A Taste of Control Theory

Kurt Bryan

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- The incubator sits in a lab with at a certain ambient temperature.
- The incubator must maintain its interior at a precise temperature using a heating/cooling element.
- How should such an element be controlled to maintain the interior temperature?

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PID control theory is very common in our technology and relies heavily on the Laplace transform.

It's a perfect application of the standard undergraduate material; the Laplace transform is indispensable here.

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The incubator will then maintain constant temperature...as long as nothing changes.

An Incubator Model: Newton's Law of Cooling

A simple model for the incubator temperature (without control) is

$$y'(t) = -k(y(t) - 70)$$

where $y(t)$ is the incubator temperature and 70 the ambient temperature. Here $k > 0$ is a constant.

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If we incorporate a heating element then

$$y'(t) = -k(y(t) - 70) + u(t)$$

where $u(t)$ quantifies a heat/cooling source under our control.

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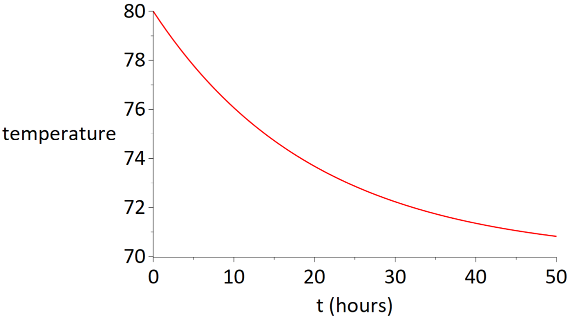

Figure: Incubator temperature $y(t) = 70 + 10e^{-t/20}$.

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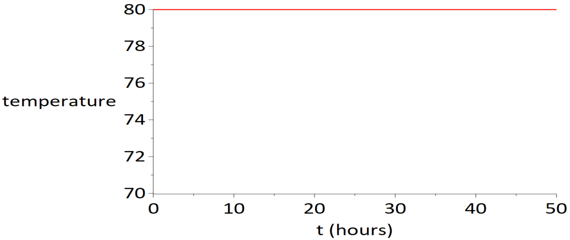

Figure: Incubator temperature with control $u(t) = 0.5$.

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But if the ambient temperature drops 5 degrees at some point:

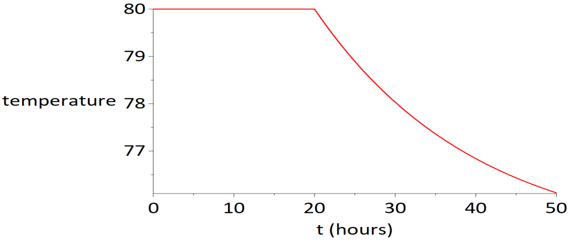

Figure: Incubator temperature if ambient drops 5 degrees at time $t = 20$.

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- The lab ambient temperature changes (someone opens the window...)
- Someone opens the incubator door to get something.
- The incubator thermal load changes (more/less stuff inside).
- The incubator itself changes over time (door seals leak, insulation degrades, heating elements age...)

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- Aeronautics—modern aircraft use control algorithms to stabilize the aircraft, augment pilot input.
- Electronics—virtually all circuits involve some kind of control, e.g., voltage, current, frequency.
- Many mechanical system employ active vibration control, e.g., certain vibration tables in labs.

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$$y'(t) = \underbrace{-0.05(y - 70)}_{\text{Newton loss}} + \underbrace{K_p(80 - y)}_{\text{control}}.$$

Proportional Closed Loop Control

Proportional control with $K_p = 1$ and a five degree drop in ambient gives the response in red (open loop $u = 0.5$ in blue):

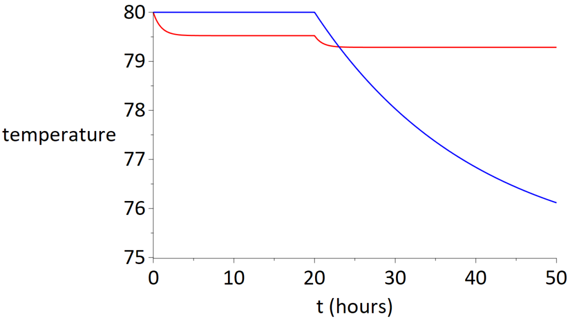

Figure: Open loop control, $u(t) = 0.5$ (blue) and proportional control $u(t) = K_p(80 - (t))$, $K_p = 1$ (red).

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This bias can be decreased by making K_p large, but this may overload the heating element (since $u = K_p(80 - y)$.)

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Looks scary. But in the Laplace s-domain it's easy to analyze.

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Solving the control integro-differential equation for $Y(s)$ is an algebra problem (just like for ODE's).

Proportional-Integral Closed Loop Control

In our case (with $K_p = K_i = 1$ and $y(0) = 80$)

$$Y(s) = \frac{80s^2 + 83.5s + 80}{s(s^2 + 1.05s + 1)}.$$

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$$Y(s) = \frac{80s^2 + 83.5s + 80}{s(s^2 + 1.05s + 1)}.$$

An inverse transform shows

$$y(t) = 80 - 0.59e^{-0.525t}\sin(0.851t).$$

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With no disturbance the temperature is

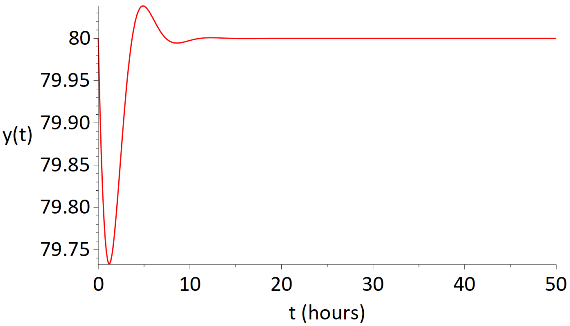
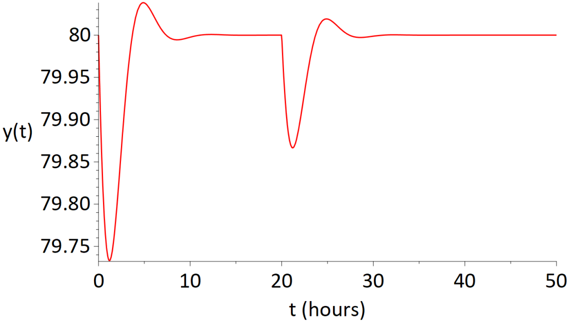


Figure: PI control, no disturbance.

Proportional-Integral Closed Loop Control

With a 5 degree drop in the ambient temperature at time $t = 20$ the PI-control incubator temperature is



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- Other fun applications include active control of a vibration isolation table, or balancing an upside-down pendulum (or Segway-type scooter).