

More Linear Algebra, Please

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These slides are available at <http://gvsu.edu/s/1BJ>

Calculus is amazing. So is linear algebra.

“Studying any kind of engine (or for that matter, anything that moves), solid-state electronic device (from transistors to antennae) or power distribution network and doing any kind of design about them is literally impossible without calculus.”

<https://news.ycombinator.com/item?id=9189553>

“Its (linear algebra’s) applications touch many more students than calculus. We are in a digital world now.”

<http://web.mit.edu/18.06/www/Essays/too-much-calculus.pdf>



What's amazing about linear algebra

With special thanks to David Austin

The simplest of starting points*

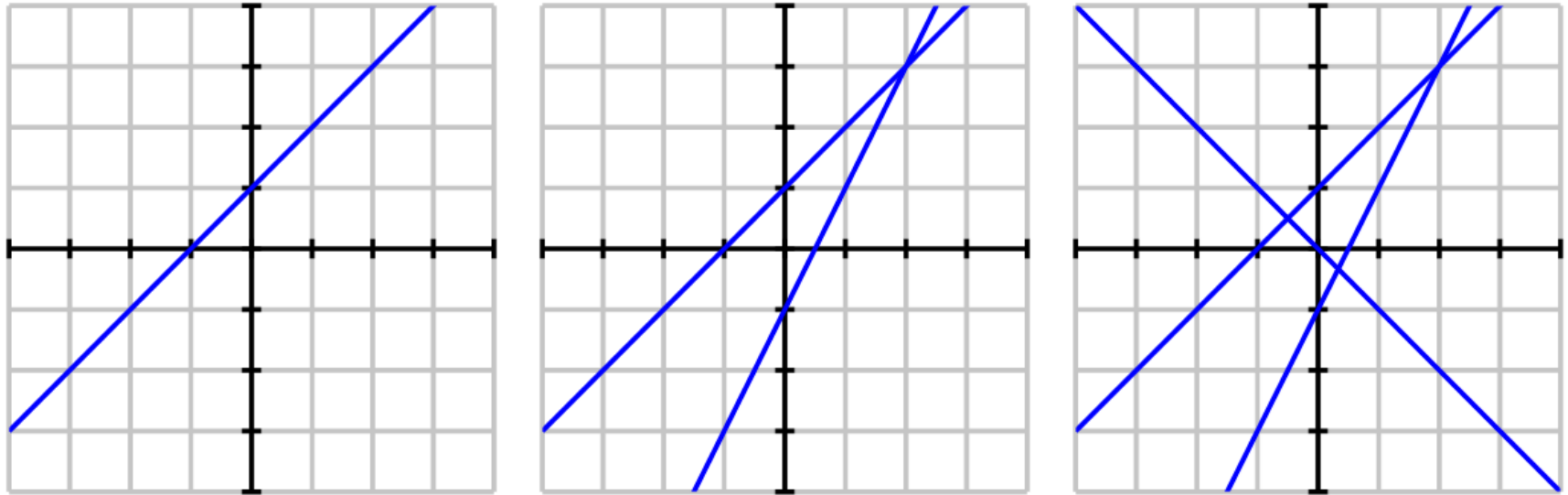


Figure 1.1.1. Three possibilities for collections of linear equations in two unknowns.

* most of the figures in this talk come from [*Understanding Linear Algebra*](#)

One idea, many perspectives

Example 2.2.5. Describe the solution space of the equation

$$\begin{bmatrix} 2 & 0 & 2 \\ 4 & -1 & 6 \\ 1 & 3 & -5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

By [Proposition 2.2.4](#), the solution space to this equation is the same as the equation

$$x_1 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix},$$

$$\begin{aligned} 2x_1 + 0x_2 + 2x_3 &= 0 \\ 4x_1 - 1x_2 + 6x_3 &= -5 \\ 1x_1 + 3x_2 - 5x_3 &= 15 \end{aligned}$$

which is the same as the linear system corresponding to

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 4 & -1 & 6 & -5 \\ 1 & 3 & -5 & 15 \end{array} \right].$$



One idea, many perspectives

Proposition 2.2.4. If $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then the following

are equivalent.

- The vector \mathbf{x} satisfies $A\mathbf{x} = \mathbf{b}$.
- The vector \mathbf{b} is a linear combination of the columns of A with weights x_j :

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n = \mathbf{b}.$$

- The components of \mathbf{x} form a solution to the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{b} \end{array} \right].$$

The role of basis

Strang: "We know everything when we know what happens to a basis."

Suppose that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

What is the matrix A ?



The role of basis

Strang: "We know everything when we know what happens to a basis."

Proposition 2.5.4. *If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation given by $T(\mathbf{x}) = A\mathbf{x}$, then the matrix A has columns $T(\mathbf{e}_j)$; that is,*

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)].$$

Proposition 4.3.3. *If A is an $n \times n$ matrix and there is a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ consisting of eigenvectors of A having associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then A is diagonalizable. That is, we can write $A = PDP^{-1}$ where D is the diagonal matrix whose diagonal entries are the eigenvalues of A*

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

and the matrix $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n]$.

Unreasonably effective

“... the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.”

- Eugene Wigner, 1960

“How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?”

- Albert Einstein, 1879-1955

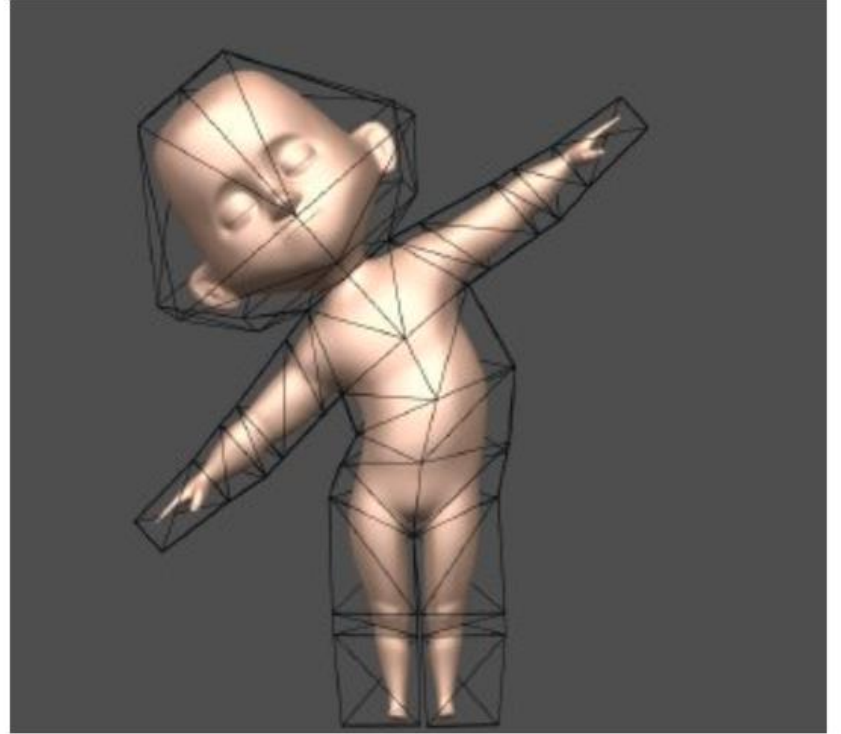
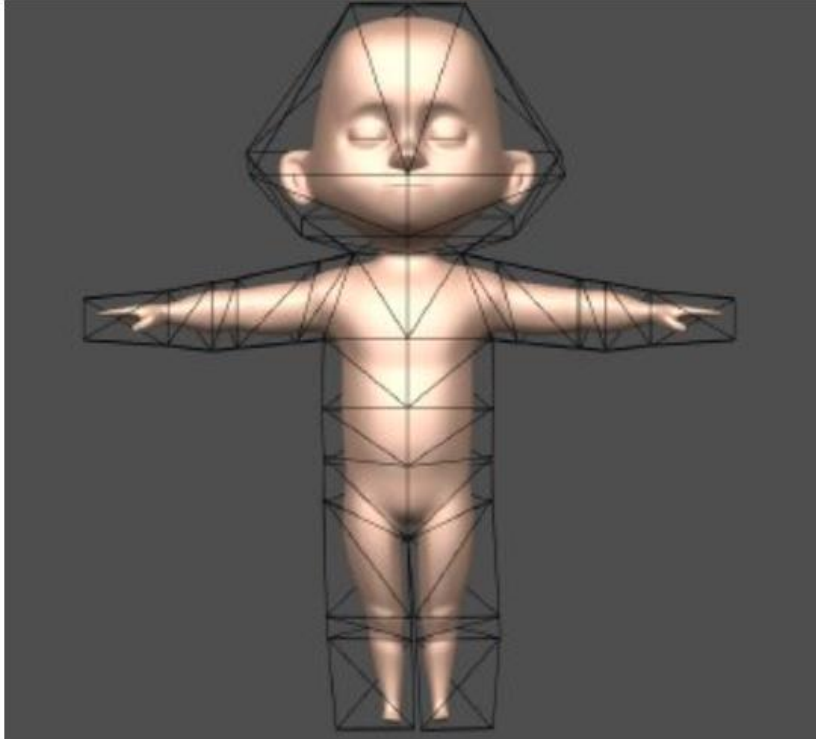
Brian Macdonald, on data science in sports analytics: “develop multivariable thinking!”

Dave Kung, via Pod Save America: “sell the brownie, not the recipe.”



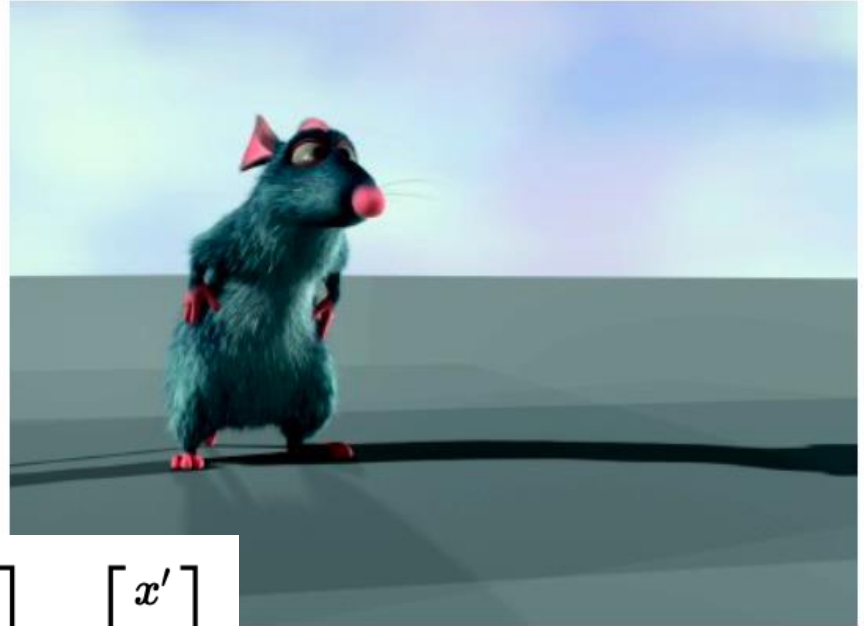
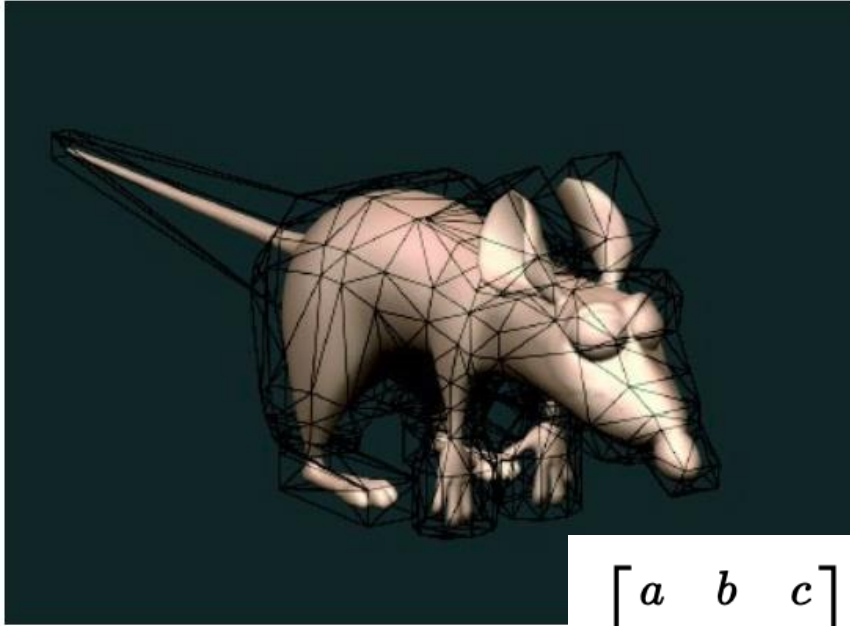
Some amazing applications

Computer graphics



[The Pixar Story](#)

Computer graphics



$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}.$$

<https://youtu.be/mX0NB9IyYpU?t=9>

Digital Images



Figure 3.2.10. A canyon wall in Capitol Reef National Park and the result of an edge detection algorithm.

Image Compression - the JPEG Compression Alg

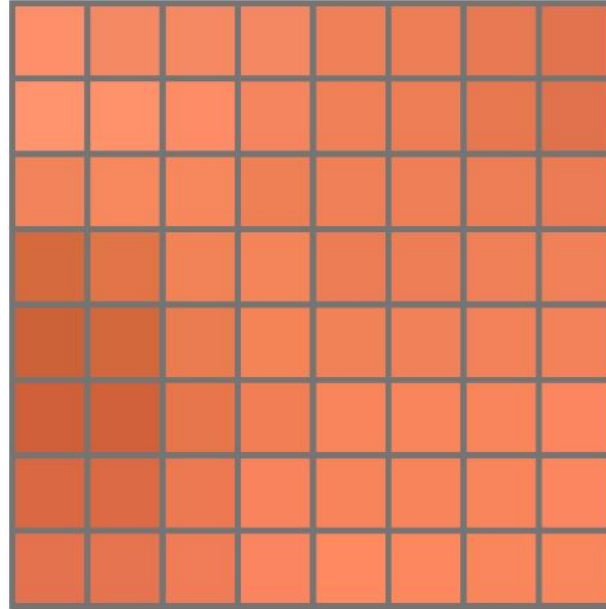
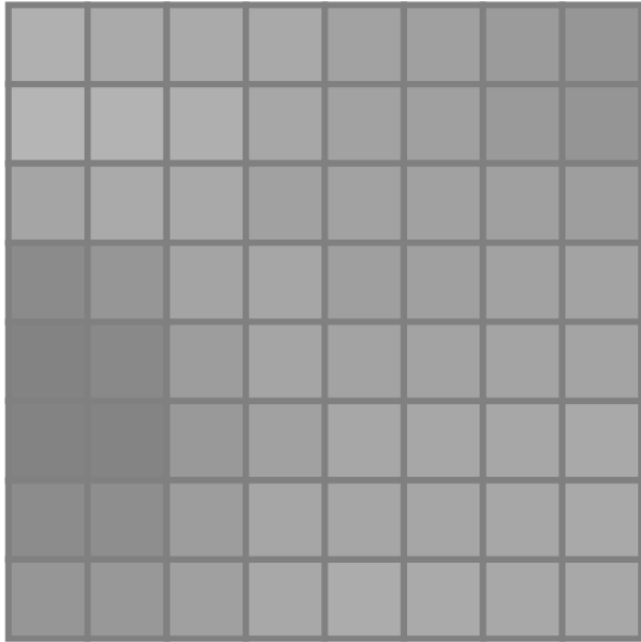


Figure 3.3.1. An image stored as a 1440×1468 array of pixels along with a close up of a smaller 8×8 array.

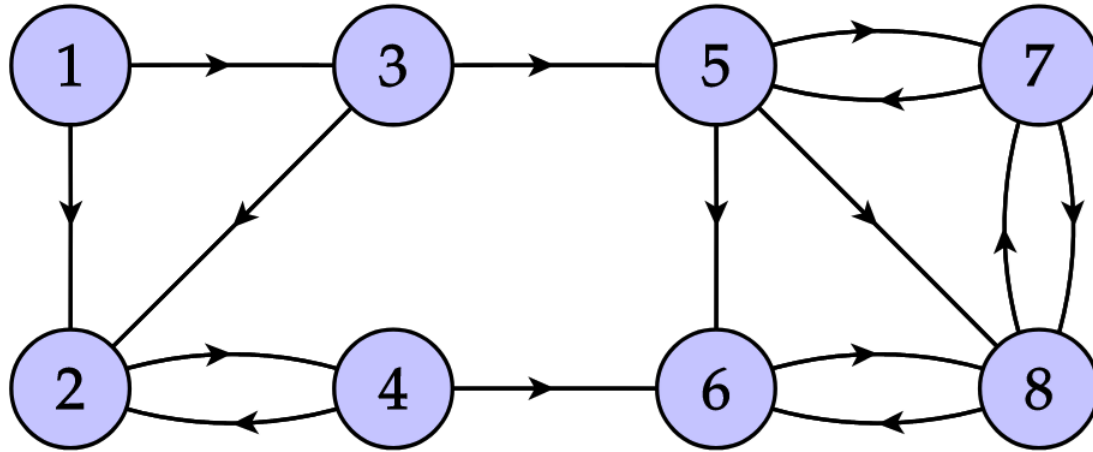
Images - the JPEG Compression Algorithm



176	170	170	169	162	160	155	150
181	179	175	167	162	160	154	149
165	170	169	161	162	161	160	158
139	150	164	166	159	160	162	163
131	137	157	165	163	163	164	164
131	132	153	161	167	167	167	169
140	142	157	166	166	166	167	169
150	152	160	168	172	170	168	168

Figure 3.3.6. The luminance values in this block.

The Google PageRank Algorithm



x_5 “gets” value from x_3
and x_7

x_5 “gives” value to x_6 ,
 x_7 , and x_8

$$x_5 = x_3/2 + x_7/2$$

Figure 4.5.10. Another model of the Internet with eight web pages.

There are currently about 2 billion websites, with fewer than 400 million of them “active”.

The Google PageRank Algorithm

$$x_1 = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

$$x_2 = \frac{1}{2}x_1 + 0x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

$$x_3 = \frac{1}{2}x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

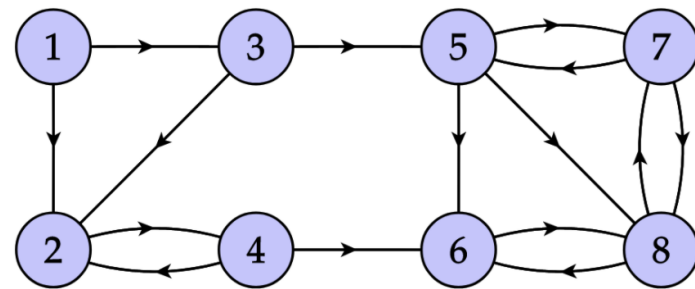
$$x_4 = 0x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

$$x_5 = 0x_1 + 0x_2 + \frac{1}{2}x_3 + 0x_4 + 0x_5 + 0x_6 + \frac{1}{2}x_7 + 0x_8$$

$$x_6 = 0x_1 + 0x_2 + 0x_3 + \frac{1}{2}x_4 + \frac{1}{3}x_5 + 0x_6 + 0x_7 + \frac{1}{2}x_8$$

$$x_7 = 0x_1 + 0x_2 + 0x_3 + 0x_4 + \frac{1}{3}x_5 + 0x_6 + 0x_7 + \frac{1}{2}x_8$$

$$x_8 = 0x_1 + 0x_2 + 0x_3 + 0x_4 + \frac{1}{3}x_5 + 1x_6 + \frac{1}{2}x_7 + 0x_8$$



Theory: solve $\mathbf{x} = G\mathbf{x}$

Practice: estimate by $\mathbf{x}_{k+1} = G\mathbf{x}_k$

So we iterate

The Google PageRank Algorithm

$$x_1 = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

$$x_2 = \frac{1}{2}x_1 + 0x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

$$x_3 = \frac{1}{2}x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

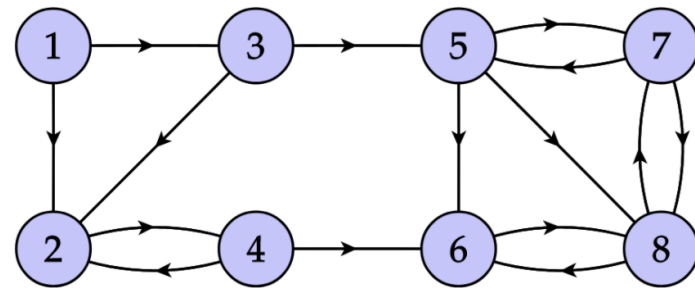
$$x_4 = 0x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8$$

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Theory: solve $\mathbf{x} = G\mathbf{x}$

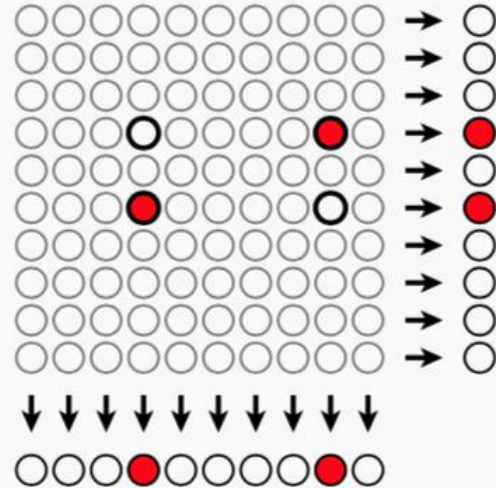
Practice: estimate by $\mathbf{x}_{k+1} = G\mathbf{x}_k$

So we iterate

Iterating is unreasonably effective!

COVID application: Dorfman Pooling

Variation: put each sample into two pools.

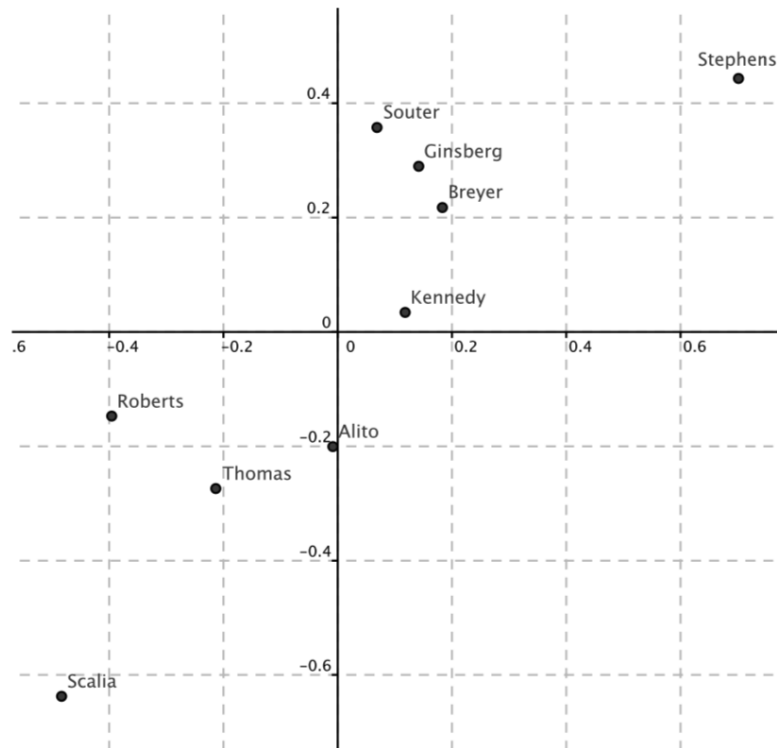


Tapestry, [a novel approach to pooled testing](#), uses more sophisticated (but still accessible) linear algebra.

The Singular Value Decomposition

Table 1. Percentage of agreement among U.S. Supreme Court justices.

	So	Ro	Gi	Th	Al	Ke	Sc	St	Br
So	100	97	83	80	86	87	87	80	82
Ro	53	100	56	91	94	100	100	63	66
Gi	87	83	100	77	77	100	80	73	86
Th	67	93	73	100	93	80	83	60	73
Al	82	93	82	82	100	100	82	75	82
Ke	61	68	64	46	68	100	57	57	73
Sc	50	81	53	95	87	82	100	45	58
St	81	59	75	59	56	75	56	100	69
Br	78	88	78	66	81	88	66	75	100





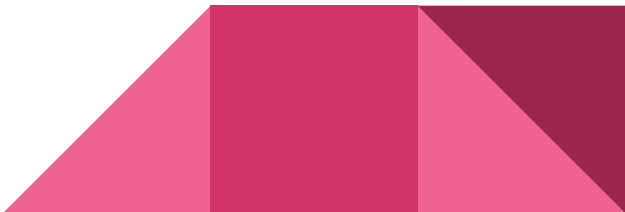
More Linear Algebra - please

For students

- + However much linear algebra you have learned: learn more.
- + The Singular Value Decomposition is one of the most valuable tools in all of linear algebra and finds a wide range of applications.

*“I went on to work in Data Science in Silicon Valley after graduating. Between working with LinkedIn and the company I started, I was always applying what I learned in your class, including holding my own with experienced researchers. **Every problem, every dataset, every domain required framing in matrix terms and applying SVDs.** I kept your notes for reference material, and I'm glad I did. More recently, I've been lecturing on machine learning with an academy, and your notes have once again proven invaluable, even 6 years into industry and teaching new students at an accelerated rate.”*

- + Read [David Austin's book](#)



For instructors

- + However much linear algebra your institution teaches, teach more -- especially applications. “Sell the brownie, not the recipe.”
- + The Singular Value Decomposition is one of the most valuable tools in all of linear algebra and finds a wide range of applications.
- + My colleagues and I wrote [an MAA FOCUS article](#) on the changes we’ve made to our linear algebra curriculum and how we’ve worked to bring key applications of linear algebra to our students early in their experience.



Other great reading

D. Austin, Moving Remy in Harmony: Pixar's Use of Harmonic Functions,

<http://www.ams.org/publicoutreach/feature-column/fcarc-harmonic>

D. Austin, How Google Finds Your Needle in the Web's Haystack,

<http://www.ams.org/publicoutreach/feature-column/fcarc-pagerank>


D. Kalman, A Singularly Valuable Decomposition

<https://www.tandfonline.com/doi/abs/10.1080/07468342.1996.11973744>

J. Remski et al, Singular Vectors' Subtle Secrets

<https://www.jstor.org/stable/10.4169/college.math.j.42.2.086?seq=1>

An [interview with Simon Funk](#) about using the SVD in the Netflix Prize Competition



Thank you

Huge thanks to David Austin, who has taught me a lot about linear algebra and created so many materials so others can learn more about it, too.

Thank you so much for attending today.

If you have additional questions, I'd welcome hearing from you directly at boelkinm@gvsu.edu.

And, these slides will remain available at <http://gvsu.edu/s/1BJ>

