWARE APPROXIMATES THE SOLUTION TO THE EQUATIONS OF CONTINUUM MECHANICS

CONTINUUM MECHANICS IS A BRANCH OF PHYSICS WHERE MATTER IS CONSIDERED TO BE CONTINUOUSLY DISTRIBUTED THROUGHOUT THE SPACE IT OCCUPIES IN CONTRAST TO A PARTICLE PICTURE OF MATTER.

THE CONCEPT OF A CONTINUUM IS A **USEFUL** ABSTRACTION OF REALITY





THINGS WE QUANTIFY IN CONTINUUM MECHANICS AND HOW TO RELATE THEM

Quantity of Interest	Number of Components
Displacement	3
Velocity	3
Mass Density	1
Stress	9
Temperature	1
Heat Flux	3
Internal Energy (per unit mass)	1
Entropy (per unit mass)	1

22 Quantities

Mathematical or Physical Law (Axioms)	Number of Equations
Kinematic Relationships	3
Balance of Mass	1
Balance of Linear Momentum	3
Balance of Angular Momentum	3
Balance of Energy	1

11 Equations

- We augment our 11 equations with **Empirical Laws** for specific materials called Constitutive Equations.
- A full set of Constitutive Equations includes expressions for Stress (6 equations), Heat Flux (3 equations), Internal Energy (1 equation), and the Entropy (1 equation) as dependent on the other quantities (and possibly their derivatives and histories).
- Constitutive Equations complete the model of a particular continuum.

THE DIFFERENTIAL EQUATIONS (AND ONE INEQUALITY) OF CONTINUUM MECHANICS

1. BALANCE OF MASS (VELOCITY AND MASS DENSITY)

$$\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho} \overline{\boldsymbol{v}}) = \mathbf{0}$$

2. BALANCE OF LINEAR MOMENTUM (MASS DENSITY, VELOCITY, AND STRESS)

$$\boldsymbol{\rho}\left[\frac{\partial \overline{\boldsymbol{v}}}{\partial \boldsymbol{t}} + (\boldsymbol{\nabla}\overline{\boldsymbol{v}})\overline{\boldsymbol{v}}\right] = \boldsymbol{\nabla} \cdot \boldsymbol{S} + \boldsymbol{\rho}\overline{\boldsymbol{b}}$$

3. BALANCE OF ANGULAR MOMENTUM (STRESS)

$$S = S^T$$

4. BALANCE OF ENERGY (MASS DENSITY, INTERNAL ENERGY, STRESS, VELOCITY, AND HEAT FLUX)

$$\boldsymbol{\rho}\dot{\boldsymbol{\phi}} = \boldsymbol{S} : \boldsymbol{L} - \boldsymbol{\nabla} \cdot \overline{\boldsymbol{q}} + \boldsymbol{\rho r}$$

5. ENTROPY INEQUALITY (MASS DENSITY, ENTROPY, HEAT FLUX, AND TEMPERATURE)

$$\rho\dot{\eta} \geq \frac{\rho r}{\theta} - \nabla \cdot \left(\frac{\overline{q}}{\theta}\right)$$

COMMERCIAL FINITE ELEMENT SOFTWARE

- APPROXIMATES THE SOLUTION TO THE EQUATIONS OF CONTINUUM MECHANICS (OR A REDUCED SET OF THEM).
- COMPETITIVE INDUSTRY – A LOT OF SOFTWARE COMPANIES (ABAQUS, ANSYS, AUTODESK, COMSOL, IMPETUS, **LS-DYNA**, NASTRAN, STRESSCHECK, ETC.)
  - COST OF COMMERCIAL FE SOFTWARE FOR INDUSTRY USERS IS $1000S/NODE/YEAR
  - LARGE COMPANY WILL HAVE 100S OR 1000S OF USERS – IT ADDS UP FAST.
- FINITE ELEMENT INDUSTRIAL COMPLEX HAS 3 PRIMARY POPULATIONS:
  1. ANALYSTS/USERS (AUTOMOTIVE, AEROSPACE, DEFENSE, INFRASTRUCTURE, HVAC, ETC.)
  2. SOFTWARE ENGINEERS (NUMERICAL LINEAR ALGEBRA, MESH GENERATION, USER INTERFACE)
  3. LIAISON BETWEEN SOFTWARE COMPANY AND USER COMMUNITY (EMPLOYED BY SOFTWARE COMPANY)
     - TEACH CLASSES, HELP-DESK, COMMUNICATES USER NEEDS TO SOFTWARE PEOPLE

FIRST DAY ON THE JOB READING MATERIAL*

1 of 3186

1 of 1613

1 of 351

Need to know how to use these two

LS-DYNA®
KEYWORD USER'S MANUAL
VOLUME I

LS-DYNA®
KEYWORD USER'S MANUAL
VOLUME II
Material Models

LS-DYNA®
KEYWORD USER'S MANUAL
VOLUME III
Multi-Physics Solvers
LS-DYNA R11
10/16/18 (r:10577)

Getting Started
GETTING STARTED
DESCRIPTION OF KEYWORD INPUT

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*MAT_001
*MAT_ELASTIC_{OPTION}

• • •

*MAT_296
*MAT_ANAND_VISCOPLASTICITY

Available Constitutive Laws

This one might be useful later

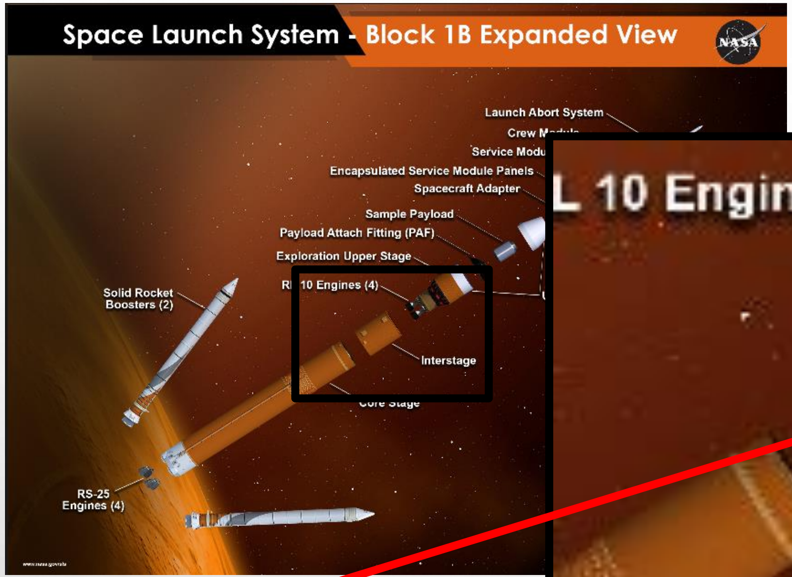
* *From here on I will focus on one particular software (LS-Dyna) because it is what I use.*

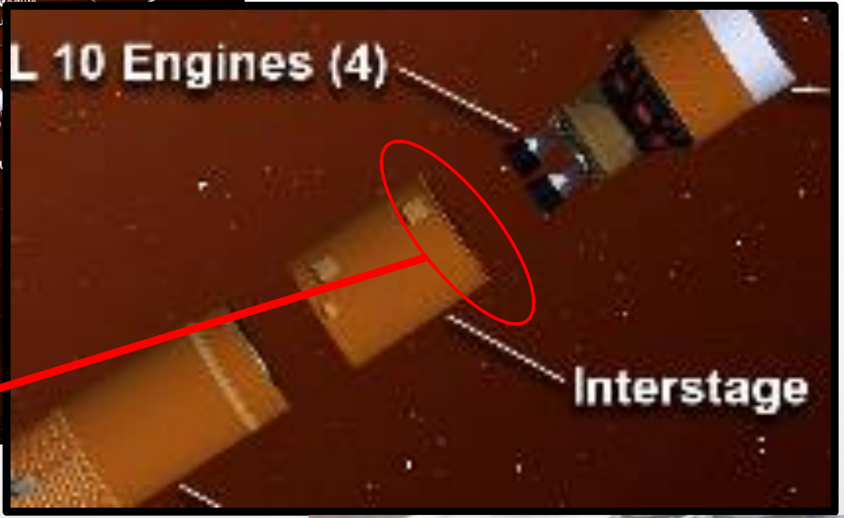
SOME NOTABLE DIFFERENCES BETWEEN THE INTRODUCTION TO THE FINITE ELEMENT METHOD CLASS AND USE OF COMMERCIAL FE SOFTWARE

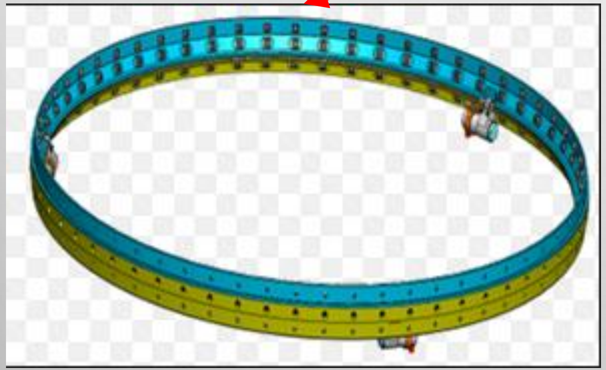
	Introductory FE class	Commercial Use of FE Software
1.	This is a math class on how to approximate solutions to DEs.	This is an engineering endeavor – Someone who knows CAD/CAE software will comfortably adapt to using FE software more than someone who took the FE class.
2.	Class focus is translation of specific DEs into simple FE code.	Knowledge of equations being solved (CM equations) is not necessary, and seldom even possessed by users commercial FE software.
3.	Class uses simplified equations, and students write a new program for each equation/BC.	Software uses the fixed set of CM equations, and assumes BCs where they are not explicitly given.

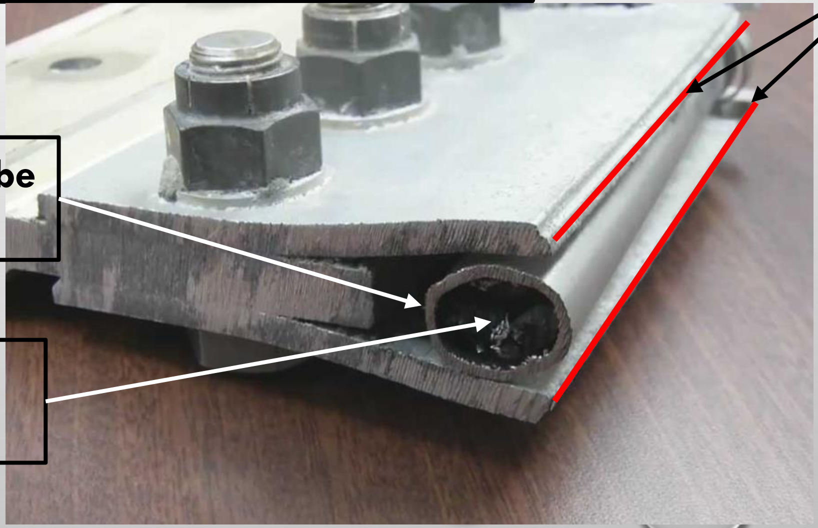
AN EXAMPLE OF USING THE FINITE ELEMENT METHOD

FINITE ELEMENT ANALYSIS OF A FRANGIBLE JOINT









Fracture surfaces

Expanding metal tube

Explosive line charge

RELIABILITY OF A FRANGIBLE JOINT

Quantifying Performance of a Frangible Joint as a Function of Design Variables

Design Variables are divided into Geometric and Material

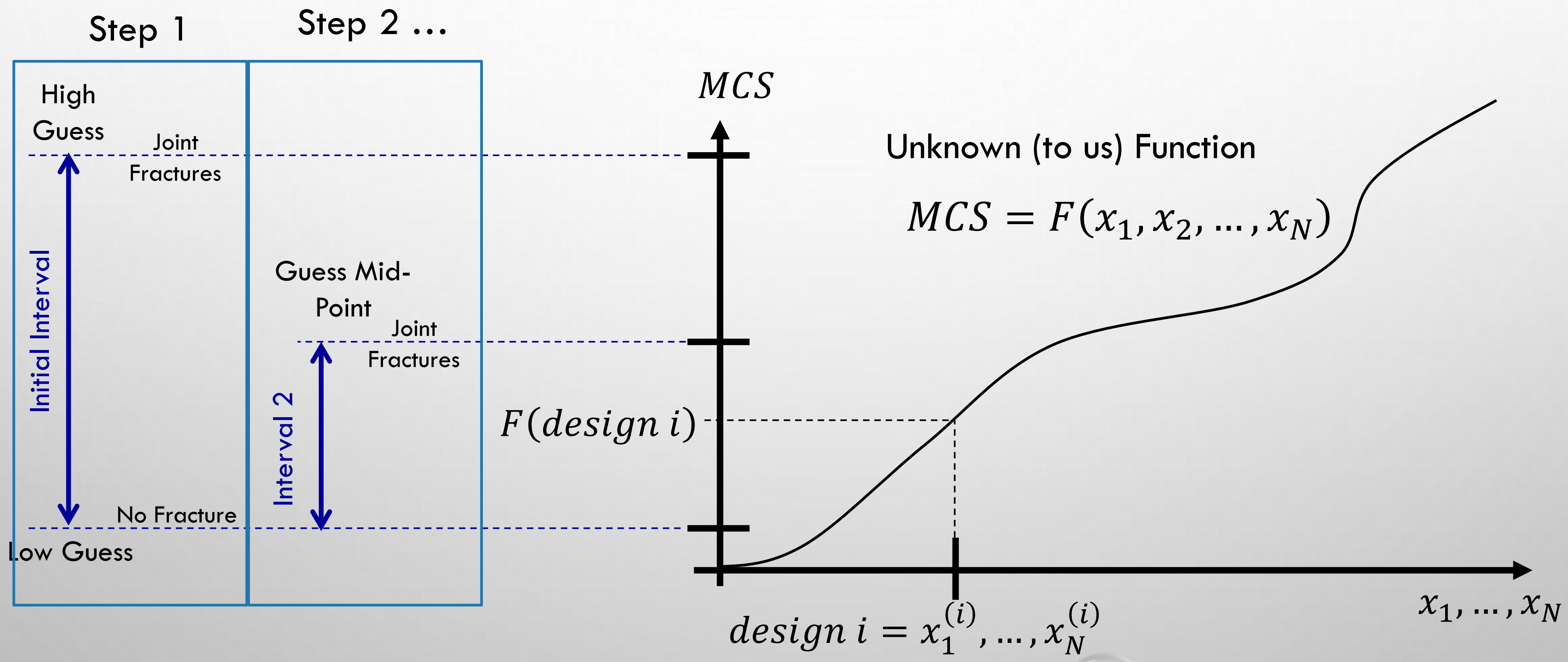
- Geometric
  - Metal thickness
  - Component Position
  - …
- Material
  - Yield Stress
  - Elongation at break
  - …
- ***"Reasonable"*** assumptions are made about the range of each variable.

A Finite Element Model is created of the joint for each design.

The design's performance is quantified by the **Minimum Charge Size** (MCS) that will fracture the joint (found iteratively).

RELIABILITY OF A FRANGIBLE JOINT

Determining the relationship between MCS and any single design - Bisection



Continue Steps until the interval is smaller than our tolerance for error

RELIABILITY OF A FRANGIBLE JOINT

Instruction: “Find the **Corners of the Box**” (Appropriate reaction: “Hah! Good one.”)

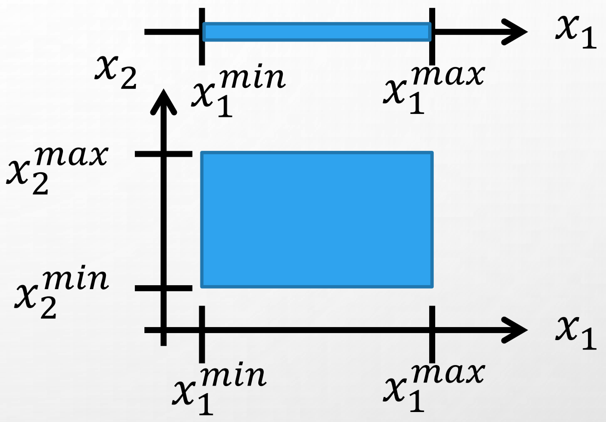
Initially, 20 variables were identified, each variable has an interval of possible values (usually centered at nominal)

One variable box: 2 corners: max and min values

Two variable box: $2^2 = 4$ corners

Three variable box: $2^3 = 8$ corners

…



Twenty variable box: $2^{20} > 10^6$ corners

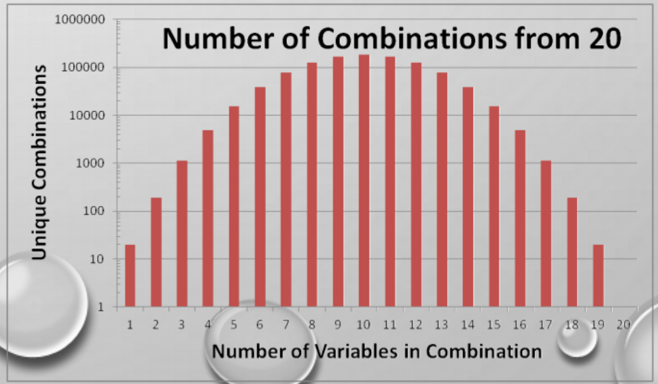
Luckily (somewhat), performance measured by MCS is typically monotonic w.r.t. each variable. Replace Corners of the Box with **Worst Case Scenarios**

One variable at worst case: 20 variables ➔ 20 worst case scenarios

Two variables at worst case: $\binom{20}{2}$= 190 worst case scenarios

…

N variables at worst case: $\binom{20}{N}$ worst case scenarios



RELIABILITY OF A FRANGIBLE JOINT

Create a **Surrogate Response Surface** to predict MCS

There exists a function F that will give the MCS, for any values of the design variables:

$$MCS = F(x_1, x_2, \dots, x_N)$$

Given a design, we can approximate the MCS via Bisection and FEM simulation:

$$\text{MCS for } i^{th} \text{ design} = F\left(x_1^{(i)}, x_2^{(i)}, \dots, x_N^{(i)}\right) = F^{(i)}$$

Repeat for as many designs as is practical:

$$F\left(x_1^{(1)}, x_2^{(1)}, \dots, x_N^{(1)}\right) = F^{(1)}$$
$$F\left(x_1^{(2)}, x_2^{(2)}, \dots, x_N^{(2)}\right) = F^{(2)}$$
$$\vdots$$
$$F\left(x_1^{(P)}, x_2^{(P)}, \dots, x_N^{(P)}\right) = F^{(P)}$$

Approximate F with a Linear Response Surface (hyperplane):

$$F(x_1, x_2, \dots, x_N) \approx L(x_1, x_2, \dots, x_N)$$
$$F(x_1, x_2, \dots, x_N) \approx B_0 + B_1(x_1 - x_1^{nom}) + B_2\,(x_2 - x_2^{nom}) + \cdots + B_N(x_N - x_N^{nom})$$
$$\tilde{F}(x_1, x_2, \dots, x_N) = B_0 + \sum_{i=1}^{N} B_i(x_i - x_i^{nom})$$

RELIABILITY OF A FRANGIBLE JOINT

P equations for N+1 unknowns, P > N+1

$$F^{(1)} = \tilde{F}^{(1)} = B_0 + B_1\left(x_1^{(1)} - x_1^{nom}\right) + B_2\left(x_2^{(1)} - x_2^{nom}\right) + \cdots + B_N\left(x_N^{(1)} - x_N^{nom}\right)$$
$$F^{(2)} = \tilde{F}^{(2)} = B_0 + B_1\left(x_1^{(2)} - x_1^{nom}\right) + B_2\left(x_2^{(2)} - x_2^{nom}\right) + \cdots + B_N\left(x_N^{(2)} - x_N^{nom}\right)$$
$$\vdots$$
$$F^{(P)} = \tilde{F}^{(P)} = B_0 + B_1\left(x_1^{(P)} - x_1^{nom}\right) + B_2\left(x_2^{(P)} - x_2^{nom}\right) + \cdots + B_N\left(x_N^{(P)} - x_N^{nom}\right)$$

Overdetermined Linear System (potentially non-existing solution):

$$\underbrace{\begin{bmatrix} \tilde{F}^{(1)} \\ \tilde{F}^{(2)} \\ \vdots \\ \tilde{F}^{(P)} \end{bmatrix}}_{\bar{f}_{P\times 1}} = \underbrace{\begin{bmatrix} 1 & \left(x_1^{(1)} - x_1^{nom}\right) & \cdots & \left(x_N^{(1)} - x_N^{nom}\right) \\ 1 & \left(x_1^{(2)} - x_1^{nom}\right) & \cdots & \left(x_N^{(2)} - x_N^{nom}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \left(x_1^{(P)} - x_1^{nom}\right) & \cdots & \left(x_N^{(P)} - x_N^{nom}\right) \end{bmatrix}}_{X_{P\times(N+1)}} \underbrace{\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_N \end{bmatrix}}_{\bar{c}_{(N+1)\times 1}}$$

RELIABILITY OF A FRANGIBLE JOINT

Improve the Surrogate Surface by adding ***some*** quadratic terms

P equations for N+1+N*(N-1)/2 unknowns, P > N+1 + N*(N-1)/2

Number of Coefficients for Linear Terms

Number of Coefficients for Non-Linear Terms

Quadratic terms only include interaction terms, such as x*y, not x^2

$$F(x_1, x_2 \dots, x_N) \approx B_0 + \sum_{i=1}^{N} B_i(x_i - x_i^{nom}) + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} C_{ij}(x_i - x_i^{nom})(x_j - x_j^{nom})$$

Again, overdetermined Linear System:

$$\begin{bmatrix} F^{(1)} \\ F^{(2)} \\ \vdots \\ F^{(P)} \end{bmatrix} = \begin{bmatrix} 1 & \left(x_1^{(1)} - x_1^{nom}\right) & \cdots & \left(x_N^{(1)} - x_N^{nom}\right) & \left(x_1^{(1)} - x_1^{nom}\right)\left(x_2^{(1)} - x_2^{nom}\right) & \cdots & \left(x_{N-1}^{(1)} - x_{N-1}^{nom}\right)\left(x_N^{(1)} - x_N^{nom}\right) \\ 1 & \left(x_1^{(2)} - x_1^{nom}\right) & \cdots & \left(x_N^{(2)} - x_N^{nom}\right) & \left(x_1^{(2)} - x_1^{nom}\right)\left(x_2^{(2)} - x_2^{nom}\right) & \cdots & \left(x_{N-1}^{(2)} - x_{N-1}^{nom}\right)\left(x_N^{(2)} - x_N^{nom}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 1 & \left(x_1^{(P)} - x_1^{nom}\right) & \cdots & \left(x_N^{(P)} - x_N^{nom}\right) & \left(x_1^{(P)} - x_1^{nom}\right)\left(x_2^{(P)} - x_2^{nom}\right) & \cdots & \left(x_{N-1}^{(P)} - x_{N-1}^{nom}\right)\left(x_N^{(P)} - x_N^{nom}\right) \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_N \\ C_{12} \\ \vdots \\ C_{N-1\,N} \end{bmatrix}$$

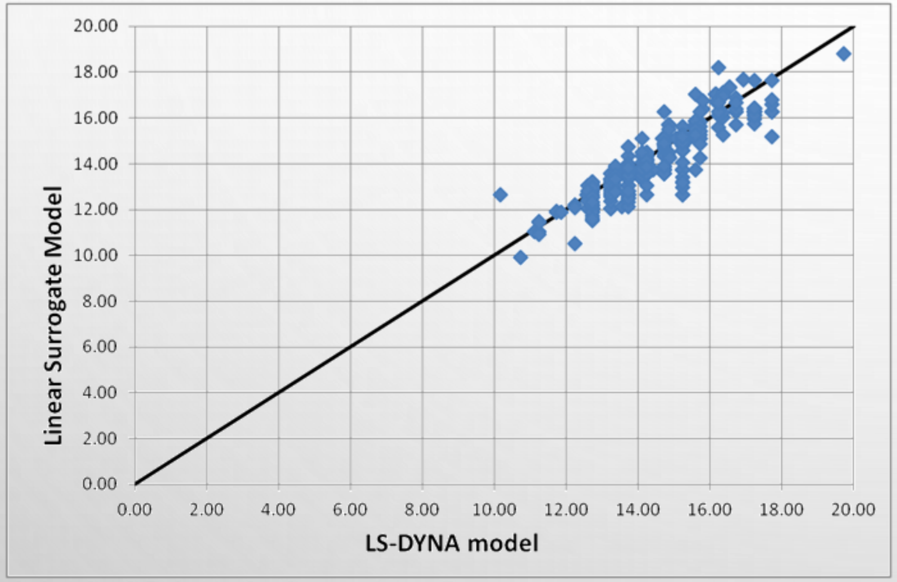
Submatrix for Linear Coefficients

Submatrix for Quatdratic Coefficients

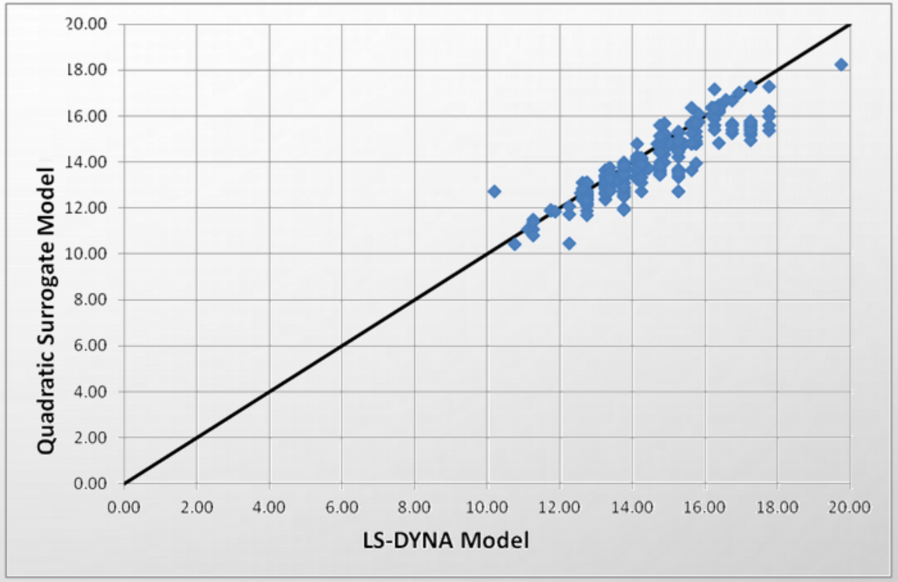
RELIABILITY OF A FRANGIBLE JOINT

How well do Surrogate Surfaces represent the actual (FEM) MCS?

Only Linear Terms



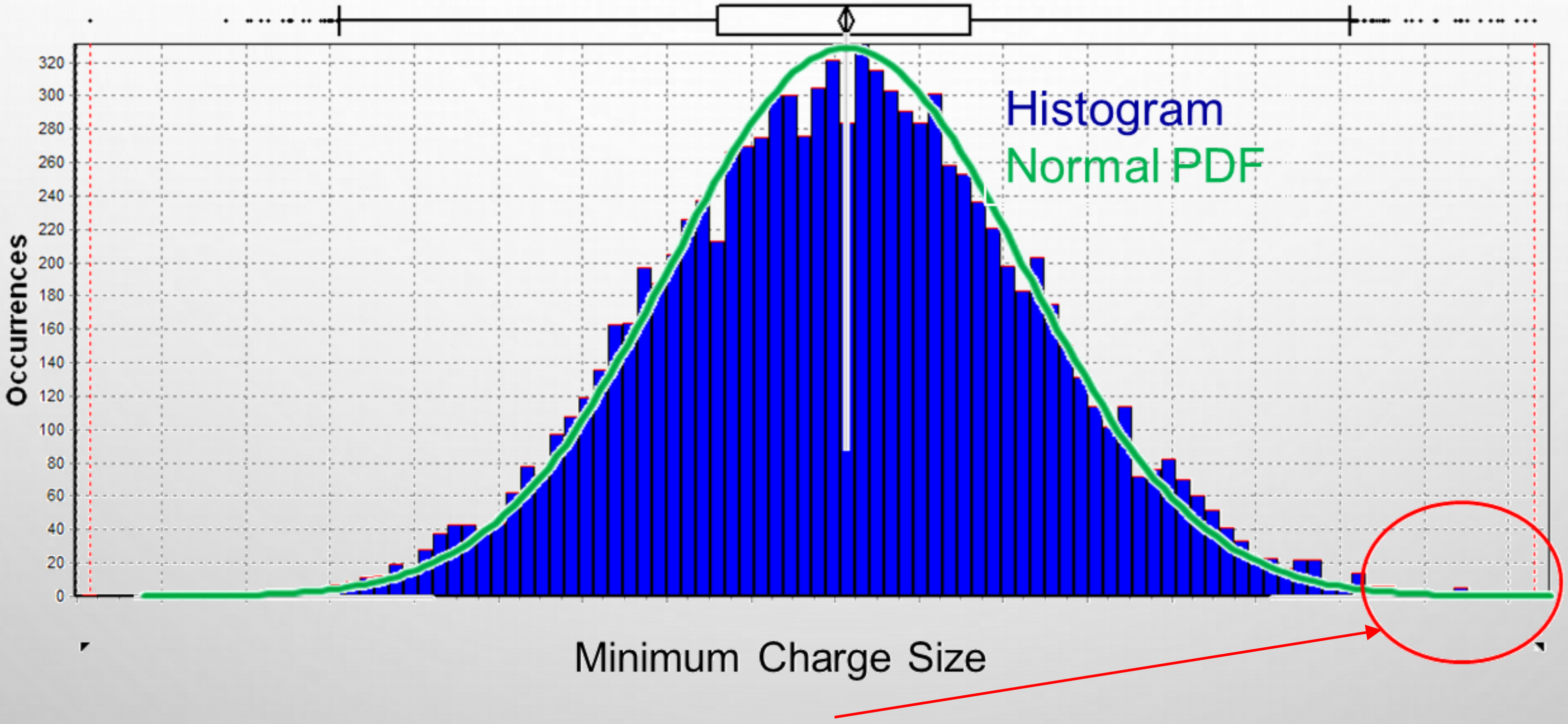
Includes Quadratic Interaction Terms



Quadratic Model is a slight improvement over the Linear Model

RELIABILITY OF A FRANGIBLE JOINT

Monte Carlo Simulation of the required strength of explosive (MCS) to effectively fracture the joint using Quadratic Surrogate Surface ***in lieu*** of FEM simulation



These randomly generated designs warrant investigation into their probability of occurrence since they require so much more explosive to fracture the joint.


QUESTIONS?
CONCERNS?
COMMENTS?