Existence Theorems and their Applications

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https://scholar.google.com/citations?user=aAHRdw0AAAAJ&hl=en&oi=ao

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Existence Theorems

- *A theorem stating the existence of an object, such as the solution to a problem or equation.*
- Generally, the existence theorems are of three types:
- (Type 1). Existence theorems that give explicit formulas for solutions, for example:

Cramer's Rule:

- Cramer's Rule (studied in linear algebra) is a method that uses determinants to solve systems of equations that have the same number of equations as variables.) It gives a condition for existence of the unique solution to system of linear equations as well as formulas to find this solution.

- *Cramer, Gabriel (1750). "Introduction à l'Analyse des lignes Courbes algébriques" (in French) . Geneva: Europeana. pp. 656–659. Retrieved 2012-05-18*

- Introduction to Line Analysis Algebraic Curves (English)

Existence Theorems

Wronskian: $W[f,g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

The second order differential equations of the type:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x) \;\ldots\ldots(1)$$

where P(x), Q(x) and f(x) are functions of x. The variation of parameters method states that:

if $W[f,g] \neq 0$*, Then (1) has exactly two linearly independent solutions. These solutions can be worked out by using Cramer's Rule.*

[1] Coddington, Earl A.; Levinson, Norman (1955). Theory of Ordinary Differential Equations. McGraw-Hill.

[2] Boyce, William E.; DiPrima, Richard C. (2005). Elementary Differential Equations and Boundary Value Problems (8th ed.). Wiley. pp. 186–192, 237–241.

Existence Theorems

(Type 2). ***The theorems which only state the condition(s) for the existence of the objects , for example:***

Bolzano-Weierstrass Theorem:

Version 1. Every bounded sequence of real numbers has a convergent subsequence

Version 2. Every bounded, infinite set of real numbers has a limit point

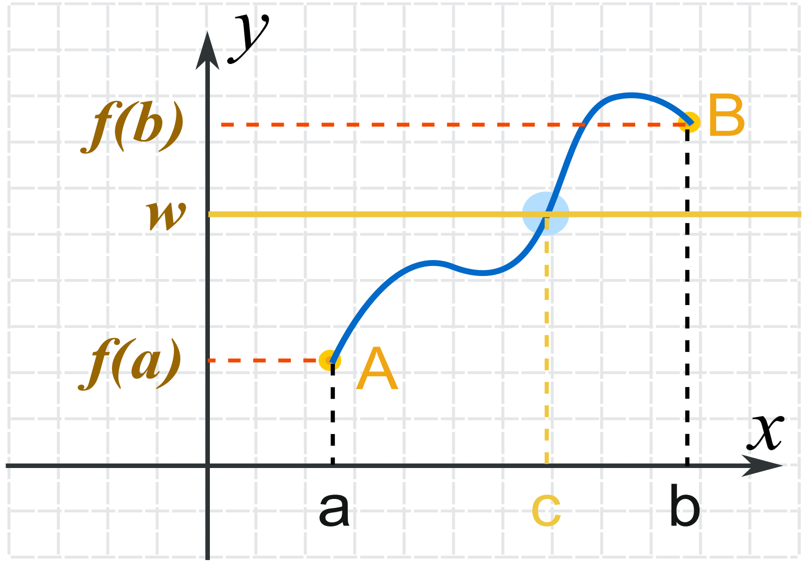
[1] C. Pugh, Real mathematical analysis. Second edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2015.

[2] K. Ross, Elementary analysis. The theory of calculus. Second edition. In collaboration with Jorge M. Lopez.

Undergraduate Texts in Mathematics. Springer, New York, 2013.

Existence Theorems

Intermediate Value Theorem

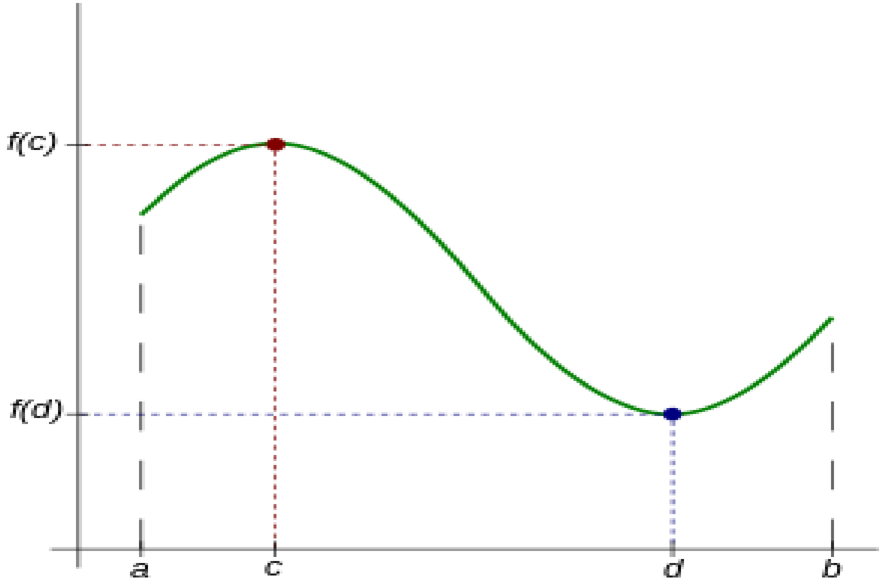


It states that if the real valued function f is continuous on $[a, b]$ and $f(a) < y < f(b)$. Then there exists$a < c < b$ such that

$$y = f(c).$$

[1] *O'Connor, John J.; Robertson, Edmund F., "Intermediate value theorem", MacTutor History of Mathematics archive, University of St Andrews*

Existence Theorems



Extreme Value Theorem

The extreme value theorem states that if a real-valued function f is continuous on the closed interval $[a, b]$, then f must attain a maximum and a minimum value, each at least once.

Rudin, Walter (1976). *Principles of Mathematical Analysis*. New York: McGraw Hill. pp. 89–90. ISBN 0-07-054235-X.

Existence Theorems

Mean Value Theorem

It states that:

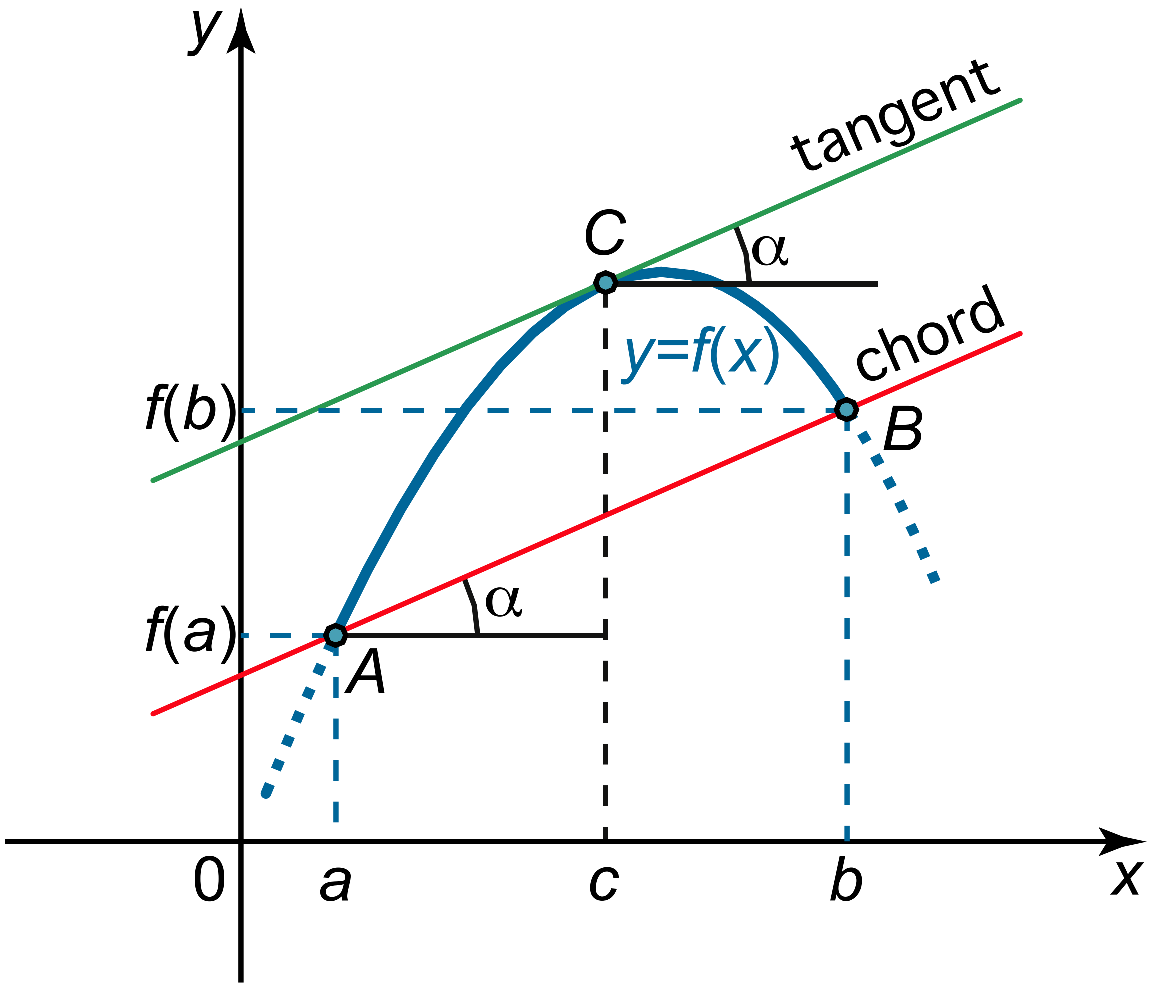
If the real-valued function $f: [a, b] \to \mathbb{R}$ is differentiable over (a, b) and continuous at $x = a$, and $x = b$. Then there exists a number $c \in (a, b)$ such that $f'(c)$ is equal to the function's average rate of change over the interval, that is

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Graphically, the theorem guarantees that an arc between two endpoints has a point at which the tangent to the arc is parallel to the secant through its endpoints.

Existence Theorems

Mean Value Theorem graphics



Existence Theorems

Mean Value Theorem applications

How do you prove positive derivative means increasing function: with the MVT.

How do you prove derivative 0 on an interval means constant function: with the MVT.

It is the first result which gives an explicit relation between values of f and f'.

Curve Sketching, the Fundamental Theorem, Taylor Series, and even Hospital's rules,

they all are refined versions of repeated applications of the MVT plus special conditions.

https://abesenyei.web.elte.hu/publications/meanvalue.pdf

Existence Theorems

Roll's theorem is a special case of mean value theorem. It states that when $\boldsymbol{f(a) = f(b)}$ then there exists a number $\boldsymbol{c \in (a, b)}$ such that $\boldsymbol{f'(c) = 0}$.

Peano existence theorem is a fundamental theorem which describe the condition for the existence of the solution to ordinary differential equation with some initial conditions. It states that the differential equations:

$$y'(x) = f(x, y(x));\ \ y(x_0) = y_0$$

has a unique solution if f is continuous.

[1] Murray, Francis J.; Miller, Kenneth S. (1976) [1954]. *Existence Theorems for Ordinary Differential Equations* (Reprint ed.). New York: Krieger

Existence Theorems

(Type 3) Existence theorems whose proofs involve iteration processes.

Lipschitz continuity

A function $f: \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous if there exists a positive real constant $k \geq 0$ such that

$$|f(x) - f(y)| \leq k|x - y| \qquad \forall\, x, y \in X.$$

The Picard-Lindelöf theorem, Picard's existence theorem, Cauchy-Lipschitz theorem, or existence and uniqueness theorem gives a set of conditions under which an initial value problem has a unique solution.

Lipschitz continuity is the central condition of the *Picard–Lindelöf theorem* which guarantees the existence and uniqueness of the solution to an *initial value problem*.

[1] Arnold, V. I. (1978). *Ordinary Differential Equations*. The MIT Press. ISBN 0-262-51018-9.

Existence Theorems

A special type of Lipschitz continuity, called *contraction* is used in the *Banach fixed-point theorem* (an existence theorem). It states that:

Let (X,d) be a complete metric space and $T: X \to X$ *satisfies*

$$d(Tx, Ty) \leq K\, d(x, y);\ \ 0 \leq K < 1.$$

*Then T has unique fixed point in X***.**

It is the fundamental existence theorem in the metric fixed-point theory. It appears to be a powerful tool for the existence of the solutions of mathematical models involving differential equations and others representing real world phenomena.

[1] Ciesielski, Krzysztof (2007). "On Stefan Banach and some of his results" (PDF). *Banach J. Math. Anal.* **1** (1): 1–10. doi:10.15352/bjma/1240321550.

Thank You