Modeling the Velocity of a Pull-Back Toy

Modeling Scenario Workshop - EXPO '22

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Abstract

This workshop focuses on a modeling scenario that examines the motion of a simple pull-back toy using a second-order linear differential equation [2]. Students can collect their own data by racing pull-back toys, or use provided data, and carry out parameter estimation by comparing a numerical solution to the experimental data. There are three goals for the workshop. First, participants will learn a bit about how the project progressed from a rough idea to a published modeling scenario. Second, participants will have the opportunity to play the role of students by implementing one of the two proposed models for the pull-back toy with real data. Finally, participants will hear about an implementation of the modeling scenario in two levels of calculus, including students' perspectives on the project.

How did we get involved with SIMIODE?

Tova Brown is an Assistant Professor of Mathematics at Wisconsin Lutheran College in Milwaukee, Wisconsin. Tova participated in the SIMIODE Practitioner's Workshop (MINDE) in June 2021.

Brody Johnson is an Associate Professor of Mathematics at Saint Louis University in St. Louis, Missouri. Brody participated in the SIMIODE Developer's Workshop (DEMARC) in June 2021.

Agenda for the Workshop

1. **Overview of the Creation Process**
2. **Hands-On Modeling**
3. **Report on Classroom Implementation**

Part 1:

Overview of the Creation Process

What Makes an Effective Modeling Scenario?

A **modeling scenario** is a pedagogical tool "in which a modeling situation, rich in detail, motivates the study of differential equations."[1]

Modeling scenarios...

- range from short in-class activities to culminating end-of-semester projects.
- are easier to implement when they blend in with the instruction of course content.
- are more engaging when they allow students to collect and/or use real data.
- are more compelling when they involve parameter estimation or open-ended questions that illustrate the usefulness of differential equations.

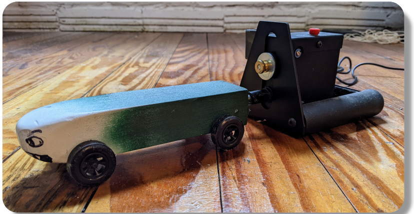
[1] `https://www.simiode.org/resources`

Origins of the Pull-Back Toy Modeling Scenario

Brody's application for DEMARC included a sample modeling scenario that was used as a course project in his six-week differential equations class in Summer 2010.

CO_2 Powered Pinewood Derby Car

The primary goal of the project is to estimate the maximum speed of the car due to the thrust provided by the high-pressure expelled gas.



Drawbacks:

- Special equipment is required to launch the car.
- A lot of space (as well as permission) is needed to conduct experiments.
- The discharge of gas from the cartridge is complicated.

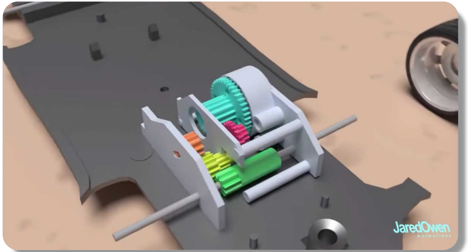
The Pull-Back Toy

Pull-back toys are powered by a clever wind-up motor and have the advantage of being inexpensive, widely available, and easy to use.

Brody used a pull-back toy (left) that was acquired as cereal box prize in the early 2000s. Tova sourced pull-back toys (right) for her entire class at unit cost of about 25¢.

The Pull-Back Motor

Anyone who likes toys is encouraged to check out the animation of a pull-back motor created by Jared Owen on YouTube [1].



Two sets of gears work inside the motor.

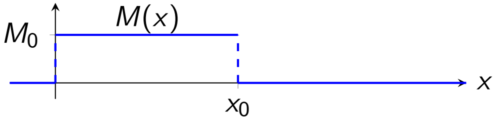
- One set is responsible for winding up the torsional spring.
- Another set transfers stored energy back to the wheels.

The ingenius aspect of the pull-back motor is how the different gear trains engage or disengage based on the direction of rotation. This is critical to the performance of the pull-back toy.

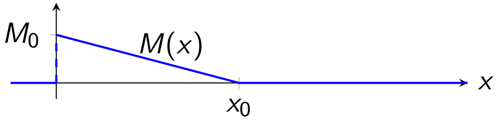
The Goal of the Modeling Scenario

Question:

How does the torque provided by the pull-back motor change as the torsional spring unwinds?



Constant Torque Model



Linear Torque Model

Here, x denotes the distance traveled and $M(x)$ is the torque provided by the motor. The constant x_0 is called the wind-down distance and the constant M_0 quantifies the maximum torque provided by the motor.

The Mathematical Model

Equation of Motion:

The position $x(t)$ of the pull-back toy is governed by the second order initial value problem

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = M(x), \quad x(0) = x'(0) = 0.$$

The equation of motion is derived in both the Student and Teacher versions of the modeling scenario[2] under standard simplifying assumptions:

- The wheels of the pull-back toy experience pure rolling (no slipping).
- Frictional losses are assumed to be proportional to the velocity of the pull-back toy.

[2] https://www.simiode.org/resources/8453

Concept 1: 2^{nd} Order ODE $\longrightarrow$ 1^{st} Order System

Introduce $v = \frac{dx}{dt}$ to represent the velocity of the pull-back toy. Then, the second order initial value problem for the pull-back toy,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = M(x), \quad \text{subject to} \quad x(0) = x'(0) = 0,$$

is equivalent to the first order initial value problem,

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = M(x) - kv \quad \text{subject to} \quad x(0) = v(0) = 0.$$

Concept 2: Numerical Methods

Euler's Method

Fix $\Delta t > 0$ and let $t_j = j\Delta t$. An approximate solution of the system

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = M(x) - kv \quad \text{subject to} \quad x(0) = v(0) = 0$$

is obtained by defining $x_0 = 0$, $v_0 = 0$ and implementing the following recursive equations:

$$x_{j+1} = x_j + v_j \Delta t \qquad v_{j+1} = v_j + (M(x_j) - kv_j)\Delta t.$$

Here, x_j and v_j represent approximations of $x(t_j)$ and $v(t_j)$, respectively.

Note: The use of a more sophisticated numerical method such as the improved Euler method or the Runge-Kutta method is recommended, if possible.

Hands-On Components

Students can complete three separate hands-on tasks:

- **Measure x_0:** The car is wound up fully and allowed to make a series of short advances until the motor has wound down. The change in position is x_0.
- **Record position data:** The car is wound up and released on a track with markings for distance traveled. By recording a video of the experiment, students can extract position [and thus velocity] data as a function of time.
- **Parameter estimation:** The values of M_0 and k can be estimated by comparing the numerical solution to the measured data. A solver can be used to find optimal values.

Note: The published modeling scenario provides much more information about each task. Instructors also have the option to focus on the last task by using the provided data.

Part 2:

Hands-On Modeling

What Does the Modeling Activity Entail?

Participants will be given access to

- a video that shows a pull-back car racing alongside a meter stick.
- an Excel spreadsheet that implements Euler's method for the solution of the equations of motion.

Participants will have the opportunity to

- extract position data from the video to fill in missing information in the spreadsheet.
- adjust the unknown parameters M_0 and k to tune the model for a better fit with the experimental data.

How Do I Participate?

1. Participants may download the following files from our shared Dropbox folder:
   - **Instructions.pdf**
   - **EXPO-22.xlsx**
   - **EXPO-22.mp4**
2. Participants will be invited to join one of two Zoom Breakout Rooms:
   - **Room 1:** Participants will work with the Constant-Torque Model.
   - **Room 2:** Participants will work with the Linear Torque Model.
3. Participants are then welcome to open the instructions and work for 15-20 minutes.
4. Participants will then come back to the main session to discuss the findings.

Data Extraction

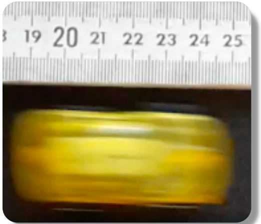
k	t_k	x_k
0	0.000	14.250
1	0.033	14.750
2	0.067	15.125
3	0.100	15.875
4	0.133	16.875
5	0.167	17.625
6	0.200	18.875
7	0.233	20.250
8	0.267	21.625
9	0.300	23.125
10	0.333	24.750
11	0.367	26.625
12	0.400	28.500
13	0.433	30.625
14	0.467	32.750
15	0.500	35.000
16	0.533	37.375
17	0.567	39.875
18	0.600	42.250
19	0.633	45.000
20	0.667	47.625
21	0.700	50.500
22	0.733	53.500
23	0.767	56.125
24	0.800	59.250
25	0.833	62.000
26	0.867	64.750

Extracted Data

The completed position data should resemble the table shown at left.

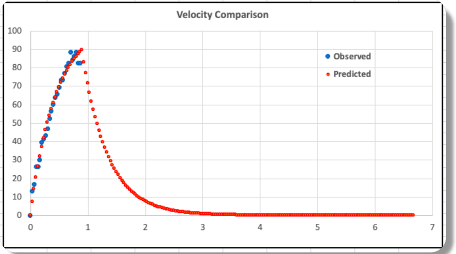
Still Frames

The still frames at right correspond to $k = 0$ and $k = 10$.



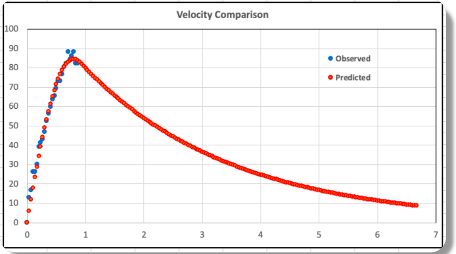
Results

Both models provide a good fit to the observed velocity data. The Linear Torque model yields a smaller estimate for the frictional constant k and thus a larger distance traveled.



Constant Torque Model

$M_0 \approx 220 \quad k \approx 2.1$



Linear Torque Model

$M_0 \approx 230 \quad k \approx 0.4$

It turns out that the total distance traveled is more accurately predicted by the Constant Torque Model. (This small toy car has a lot of friction!)

Part 3:

Report on Classroom Implementation

Fall 2021 Modeling Projects in Calculus I & II

memo

123 Consulting

To: Calculus Junior Consultants

From: Tova Brown

Date: 8/30/21

Re: Pull-back toy car project

My client needs some analysis done on these little toy cars. They want to know things like the top racing speed of the vehicles.

There's not a large budget for this project, but I do have some meter yardsticks if you want to stop by to borrow them. The plan for the project is we'll take position and time measurements; we can get the rest of what we need from the math, without having to take fancy and expensive measurements. A cell phone camera could record video of the car moving, with a length scale in view, and time data can be extracted from the frame rate of the video.

Please do your analysis in a spreadsheet, and get me a report sometime this semester. I'll need one report that can be passed on directly to the client, and also a detailed technical report that the senior consultant can use if needed. We'll have periodic team meetings as well.

Acknowledgement:

Inspired by [3].

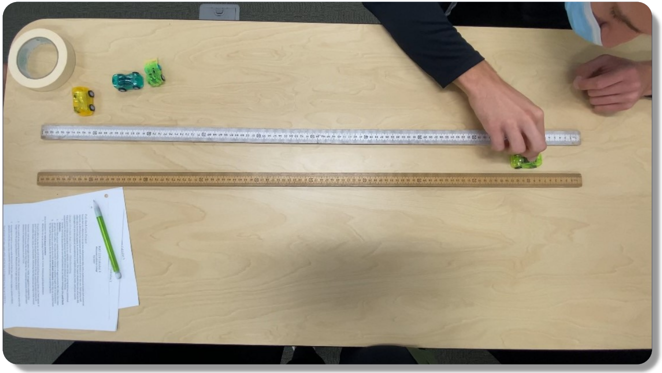
Calculus I: Hands-On Opportunities in Class

First Day:

- **Measure** x_0, the wind-down distance
- Video recording of **position data**

Second Day:

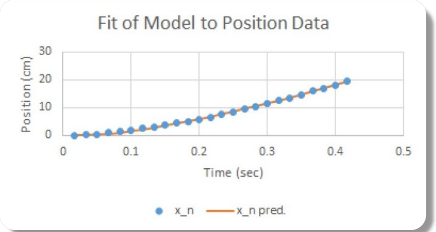
- **Parameter estimation** in a pre-programmed spreadsheet



Calculus I: Follow-up Writing

Student teams were asked to write two reports:

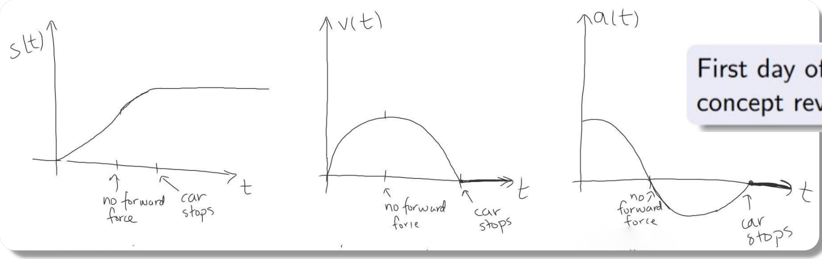
- **Client Report:** write about experiment and results in non-technical language for a general audience
- **Internal Technical Report:** write about analysis and results in more technical language for an expert audience



Excerpt from a student team's Client Report

The graph above is a model of our recorded position and time (blue dots) and the predicted position and time (orange line) given by our in-house software. The graph has an upward slope and this means it moves forward over time, which is good because that is what the car is designed to do.

Calculus II: A More Independent Team Project



First day of class: setting notation, concept review, brainstorming.

Hands-On Experiments at Home:

- Measure wind-down distance x_0
- Record position data with video
- **Implement Euler's Method**
- Parameter estimation

In-Class Presentations of Results:

- No written report (logistic reasons)
- 10 minute, 5 slide maximum rule
- Present results
- Overview methods of analysis

Both Classes: Follow-Up Reflections

After the project, students were asked to individually respond in writing to three questions:

- **Applied Math:** What did you learn about applied math, or how did your perspective about math change, by working through this project?
- **Group Dynamics:** Is there anything I need to know?
- **Value:** Would you recommend I assign this project in this future? Why or why not? Do you have any advice for improving the project?

Reflections: Perspective on Applied Math

This made it feel like we were in the real world solving real-world problems.

I've always kindof hated calculus because I didn't think there were any practical, real world applications, but this helped to kind of flip that mindset. It was interesting to learn about the relationships between acceleration, velocity, and distance and then see all of those ideas play out with our little car.

Applied math puts into perspective that Calculus 1 is still just the tip of the iceberg when it comes to how real math is used all around us.

This project opened my eyes to the intricacy and vast extent of math behind even such a simple movement as that of a little toy car. This project shows how math is quite amazing and just how much of it is used or explains things without us even knowing.

Reflections: Future Use of Project

I honestly would recommend this project to future classes because of how realistic it felt. We were given a problem that needed an answer, and through applied math we were able to experiment and calculate a solution with prior knowledge we learned in class.

I would most definitely recommend that future classes perform this experiment. However... I believe that more time should be spent talking over the particular calculations that are used in the Excel spreadsheets along with how exactly those calculations are processed.

I would not recommend this exact project to future classes, because although we were collecting all of the data it still felt like we had training wheels on for this assignment, especially when we worked with excel.... Instead of having the complex spreadsheets, find an easier way to include the students in the actual making of the spreadsheets.

What Makes an Engaging Modeling Project?

Modeling projects...

- are easier to implement when they blend in with the instruction of course content.

> Calc II: differential equations and Euler's Method were already course topics.
> Calc I: only the position-velocity-acceleration application of derivatives.

- are more engaging when they allow students to collect and/or use real data.

> Calculus students didn't see just toys; they saw real-world moving objects!

- are more compelling when they involve parameter estimation or open-ended questions that illustrate the usefulness of differential equations.

> Calc I students were curious about the numerical methods and parameter estimations; you don't have to minimize or hide these ideas from them!

Do You Have an Idea for a Modeling Scenario?

Probably each of you has an idea that could serve as the basis for a new modeling scenario. Here are a few things to think about as you engage in the creation process:

- **Accessibility:** Does the modeling scenario make use of ideas, concepts, and materials that make sense for a typical differential equations class?
- **Prior Work:** Has anyone written a modeling scenario on a similar topic? If so, is there a way to build on the prior work or take the idea in a new direction?
- **Student/Teacher Versions:**
    - Student Version: Think of this version as being similar to the assignment description you would give students in your course.
    - Teacher Version: Longer version that may include advice about implementation, sample data, written solutions, code, spreadsheets, etc.

References

[1] YouTube video: ‘How does a Pull-Back Toy Car work?’
https://www.youtube.com/watch?v=QdvfiVebb_s. Accessed 15 June 2021.

[2] Johnson, Brody Dylan. 2021. 3-099-S-PullBack,
https://www.simiode.org/resources/8453.

[3] Winkel, Brian J. 1990. First year calculus students as in-class consultants, International Journal of Mathematical Education in Science and Technology, 21:3, 363-368,
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