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Minicourse M-R1 Introduction to Differential Equations of Stochastic Processes

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www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

Modus operandi

We will proceed through the slides and activities and ask you to think about the issues, pausing for some reflection with perhaps some contributions from you. Please use the Chat to ask questions.

We will have time for conversation at the end **or** any time during the conference - check out my avatar **or** through email, Director@simiode.org.

Outline of Minicourse

- Pose a real problem
- Introduce probability concepts
- Share notions and mathematics of Stochastic Processes
- Investigate role of differential equations in Stochastic Processes
- Special case - Poisson Process
- Demonstrate applications of Poisson Process
- Return to our real problem.

All materials/references are available at www.simiode.org and in the Conference Program for this Minicourse.

Source: Winkel, B. 2013. 1-027-T-StochasticProcesses-TeacherVersion. **https://www.simiode.org/resources/802**.

Real Problem

We have a process (human or machine processor or server) which has an hourly cost to operate (depends on skill and speed) and for which we make money for each customer process we complete.

Our customer is a human or a machine. Arrivals enter a finite line to be served and we make money off each service.

We wish to know **what level of service we should provide so as to maximize our profit**, knowing that we will lose customers who are turned away when our waiting line is full.

We know the hourly costs of our service, depending on skill and speed, and the charge for each of our completed services.

Mathematical Formulation of Real Problem

1. Suppose the cost to provide a service at an average rate of μ customers per hour is $c \cdot \mu$ dollars/hour.
2. We gross A dollars for every customer served.
3. Our system or line has a capacity of N customers, i.e. when there are N customers in line potential customers will walk away and we will lose their customer dollars.
4. Suppose our customers arrive on average at a rate of λ customers per hour. This is known from experience or design.
5. What service rate, μ, will provide maximum hourly profit?
6. That is, we seek to maximize money coming in from servicing customer less money going out for actual servicing effort.

What would a profit function look like?

(Money In!) Money we take in each hour by customers arriving, being able to get in line, getting serviced, and paying the amount A dollars each.

LESS

(Money out!) Money we spend on serveer in each hour.

What would a profit function look like?

(Money In!) Money we take in each hour by customers arriving, being able to get in line, and paying the amount A dollars each.

LESS

(Money out!) Money we spend on server in each hour.

Suppose we know the percentage of time the line is full, p_N, (we have N customers in the line) and we have to turn away customers.

Then what does $1 - p_N$ mean?

Recall variables:

1. μ is service rate of average number of customers per hour.
2. λ is arrival rate of average number of customers per hour who/which arrive at our line.
3. c the cost per unit of server (ability and speed).
4. N the capacity of our line above which our customers are turned away.
5. p_N the percentage of time the line is full.
6. A the revenue in dollars for each customer served.

Which of these are good models for money in and money out? Number? In or Out?

(1) $\lambda \cdot A$, **(2)** $c \cdot \mu$, **(3)** $A \cdot \mu$
(4) $\lambda \cdot A \cdot p_N$, **(5)** $c \cdot \mu \cdot p_N$
(6) $\lambda \cdot A \cdot (1 - p_N)$, **(7)** $c \cdot \mu \cdot (1 - p_N)$

How does this look for profit as a function of service rate, μ?

$$\mathcal{P}(\mu) = \lambda \cdot (1 - p_N) \cdot A - c \cdot \mu$$

What will we need in order to have a function of μ to optimize?

Let's go after it then! But first ...

What does it mean to be a random event? Examples and note one:

- **Number of names on nightly police blotter**.
- Number of cars passing spot on an interstate in time interval?
- Number of channel switches per minute performed on the television by the resident “couch potato” in the dorm lounge.
- Number of particle emissions per second from a isotope.
- Number of active phone calls at a given three digit exchange.
- Number of hits to your Facebook page per hour.
- Number of no-hitters per season in Major League Baseball.
- Number of people in the array of WalMart checkout counters.
- Number of requests at Amazon Service Desk per hour.
- Number of points scored per minute in college football game.
- Number of paramecia in a Petri dish each hour of a lab “run.”

- Consider, arrivals of telephone calls at a switching station in a given interval of time.
- Examine system over period of time $[t, t + h]$
- "State" is a common word with a technical definition which we will see often in our study now. This system is in state E_n at time t if exactly n messages (or more generally, events have taken place) have arrived in time interval $[0, t]$.
- We seek to build a mathematical model for $P_n(t)$, the probability that the system is in state E_n at time t.

Take one of the possible random events described in the list above:

[Consider "Number of names on nightly police blotter."]

1. Define what the state, E_n, might be for each n.
2. What is reasonable time frame over which to study this?
3. Explain what $P_n(t)$ means in each case for $n = 0, 1, 2, 3 \ldots$.
4. Attempt to sketch $P_n(t)$ for $n = 0, 1, 2$, and 100, on time interval $t \in [0, T]$. Think what $P_n(t)$ means for each n.
5. What does $P_0(0)$ mean and what value would it often take?

Offer reasonable assumptions on this random process.

Pertinent/relevant assumptions as well.

Formalize assumptions into mathematics about a system which moves from state to state randomly.

C1) If in time interval, $[t, t+h]$, h is very small, not too much can happen. Perhaps move from one state, say E_n, to an adjacent state $E_{n\pm1}$ only. We only have states, E_n, for integers $n \geq 0$.

C2) If in state E_n what happens in time interval $[t, t+h]$ depends only on current state E_n and the length of time interval h; **not** on current time or previous history.

C3) If in state E_n the probability that we move to a non-adjacent state, $E_{n\pm k}$, $k > 1$ in time interval $[t, t+h]$ is very small. Not too much can happen in tiny interval.

C4) It does not matter what time it is as to system changing state. Movements between states is independent of time.

(C4) indicates that actual time has no effect on the events. If time shapes events then we have a more complicated situation. But we are comfortable (are you?) with what we have going for us already.

These assumptions translate into axioms for a *stochastic process*.

Our system can be in any of the states: $E_0, E_1, E_2, E_3, \ldots, E_n, \ldots$

A process which satisfies the following axioms is a stochastic process and is a *general birth and death process*.

1. System changes only to its neighbors, E_n to E_{n+1} or E_{n-1} for $n \geq 1$, E_0 to E_1 only.

2. If at any time t the system is in state E_n, the probability that during the time interval $[t, t+h]$ the transition E_n to E_{n+1} occurs equals $\lambda_n h + o(h)$, and the probability that during the time interval $[t, t+h]$ the transition E_n to E_{n-1} occurs equals $\mu_n h + o(h)$. Here λ_n and μ_n say that these state transition probabilities might be a function of the state, E_n itself.

   $o(h)$ is very small when compared to h. So, the probabilities may not be exact, but error is very small when compared to h. Formally, $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

3. The probability that during the time interval $[t, t+h]$ more than one change occurs is $o(h)$.

4. The system's change is independent of t.

We have interest in $P_n(t)$, the probability a system is in state E_n at time t.

We have other interests, e.g., $\sum_{n=0}^{\infty} n \cdot P_n(t)$, the *mean state* (average value or expected value of the state) of the system at time t.

We formulate a mathematical model for $P_n(t)$, the probability that the system is in state E_n at time t.

We first obtain some information about $P_n(t+h)$, i.e. the probability that the system is in state E_n at time $t+h$. We derive this from the condition of the system at time t.

At time $t + h$ the system can be in state E_n if only if one of the following mutually exclusive events is true:

1. at time t the system was in state E_n and in the interval $[t, t+h]$ no change occurred;
2. at time t the system was in state E_{n-1} and a transition to state E_n occurred in the time interval $[t, t+h]$;
3. at time t the system was in state E_{n+1} and a transition to state E_n occurred in the time interval $[t, t+h]$; or
4. at time t the system was in state E_k where $|k-n| > 1$, i.e. more than two steps from E_n and two or more transitions occurred in the time interval $[t, t+h]$.

Since the events described in (1) - (4) above are mutually exclusive we can add their probabilities to get the probability that at time $t + h$ the system can be in state E_n.

We obtain the equations below for $P_n(t + h)$ for $n > 0$ by adding the probabilities of these mutually exclusive and exhaustive events.

Incidentally, the product of two probabilities, e.g., $P_{n-1}(t) \cdot \lambda_{n-1} h$ signifies two independent events, i.e. one does not portend information about the other, so we just multiply their probabilities.

Consider, $P_{n-1}(t) \cdot \lambda_{n-1} h$. First, the probability that the system is in state E_{n-1} at time t and second, the probability that the system moves from state E_{n-1} to state E_n in the time interval $[t, t + h]$.

$$
\begin{aligned}
P_n(t+h) &= P_n(t)(1-\lambda_n h-\mu_n h) \\
&\quad +P_{n-1}(t)\lambda_{n-1}h \\
&\quad\quad +P_{n+1}(t)\mu_{n+1}h+o(h)\,.
\end{aligned}
$$

For $n = 0$, we have

$$
\begin{aligned}
P_0(t+h) &= P_0(t)(1-\lambda_0 h-\mu_0 h) \\
&\quad\quad +P_1(t)\mu_1 h+o(h)\,.
\end{aligned}
$$

We call these general *birth and death equations*.

We move to a set of differential equations (goodie!) by examining first difference quotients,

$$
\begin{aligned}
P_n(t+h) - P_n(t) \quad &= \quad -P_n(t)(\lambda_n h + \mu_n h) + P_{n-1}(t)\lambda_{n-1} h \\
&\qquad + P_{n+1}(t)\mu_{n+1} h + o(h)\,,
\end{aligned}
$$

and then divide by h and take the limit as h goes to 0.

And so we have (recall $\lim_{h\to 0} \frac{o(h)}{h} = 0$) differential equations for $P_n(t)$ for integer values $n > 0$:

$$
P_n'(t) = \frac{dP_n(t)}{dt} = -P_n(t)(\lambda_n + \mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\mu_{n+1}\,.
$$

Using a similar process for $n = 0$ we can show that

$$P_0'(t) = \frac{dP_0(t)}{dt} = -P_0(t)(\lambda_0 + \mu_0) + P_1(t)\mu_1 \,. \tag{1}$$

And from above for integer values $n > 0$:

$$P_n'(t) = \frac{dP_n(t)}{dt} = -P_n(t)(\lambda_n + \mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\mu_{n+1} \,. \tag{2}$$

This is an infinite sytem of first order, ordinary differential equations to solve. Egads!

Let us consider some situations.

Pure birth process, i.e. $\mu_n = 0$ for all n says there is no death or loss in state value and $\lambda_n = \lambda \cdot n$ says that growth depends on the state E_n, as in birth rate proportional to population size.

For the growing police blotter we have a process called a *Poisson process*, i.e. $\mu_n = 0$ for all n says there are no erasures of names from the police blotter. Only names get added to the list and $\lambda_n = \lambda$ says that growth of the list DOES NOT depend on how many names are on the list. This Poisson process is used to test randomness in many phenomena as we shall see shortly.

Let us examine the Police Blotter or Roster situation.

Suppose our system is a Poison process, which involves the number of names on the police blotter or roster for the evening shift, i.e. we wish to model the number of names of persons "checked-in" or who called in about something (dog barking!) at the police station during the course of the evening, beginning at midnight ($t = 0$).

Our system will be in state E_n if there are n names on the roster and $P_n(t)$ is the probability that n names are on the roster at time t (say in minutes past midnight).

You can make an argument as to whether or not such events are random. Let us assume these arrivals on the police roster occur randomly throughout the night.

Here are some questions to consider in this situation:

- What does λ signify or mean? Let us see if we can find out.
- If $t = 0$ signifies the beginning of the night shift at midnight with clean slate, what is $P_0(0)$? What are $P_n(0)$ for all other $n > 0$?

  Say out loud to your self what $P_n(0)$ describes.

Pause.

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- If $t = 0$ signifies the beginning of the night shift at midnight with clean slate, what is $P_0(0)$? What are $P_n(0)$ for all other $n > 0$?

  Say out loud to your self what $P_n(0)$ describes.

$P_n(0)$ is the probabillty the system is in state E_n (or has n names on the roster) at time t and since the sheet is blank to start, we have $P_0(0) = 1$ while $P_n(0) = 0$ for $n > 0$.

Write out (1) in this case with initial condition $P_0(0) = 1$.

$$P_0'(t) = \frac{dP_0(t)}{dt} = -P_0(t)(\lambda_0 + \mu_0) + P_1(t)\mu_1 = -\lambda\lambda\,,$$

for in this case $\lambda_n = \lambda$ while $\mu_n = 0$ for all integers $n \geq 0$. OR Just

$$P_0'(t) = -\lambda \cdot P_0(t) \quad \text{with} \quad P_0(0) = 1\,.$$

Can we solve this? If so, what is the solution?

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$$P_0'(t) = \frac{dP_0(t)}{dt} = -P_0(t)(\lambda_0 + \mu_0) + P_1(t)\mu_1 = -\lambda P_0(t)\lambda\,,$$

for in this case $\lambda_n = \lambda$ while $\mu_n = 0$ for all integers $n \geq 0$. OR Just

$$P_0'(t) = -\lambda \cdot P_0(t) \quad \text{with} \quad P_0(0) = 1\,.$$

Can we solve this? If so, what is the solution? We sure can!

This is the good olde exponential decay, first order differential equation, so

$$P_0(t) = P_0(0)e^{-\lambda \cdot t} = e^{-\lambda \cdot t}\,.$$

Again, pause to say out loud and think about what the solution

$$P_0(t) = P_0(0)e^{-\lambda \cdot t} = e^{-\lambda \cdot t}\,,$$

to this differential equations

$$P_0'(t) = -\lambda \cdot P_0(t) \quad \text{with} \quad P_0(0) = 1\,.$$

really says in the context of the police blotter as the night moves on. What does the plot of the solution look like and does it make sense?

Pause.

Now, write out (2) for $n = 1$ first (and then we can do the same for $n = 2, 3, 4$, etc.)

$$P_n'(t) = \frac{dP_n(t)}{dt} = -P_n(t)(\lambda_n + \mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\mu_{n+1}\,.$$

Recall, $\lambda_n = \lambda \quad \text{and} \quad \mu_n = 0$. Thus, we have,

$$P_1'(t) = \frac{dP_1(t)}{dt} = -P_1(t)\lambda + P_0(t)\lambda\,.$$

Here if we knew $P_0(t)$ (and we do) from $P_0'(t) = -\lambda P_0(t)$ above, we could solve for $P_1(t)$.

$$P_1'(t) = \frac{dP_1(t)}{dt} = -\lambda P_1(t) + \lambda e^{-\lambda \cdot t} \quad \text{with} \quad P_1(0) = 0\,.$$

What solution strategy or technique would we use?

$$P_1'(t) = \frac{dP_1(t)}{dt} = -\lambda P_1(t) + \lambda e^{-\lambda \cdot t} \quad \text{with} \quad P_1(0) = 0\,.$$

What solution strategy or technique would we use?

Integrating factor? Separation of Variables? Mathematica? Maple? SAGE?

We obtain

$$P_1(t) = e^{-\lambda \cdot t}(\lambda \cdot t)\,.$$

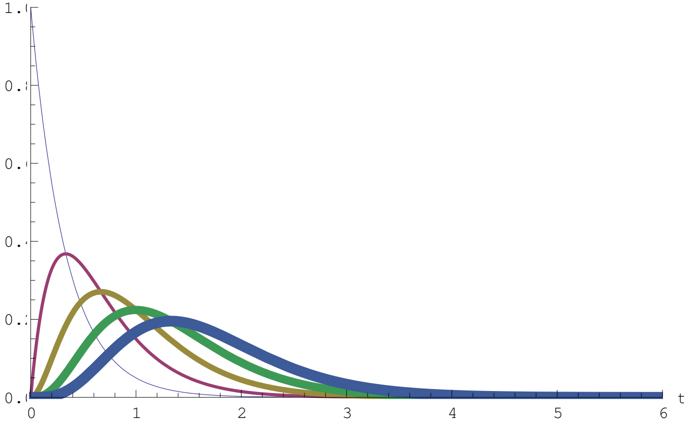
We can proceed to solve for

$$P_2(t) = \frac{1}{2} e^{-\lambda \cdot t}(\lambda \cdot t)^2\,,$$

and either through seeing a pattern or formal induction arrive at

$$P_n(t) = \frac{1}{n!} e^{-\lambda \cdot t}(\lambda \cdot t)^n\,.$$

What do these plots of $P_n(t)$'s tell us? Does this make sense?



Plots of $P_n(t)$ for $n = 0, 1, 2, 3, 4$ with $\lambda = 3$ in the Poisson process. Thinest plot for $n = 0$ and thickest for $n = 4$.

Always touch base with the modeled activity, so consider this.

A kind of reality check. What does $F(t) = 1 - P_0(t) = 1 - e^{-\lambda \cdot t}$ represent? Look like?

Always touch base with the modeled activity, so consider this.

A kind of reality check. What does $F(t) = 1 - P_0(t) = 1 - e^{-\lambda \cdot t}$ represent? Look like?

$F(t)$ is the probability that we have at least one arrival in our blotter in the time interval $[0, t]$ and we see that this becomes more and more likely as the evening moves on. Indeed it approaches 1.

Now we turn to ascertaining several things

1. mean or average state value of our system at time t,
2. the meaning and significance of our λ, and the
3. the long term probability or percentage of time our system spends in each state, E_n.

First, consider the expression for the mean or average state value, $M(t)$, of the system at time t

$$M(t) = \sum_{n=0}^{\infty} n \cdot P_n(t) = \sum_{n=0}^{\infty} n \cdot \left(\frac{1}{n!} e^{-\lambda \cdot t} (\lambda \cdot t)^n \right) .$$

Using basic series for e^x we can show that $M(t) = \lambda \cdot t$ and more importantly we can see that $M(1) = \lambda$.

So we now see λ is the average number of arrivals per minute.

Let us examine and then use the notion of the long term probability or percentage of time our system spends in each state, E_n.

$$M(t) = \sum_{n=0}^{\infty} n \cdot P_n(t) = \sum_{n=0}^{\infty} n \cdot \left(\frac{1}{n!} e^{-\lambda \cdot t} (\lambda \cdot t)^n \right) .$$

in particular if $P_n(t) = \frac{1}{n!} e^{-\lambda \cdot t} (\lambda \cdot t)^n$, we can define

$$p_n = P_n(1) = \frac{1}{n!} e^{-\lambda} (\lambda)^n ,$$

as the the percent of time in a 1 unit time interval that we are in state E_n, in the case of the police blotter the percentage of time we have exactly $n = 0, 1, 2, \ldots$ names on the blotter.

VERY IMPORTANTLY, this value of λ can be tabulated for given data to see if observed phenomena is Poisson, i.e. random.

Consider two dice and event E_n where n is the sum of the faces of rolled two six-sided, fair dice. Each number on a die has $1/6$ probability of turning up and for each sum value n we give the probability that the sum of the two dice thrown would be n in table below. See how each entry in the second column is computed?

Sum of Dice (k)	Probability ($P(k)$)
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

What then is Average or Mean Sum, namely, $M = \sum_{k=2}^{12} k \cdot P(k)$?

V-2 Rocket hits on London in World War II - Real Application!

Source: Clarke, R. D. 1946. An Application of the Poisson Distribution. *Journal of the Institute of Actuaries*. 72: 481.

During World War II, London was assaulted with German flying-bombs on V-2 rockets. The British were interested in whether or not the Germans could actually target their bomb hits or were limited to random hits with their flying-bombs.

In [Clarke1946] the analysis which led the British to determine whether or not the Germans could target their bombs or were merely limited to random hits is presented. R. D. Clarke, the author, said,

> *During the flying bomb attack on London, frequent assertions were made that the points of impact of the bombs tended to be grouped in clusters. It was accordingly decided to apply a statistical test to discover whether any support could be found for this allegation.*
>
> ⋮
>
> *The occurrence of clustering would have been reflected . . . by an excess of squares containing either a high number of flying bombs or none at all, with a deficiency in the intermediate classed.*

This analysis is important. For if the Germans could only target randomly, then deployment throughout the countryside of various security installations would serve quite well to protect them, as random bombing over a wide range was unlikely to hit a given target. However, if the Germans could actually target their flying-bombs, then the British were faced with a more potent opponent and deployment of security installations would do little to protect them.

The British mapped off the central 24 km by 24 km region of London into 1/2 km by 1/2 km square sectors. They recorded the number of bomb hits, noting their location.

# bomb hits (k) per area	0	1	2	3	4	5 and over
# areas with k bomb hits	229	211	93	35	7	1

Tally of the number of “flying-bomb” attacks on London in WWII.

There were 576 square sectors in London and $\lambda = 0.929$ is the average number of flying bomb hits per sector.

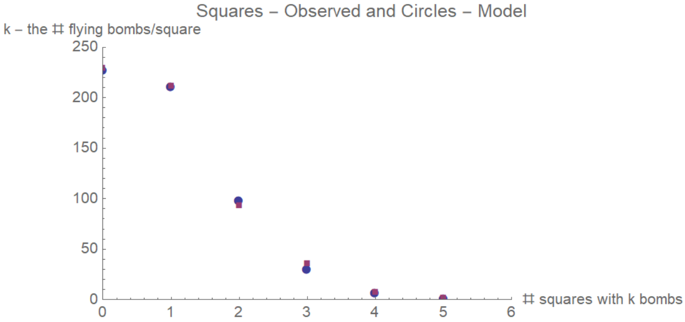
$$p_n = P_n(1) = \frac{1}{n!} e^{\lambda} (\lambda)^n \,,$$

where p_n is the average number of bombs per sector expected if the process was random.

MISSION: Perform an analysis to determine if the bombing was random or was capable of targeting. State your assumptions and offer up a complete analysis.

This is important to Winston Churchill!

$$p_n = P_n(1) = \frac{1}{n!} e^{\lambda} (\lambda)^n \,.$$



The theoretical data and the observed data are almost identical and so **the British concluded that the bombings were random and the Germans were incapable of "strict targeting."** Thus, they deployed resources widely, not clustered, for over a wide space the Germans were incapable of targeting small resource areas.

Review the problem and apply the Pure Birth and Death Process differential equations.

Real Problem

We have a process (human or machine processor or server) which has an hourly cost to operate (depends on skill and speed) and for which we make money for each customer process we complete. Our customer is a human or a machine. Arrivals enter a finite line to be served and we make money off each service.

We wish to know **what level of service we should provide so as to maximize our profit**, knowing that we will lose customers who turned away when our waiting line is full. We know the hourly costs of our service, depending on skill and speed, and the charge for each of our completed services.

Here we have $\lambda_n = \lambda$ (average arrival rate per hour) and $\mu_n = \mu$ (average service rater per hour) for $n = 0, 1, 2, \ldots, N$.

$$
\begin{aligned}
P_0'(t) &= \frac{dP_0(t)}{dt} = -P_0(t)(\lambda_0 + \mu_0) + P_1(t)\mu_1 \\
P_0'(t) &= \frac{dP_0(t)}{dt} = -P_0(t)(\lambda + \mu) + P_1(t)\mu\,.
\end{aligned}
$$

And from above for integer values $0 < n \leq N$:

$$
\begin{aligned}
P_n'(t) &= \frac{dP_n(t)}{dt} = -P_n(t)(\lambda_n + \mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\mu_{n+1} \\
P_n'(t) &= \frac{dP_n(t)}{dt} = -P_n(t)(\lambda + \mu) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu\,.
\end{aligned}
$$

This is an **finite** sytem of first order, ordinary differential equations to solve with $P_0(0) = 1$ and $P_n(0) = 0$ for $0 < n \leq N$.

One can solve them both for $P_n(t)$ the probability that the system is in state E_n at time t and in the discrete case for p_n for $0 \leq n \leq N$, the percentage of time the system is in state E_n, due to truncated nature of the processes. Finally, one can find p_N and then proceed to $\mathcal{P}(\mu)$, our profit function in terms of μ with given data λ, N, A, and c..

That is we can return to our problem of maximizing revenue with our new found (or accepted) $1 - p_N$

$$1 - p_N = \frac{\left(\left(\frac{\lambda}{\mu}\right)^N \left(1 - \frac{\lambda}{\mu}\right)\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

We seek μ to maximize $\mathcal{P}(\mu)$ for given data λ, N, A, and c.

$$\begin{aligned}
\mathcal{P}(\mu) &= \lambda \cdot (1 - p_N) \cdot A - c \cdot \mu \\
&= \lambda \cdot \frac{\left(\left(\frac{\lambda}{\mu}\right)^N \left(1 - \frac{\lambda}{\mu}\right)\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} \cdot A - c\mu\,.
\end{aligned}$$

Calculus here we come!

No-hitters in Baseball - Random Events?

Consider the phenomenon of no-hitters in baseball. A no-hitter for a pitcher is a 9 inning game in which the pitcher allows no hits! They are rare, but are they randomly distributed?

We examine the number of no-hitters (k) per season, and the number of seasons with k no-hitters. Using the Poisson analysis ascertain if no-hitters are randomly distributed among baseball seasons. Offer your analysis and defend your conclusions.

The following data relates to the number of no-hitters per season for Major League Baseball from the years 1876-1989, some 114 years of professional baseball history.

# no-hitters (k)/season	0	1	2	3	4	5	6	7+ and over
# seasons with k no-hitters	26	31	23	19	10	4	1	1

The number of seasons with k no-hitters in MLB from 1876-1989.

Question: Are no-hitters per season a random event? Offer an analysis and defend your decision on the question.

Offer a rationale for why you think your conclusion might hold, i.e. if no-hitters are random why? If not, why not?

Other questions: Do they occur more frequently near the end of the season when pitchers are at peak form or less frequently at the end of the season when pitchers are weak from throwing all season and batters or "on to their pitches"?

Rutherford's Experiments on Radioactive Disintegration

Lord Ernest Rutherford, the famous British physicist who worked in the early part of the twentieth century, was detecting radioactive disintegrations in his laboratory. His results are reported in his book.

Basically Rutherford took $N = 2608$ time intervals of 7.5 seconds each and counted the number of particles in each interval which reached a counter. Here are his data.

# particles (k)/interval	0	1	2	3	4	5	6	7	8	9	10
# intervals with k particles	57	203	383	525	532	408	273	139	454	27	16

The number of time intervals with k particles counted in these intervals from data compiled by the physicist Ernest Rutherford.

From this data, can you infer that radioactive disintegration is a random process? Write up your opinion. Defend your conclusion using the Poisson model approach.

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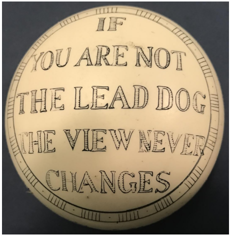

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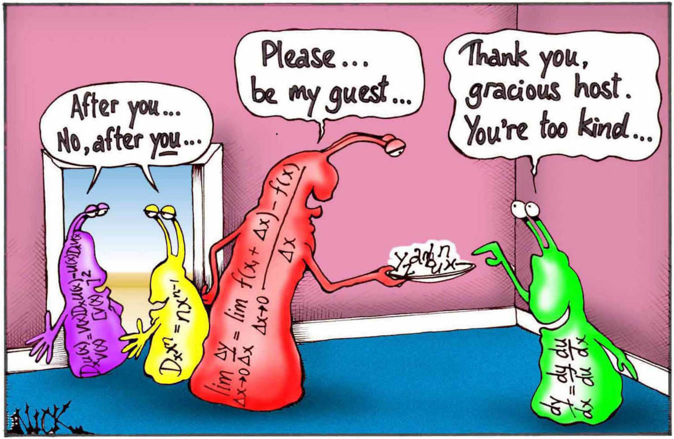
Discussions and Questions



Deferential equations.

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