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SIMIODE

A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

Workshop at SIMIODE EXPO 2022

10 February 2022, 1:30 - 2:10 PM Eastern US Time

Modeling a Falling Column of Water
Torricelli's Law

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All resources for this Workshop are available at the SIMIODE site at **https://www.simiode.org** in a reserved place at

- **Resource Guide** for Table of Contents
- **https://www.simiode.org/resources/488** for Student Version or
- **https://www.simiode.org/resources/463** for Teacher Version.
- Student Version is freely available to the public while Teacher Version is available only for teachers registered in SIMIODE which is FREE.

Example of Modeling Scenario - Let's Really Dive In

Modeling Falling Column of Water

We collect data on a falling column of water and model the height using first principles from physics with a differential equation.

1. Video, data, qualitative behavior, empirical model
2. First principles analytic model, Torricelli's Law
3. Differential equation, estimate parameters, validate model
4. Discussions

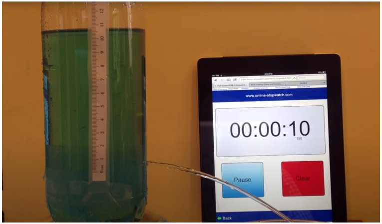
Source:
1-015-S-Torricelli **https://www.simiode.org/resources/488**
1-015-T-Torricelli **https://www.simiode.org/resources/463**

What Students Can Accomplish

Outline of modeling process

- seeing and collecting data,
- conjecturing empirical models,
- building an analytical model from scientific first principles,
- creating a differential equation model,
- solving of the differential equation,
- estimating parameter,
- comparing model with the actual data.

We use data taken from video at SIMIODE YouTube Channel



https://www.youtube.com/watch?v=NBr4DOj4OTE .

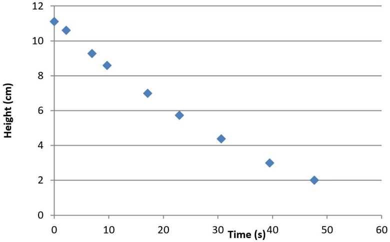
Cylindrical column (radius = 4.17 cm) of water empties through a hole (diameter = 11/16" = 0.218281 cm) in bottom of column. Exit hole at bottom of column - height is 0 cm. Others at **https://www.youtube.com/playlist?list=PLHUyzRr_S0TtP0gAUffhAS8FDccmLh0m8**. OR Just search SIMIODE YouTube.

We seek to model $h(t)$, the height of the column of water.

Here is data we collected. What do you see or notice?

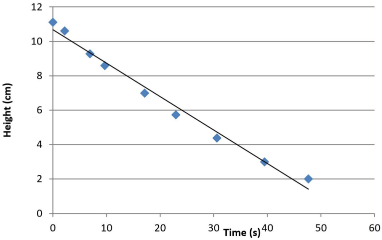
Make some observations now.

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



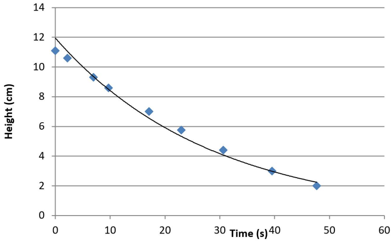
Linear Fit?

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



Exponential Decay Fit?

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



All are empirical fits with no understanding.

They just fit a function to data.

And neither line nor exponential are good.

From video of falling column we can observe.

What happens to height $h(t)$?

How fast is column of water falling? Early and later?

For large $h(t)$ the column of water falls faster.

For small $h(t)$ falls slower.

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0

Ideas about $\dfrac{dh(t)}{dt}$ - rate of change in height $h(t)$?

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0

Check out the average rate of falling of the height of the column of water in several intervals, say, $[0, 2.187]$,

$$\frac{10.6 - 11.1}{2.187 - 0} = -0.2286\,,$$

or in the interval $[39.503, 47.663]$,

$$\frac{2.0 - 3.0}{47.663 - 39.503} = -0.122549\,.$$

What do you see? What can you say about $h'(t)$?

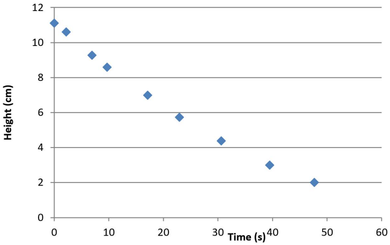
Let's find a model from some first principles.

This would be an analytic model.

NOT just fit a function to data.

NOT just "it looks like it falls faster or slower."

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



Enter Evangelista Torricelli 1608–1647, an Italian physicist and mathematician, and a student of Galileo. Best known for his invention of the barometer. Known for his wicked mustache!

Torricelli's Law to the rescue!

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}, \quad h(0) = h_0 \quad b > 0.$$

Say it out loud in sentence form.

Explain to yourself what it means.

Does Torricelli's Law agree with observations?

For large $h(t)$ the column of water DOES fall faster.

For small $h(t)$ the column of water DOES fall slower.

We build the model that IS Torricelli's Law from First Principles.

This will be an analytic model.

Basically, The Law of Conservation of Energy says that total energy is conserved.

We will apply it to a slab of water, first at the surface of the column of water and then at the bottom of the column ($h = 0$)

Total Energy is the the sum of the **potential energy** and the **kinetic energy** of a particle of mass m and this sum is constant at each instance in time, t.

Use the Law of Conservation of Energy to derive Torricelli's Law, by computing the Total Energy of a slab of water (initially) when it is at the top of the column of water and then when it is (finally) at the bottom of the column of water at the exit spout level.

$$TE_i = KE_i + PE_i = \frac{1}{2}mv_i^2 + mgh = \frac{1}{2}m0^2 + mgh = mgh\,.$$

$$TE_f = KE_f + PE_f = \frac{1}{2}mv_f^2 + mg\cdot 0 = \frac{1}{2}mv_f^2\,.$$

Now applying the Law Conservation of Energy, $TE_i = TE_f$, we build an equation - see the equal sign!!

$$TE_i = mgh = \frac{1}{2}mv_f^2 = TE_f\,,$$

We employ an effective tool in modeling: **equate two ways of computing** loss in volume of water in tank at time t - (1) Falling slab of water on the top and (2) water exiting small bore hole at bottom, to arrive at

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)} = -b\sqrt{g} \cdot (h(t))^{1/2}.$$

We have arrived at our model and now we seek to determine b and validate our model.

The parameter b contains (or "hides") information, e.g., cross sectional area of cylinder (presumed constant), size of opening spout at bottom of cylinder, viscosity, temperature, friction on the inside surface of the cylinder and spout hole.

We solve this differential equation for $h(t)$ to realize a model. What strategy/technique can we employ? What technology?

We use this solution and our data to estimate parameter b and validate our model by comparing model predictions to data.

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)} = -b\sqrt{g} \cdot (h(t))^{1/2}.$$

Separate the variables

$$(h(t))^{-1/2} \cdot \frac{dh(t)}{dt} = -b\sqrt{g}.$$

$$(h(t))^{-1/2} \cdot \frac{dh(t)}{dt} = -b\sqrt{g}.$$

OR

$$(h(t))^{-1/2} \cdot dh = -b\sqrt{g} \cdot dt.$$

Integrate both sides. (What is C?)

$$\int (h(t))^{-1/2} \cdot \frac{dh(t)}{dt}\, dt = \int -b\sqrt{g}\, dt + C\,,$$

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

Now to find C using Initial Conditions:

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

$$2(h(0))^{1/2} = -b\sqrt{g} \cdot 0 + C = C.$$

Thus we have

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + 2(h(0))^{1/2}.$$

Divide both sides by 2 and then square both sides yields:

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2. \tag{1}$$

This is model for height of the column of water, $h(t)$, at time t.

What do we know and what do we need to estimate b in (1)?

$h(0) = 11.1$ cm and $g = 980$ cm/s^2

Thus from $h(0) = 11.1$ cm and $g = 980$ cm/s^2

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2} \right)^2$$

becomes

$$h(t) = \left(-\frac{b\sqrt{980}}{2} \cdot t + (11.1)^{1/2} \right)^2 ,$$

and expanded in decimals we have

$$h(t) = (-15.6525 \cdot b \cdot t + 3.33166)^2 . \tag{2}$$

We have arrived at our model and now we seek to determine b and validate our model.

We turn to our Excel spreadsheet and seek to determine the parameter b which minimizes the sum of the squared errors between our data (h_i) and our model $(h(t_i))$ over our data points.

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

Minimize as a function of the parameter b:

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

where

- t_i is the i^{th} time observation,
- h_i is the observed height at time t_i,
- $h(t_i)$ is our model's prediction of the height at time t_i, and
- $n = 9$ is the number of data points we have.

Model Analysis in Excel Using Solver

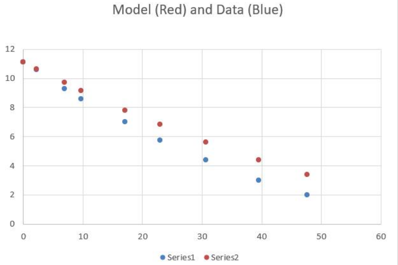
Data collected Friday, 5 August 2016 by Brian Winkel

SOURCE for Data
https://www.youtube.com/watch?v=NBr4DOj4OTE
Radius of hole 11/64" = 0.218281 cm and radius of cylinder 4.17 cm

Model h'(t) = -b Sqrt(g h(t))

Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2

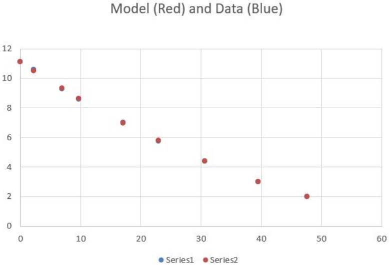
				b=	0.002
	Zeroed	**Actual**	**Model**	**SSE**	
Time (s)	**Time**	**Height (cm)**			
8.679	0	11.1	11.09995836	1.73426E-09	
10.866	2.187	10.6	10.64844791	0.0023472	
15.612	6.933	9.3	9.700872913	0.160699092	
18.396	9.717	8.6	9.165570453	0.319869938	
25.781	17.102	7	7.819192408	0.671076202	
31.647	22.968	5.75	6.825923093	1.157610501	
39.282	30.603	4.4	5.634134022	1.523086785	
48.182	39.503	3	4.389102871	1.929606788	
56.342	47.663	2	3.384016994	1.915503039	
			Total SEE	7.679799546	



We can use Excel's Solver to minimize the TOTAL SEE or SSE which is currently 7.679799546 with parameter $b = 0.002$ by **asking Solver** to minimize SEE or SSE as a function of $b = 0.002$ cell.

Parameter Estimation with Excel Solver - Results

Model h'(t) = -b Sqrt(g h(t))					
Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2					
				b=	0.002581
	Zeroed	Actual	Model	SSE	
Time (s)	Time	Height (cm)			
8.679	0	11.1	11.09995836	1.73426E-09	
10.866	2.187	10.6	10.51908638	0.006547014	
15.612	6.933	9.3	9.312232219	0.000149627	
18.396	9.717	8.6	8.638501405	0.001482358	
25.781	17.102	7	6.973871293	0.000682709	
31.647	22.968	5.75	5.778477118	0.000810946	
39.282	30.603	4.4	4.390799244	8.46539E-05	
48.182	39.503	3	3.013347567	0.000178158	
56.342	47.663	2	1.977592125	0.000502113	
			Total SEE	0.010437581	



Note: TOTAL SEE was at 7.679799546.

SUCCESS in modeling real world phenomenon!

Or let's go live to the Excel notebook in the Teacher Version of this Modeling Scenario!

Watch carefully because things can happen fast!

We explain Solver for you.
Many faculty have not seen Solver in Excel.

Live Excel Sheet for our data

KEY POINT: In traditional course the differential equation is presented and students work on techniques, often with no knowledge of where the differential equation comes from or what its purpose is.

SIMIODE motivates with modeling.

Assignment

1. Write an overview of the modeling process to obtain $h'(t)$ using first principles - not all details, just highlights. Arrive at the model

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2 .$$

2. Collect data for your team's cylinder from SIMIODE YouTube Channel or **https://www.simiode.org/resources/488**.

   What if we took many data points? Few data points? Try both by taking a subset of ALL the data points you took for "few data" points and see how your parameter b fares.

In Excel we do these steps to determine best fit parameter b:

3. Compute our model value of the height $h(t_i)$ at time t_i.
4. Take the differences between actual data and model prediction, i.e. $h_i - h(t_i)$, and square these differences, $(h_i - h(t_i))^2$.
5. Sum these square errors to obtain $SSE(b)$.
6. Use Excel's Solver to minimize $SSE(b)$.
7. Read the value of b and put it in our model as best parameter estimate of b.
8. Plot our best model values on the same axes as our data and compare.
9. Collect the parameters b for the various outflow hole sizes from different videos selected by teams and see if there is any relationship between hole size and b.

This is what Modeling Scenarios in SIMIODE are all about in using modeling to teach differential equations. In this case,

1. collecting data,
2. modeling the phenomenon with first order, nonlinear ODE,
3. solving the ODE using separation of variables,
4. determining the best parameters,
5. validating the modeling process,

and

6. communicating results.

Thank you.

Modeling LSD in the Blood Stream and Tissues

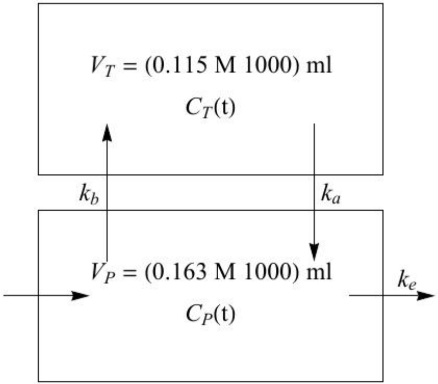
	Time (hr)	0.0833	0.25	0.5	1.0	2.0	4.0	8.0
Subject 1	Plasma Conc (ng/ml)	11.1	7.4	6.3	6.9	5.	3.1	0.8
	Perform Score (%)	73	60	35	50	48	73	97
Subject 2	Plasma Conc (ng/ml)	10.6	7.6	7.	4.8	2.8	2.5	2.
	Perform Score (%)	72	55	74	81	79	89	106
Subject 3	Plasma Conc (ng/ml)	8.7	6.7	5.9	4.3	4.4	—-	0.3
	Perform Score (%)	60	23	6	0	27	69	81
Subject 4	Plasma Conc (ng/ml)	10.9	8.2	7.9	6.6	5.3	3.8	1.2
	Perform Score (%)	60	20	3	5	3	20	62
Subject 5	Plasma Conc (ng/ml)	6.4	6.3	5.1	4.3	3.4	1.9	0.7
	Perform Score (%)	78	65	27	30	35	43	51

Summary of data collected on 5 male volunteers who were given LSD and then tested on simple addition questions.

Source: Metzler, C. M. 1969. A Mathematical Model for the Pharmacokinetics of LSD Effect. *Clinical Pharmacology and Therapeutics*. 10(5): 737-740.

SIMIODE: **5-001-S-LSDAndProblemSolving**
OR **5-001-T-LSDAndProblemSolving**

Two compartment model for LSD flow between plasma and tissue compartments in the human body.



$$
\begin{aligned}
V_P C_P'(t) &= k_a V_T C_T(t) - k_b V_P C_P(t) - k_e V_P C_P(t) \\
V_T C_T'(t) &= k_b V_P C_P(t) - k_a V_T C_T(t) .
\end{aligned}
$$

Use Mathematica to solve this system of differential equations

$$
\begin{aligned}
V_P C_P'(t) &= k_a V_T C_T(t) - k_b V_P C_P(t) - k_e V_P C_P(t) \\
V_T C_T'(t) &= k_b V_P C_P(t) - k_a V_T C_T(t) \, .
\end{aligned}
$$

with parameters we share a peek at part of the solution for $C_P(t)$.

$$
\frac{6.13497 \left(1.k_a e^{1.t\sqrt{(1.k_a+1.k_b+1.k_e)^2-4.k_b k_e}} - 1.k_b e^{1.t\sqrt{(1.k_a+1.k_b+1.k_e)^2-4.k_b k_e}} + 1.k_e e^{1.t\sqrt{(1.k_a+1.k_b+1.k_e)^2-4.k_b}}\right.}{}
$$

$\cdots$

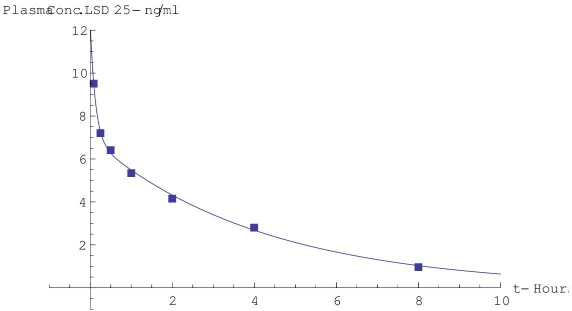
$\cdots$

$\cdots$

And we form the sum of square errors between the data and model, in terms of parameters $k_a, k_b, k_e, \ldots$

$$SSE(k_a, k_b, k_e) = \sum_{i=1}^{7} (C_P(t_i) - O_i)^2 \tag{3}$$

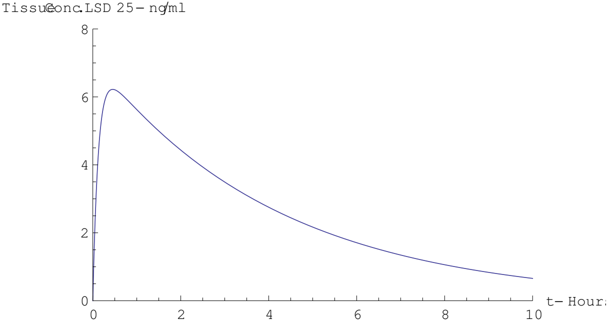
Minimizing $SSE(k_a, k_b, k_e)$ we obtain the parameters $k_a = 4.63679$, $k_b = 3.18659$, and $k_e = 0.41128$.



Plot of the observed values of the average concentration of LSD (ng/ml) (squares)in plasma and the model.

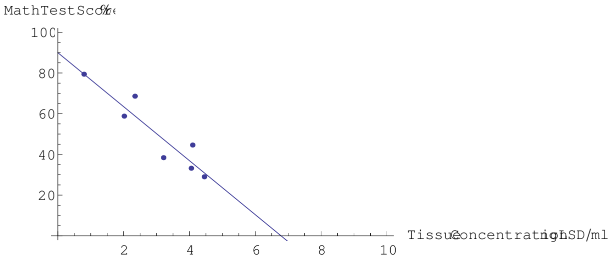
$$C_P(t) = 0.128905\left(41.2194e^{-7.99617t} + 53.9669e^{-0.238492t}\right) .$$

$$C_T(t) = 0.128905\left(55.419e^{-0.238492t} - 55.419e^{-7.99617t}\right).$$



Plot of the model of tissue concentration of LSD in ng/ml.

This means with a model we can watch the concentration of LSD in the body tissue over time from just observing the concentration of LSD in the plasma (or blood) from regular draws.



Plot of Performance Score (%) (PS) on the simple arithmetic problems vs. the model prediction of the concentration of LSD in ng/ml in the tissue compartment (CT).

This means for every ng/ml increase in LSD in the tissue compartment the score drops a little over 9 points.