An Applied Project-Driven Approach to Undergraduate Research Experiences

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Abstract

- Early in my career I was approached by a Ball State student to be an Honors Thesis advisor.
- Having no clue what constituted an appropriate honors thesis project, I gave the student an open-ended problem to consider – modeling heat flow in a thermos.
- Not only did the student complete his honors thesis, but the resulting work led to a refereed journal article and opened the door to a very successful series of collaborative undergraduate research projects.
- All of the problems have the following in common – they are simple to state, open-ended, student driven, mathematically significant, rely on student insight, and require a substantial amount of work on both the student's and my part.
- Several of these projects have led to a refereed publication that could be used to illustrate topics taught in the undergraduate curriculum.
- We will look at the process I have developed for this type of research, what works and what doesn't work, and touch on some of the topics explored, namely heat flow, cryptography, and diving boards.

Project Initiation

- Typically a student contacts me about doing some undergraduate research.
- Most students are in Ball State's Honors College.
- Part of the Honors College requirements is a senior Honors Thesis.
- Other projects have arisen as part of an Indiana Space Grant.
  - In this case I contacted some of the students involved.

Project Expectations

- A student should plan to spend at least a semester or summer on the project.
- Most projects have taken two semesters to complete.
- The first semester is spent learning requisite theory, setting up, and solving the problem.
- The second semester is spent testing the problem solution, implementing it on a computer, and writing up the work.
- Each thesis has required many revisions before final version meets my approval.
- If possible, the student should present the work.
- One venue that has worked well is our departmental undergraduate colloquium.

Project Process

- Each project involves an applied mathematical problem of interest to the student (and me).
- Problems are suggested and tailored to the student's mathematical background.
- Each requires self-study, learning, and mastery of mathematics new to the student.
- Each involves development of a mathematical problem or model.
- Each involves solution of the problem or model.
- Each involves testing the solution with data.
- All incorporate software and technology (typically Mathematica).
- Most have resulted in a refereed publication.

Sample Projects

- Heat Flow in a Thermos (James Scherschel)
- Elliptic Curve Cryptography (Amiee O'Maley)
- Displacement of a Diving Board (Brenda Skoczelas)

- Problem Statement
- Mathematical Background
- Initial attempts
- Student input/insight
- Results
- Project Dissemination

Heat Flow in a Thermos – Problem Statement

- Find a way to describe the temperature of a thermos full of hot coffee at any point in the thermos at any time after the coffee is poured into the thermos.

Heat Flow in a Thermos – Mathematical Background

- Newton's Law of Cooling
- Initial Value Problem (ODE)
- Boundary Value Problem (PDE)
- Conservation of Energy
- Fourier's Law of Heat Conduction
- Heat Equation
- Separation of Variables
- Fourier Series

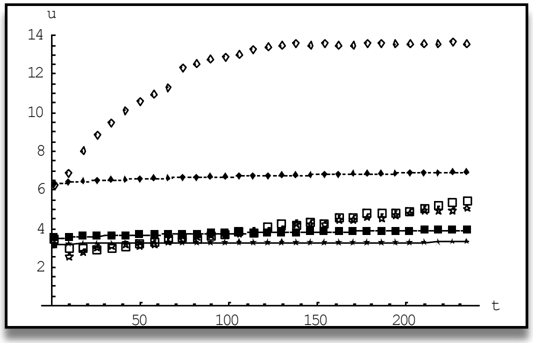
$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} - \frac{\rho c}{\kappa}\frac{\partial u}{\partial t} &= 0, && (4)\\
\frac{\partial u}{\partial y}(0,t) &= 0, && (5)\\
u(a,t) + \frac{\kappa}{h}\frac{\partial u}{\partial y}(a,t) &= T_0, && (6)\\
u(y,0) &= f(y). && (7)
\end{aligned}
$$

Heat Flow in a Thermos – Initial Attempts

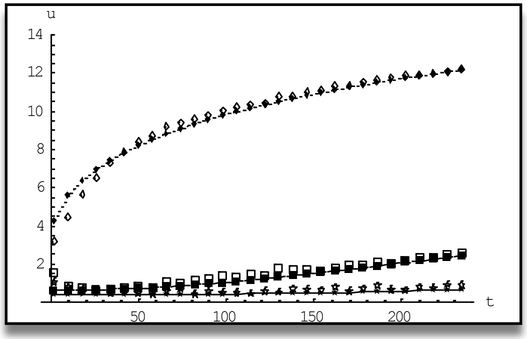
- My original idea was to use boiling water in a thermos.
- Temperature probes hooked to CBL's were put into the thermos to measure temperature at various heights in the thermos.
- Besides issues with safety, there was a question of how to cap off the thermos with temperature probes coming out the top.
- My initial attempts resulted in "failure" – no useful data was collected!

Heat Flow in a Thermos – James' Input/Insight

- James decided to use ice-water instead of boiling water.
- This is safer, allows an open top, and reduces heat loss due to steam.
- Unfortunately, the results with this new approach weren't much better than those with hot water, as seen in the top figure at right.
- Two other changes James made were to put cotton balls into the ice-water to make a "solid" instead of a liquid and to assume the initial temperature distribution has an exponential term.
- This lead to the results shown in the bottom figure which are surprisingly good!



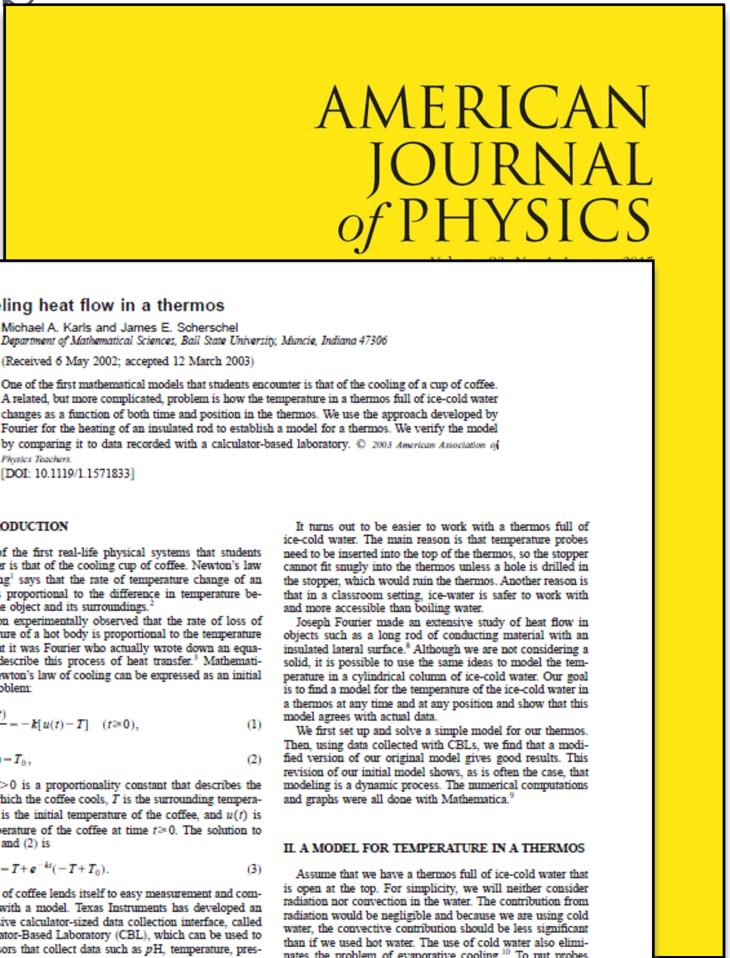
Initial Model: ____
Actual: ....



Final Model: _____
Actual: ....

Heat Flow in a Thermos – Project Dissemination

- This work resulted in a co-authored paper that appeared in the American Journal of Physics.



Modeling heat flow in a thermos

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One of the first mathematical models that students encounter is that of the cooling of a cup of coffee. A related, but more complicated, problem is how the temperature in a thermos full of ice-cold water changes as a function of both time and position in the thermos. We use the approach developed by Fourier for the heating of an insulated rod to establish a model for a thermos. We verify the model by comparing it to data recorded with a calculator-based laboratory.

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I. INTRODUCTION

One of the first real-life physical systems that students encounter is that of the cooling cup of coffee. Newton's law of cooling[1] says that the rate of temperature change of an object is proportional to the difference in temperature between the object and its surroundings.[2]

Newton experimentally observed that the rate of loss of temperature of a hot body is proportional to the temperature itself, but it was Fourier who actually wrote down an equation to describe this process of heat transfer.[3] Mathematically, Newton's law of cooling can be expressed as an initial value problem:

$$\frac{du(t)}{dt} = -k[u(t) - T] \quad (t \geq 0), \tag{1}$$

$$u(0) = T_0, \tag{2}$$

where $k > 0$ is a proportionality constant that describes the rate at which the coffee cools, T is the surrounding temperature, T_0 is the initial temperature of the coffee, and $u(t)$ is the temperature of the coffee at time $t \geq 0$. The solution to Eqs. (1) and (2) is

$$u(t) = T + e^{-kt}(-T + T_0). \tag{3}$$

A cup of coffee lends itself to easy measurement and comparison with a model. Texas Instruments has developed an inexpensive calculator-sized data collection interface, called a Calculator-Based Laboratory (CBL), which can be used to link sensors that collect data such as pH, temperature, pressure, force, or motion, with a programmable Texas Instruments calculator.[4] By using a CBL with a temperature probe, a Texas Instruments TI-85 calculator, and a program available from their website,[5] the temperature can be collected, plotted, and compared to the solution in Eq. (3).[6] Good results can be obtained with a styrofoam cup with a lid and some hot water. The lid is important to reduce evaporation and convection.[7]

Now consider an added level of complexity—what if we wish to know the temperature at any position within the cup of coffee at any particular time? One way to approach this problem is to think of the water in the cup as a cylinder of heat conducting material that is insulated on the side, top, and bottom. Because a thermos is essentially a tall, well-insulated cup, we will model the heat flow in a cylinder surrounded by air at room temperature.

It turns out to be easier to work with a thermos full of ice-cold water. The main reason is that temperature probes need to be inserted into the top of the thermos, so the stopper cannot fit snugly into the thermos unless a hole is drilled in the stopper, which would ruin the thermos. Another reason is that in a classroom setting, ice-water is safer to work with and more accessible than boiling water.

Joseph Fourier made an extensive study of heat flow in objects such as a long rod of conducting material with an insulated lateral surface.[8] Although we are not considering a solid, it is possible to use the same ideas to model the temperature in a cylindrical column of ice-cold water. Our goal is to find a model for the temperature of the ice-cold water in a thermos at any time and at any position and show that this model agrees with actual data.

We first set up and solve a simple model for our thermos. Then, using data collected with CBLs, we find that a modified version of our original model gives good results. This revision of our initial model shows, as is often the case, that modeling is a dynamic process. The numerical computations and graphs were all done with Mathematica.[9]

II. A MODEL FOR TEMPERATURE IN A THERMOS

Assume that we have a thermos full of ice-cold water that is open at the top. For simplicity, we will neither consider radiation nor convection in the water. The contribution from radiation would be negligible and because we are using cold water, the convective contribution should be less significant than if we used hot water. The use of cold water also eliminates the problem of evaporative cooling.[10] To put probes into our thermos to collect temperature data, the top of the thermos must be left open to the air.

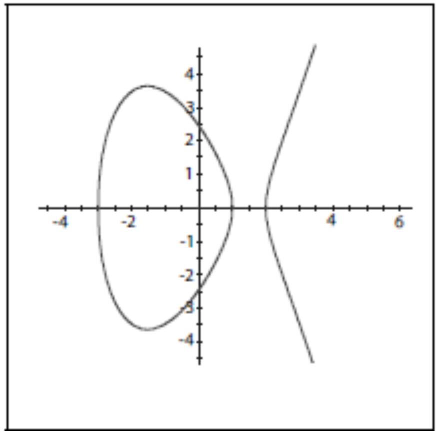
We can think of the ice-cold water as a cylinder of conducting material that is insulated at the bottom and the side; heat is lost to the air by convection at the top surface of the water.[11] We also will assume that at any time the temperature is the same at any point of a cross-sectional slice of water. Let $u(y,t)$ be the temperature of a cross-sectional slice of the water in the thermos at position $0 \leq y \leq a$ and time $t \geq 0$, where $y = 0$ and $y = a$ correspond to the bottom and top of the thermos, respectively. Assume that the air surrounding the thermos is at constant temperature T_0, and the initial temperature distribution is given by some sectionally smooth function $f(y)$, that is, $f'(y)$ exists and is continuous on $[0,a]$, except possibly at a finite number of jumps or remov-

Elliptic Curve Cryptography – Problem Statement

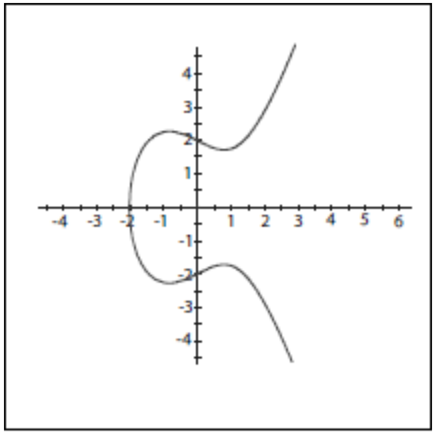
- Find a way to implement a cryptographic scheme known as Elliptic Curve Cryptography via software.

Elliptic Curve Cryptography – Mathematical Background

- Groups, Abelian Groups, and Fields
- Elliptic Curves Over the Real Numbers
- Elliptic Curve Groups
- Elliptic Curves Over a Finite Field
- Elliptic Curve Cryptography
- Discrete Logarithm Problem
- ElGamel Method



Elliptic curves $y^2 = x^3 - 7x + 6$ (left) and $y^2 = x^3 - 2x + 4$ (right)



Elliptic Curve Cryptography – Initial Attempts

- Starting with some background material on elliptic curves from a paper by Ezra Brown, my initial plan was to implement ECC in the programming language C.
- A great resource for this is the book Implementing Elliptic Curve Cryptography by Michael Rosing.
- This proved to be much harder than expected, especially since not only would we need to learn ECC, but also need to learn much more C than either of us knew.

Three Fermat Trails to Elliptic Curves

Ezra Brown

Ezra (Bud) Brown (brown@math.vt.edu) professes mathematics at Virginia Tech, where he has been since 1969. The elliptic curve bug first bit him while he was in graduate school at Louisiana State, and has never really gone away. Although his main research has been in quadratic forms and algebraic number theory, he once wrote a paper with a sociologist. He loves to talk about mathematics and its history with anyone, especially students. He occasionally sings in operas, plays jazz piano just for fun, and bakes biscuits for his classes. He and his mathematical grandfather, L. E. Dickson, have

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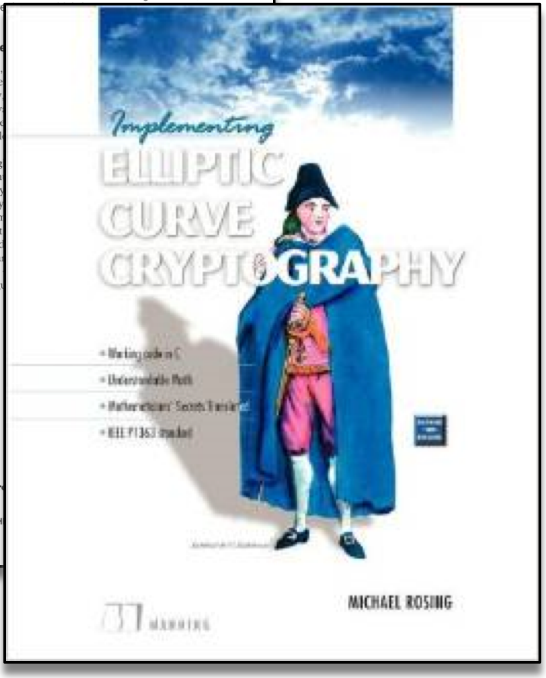
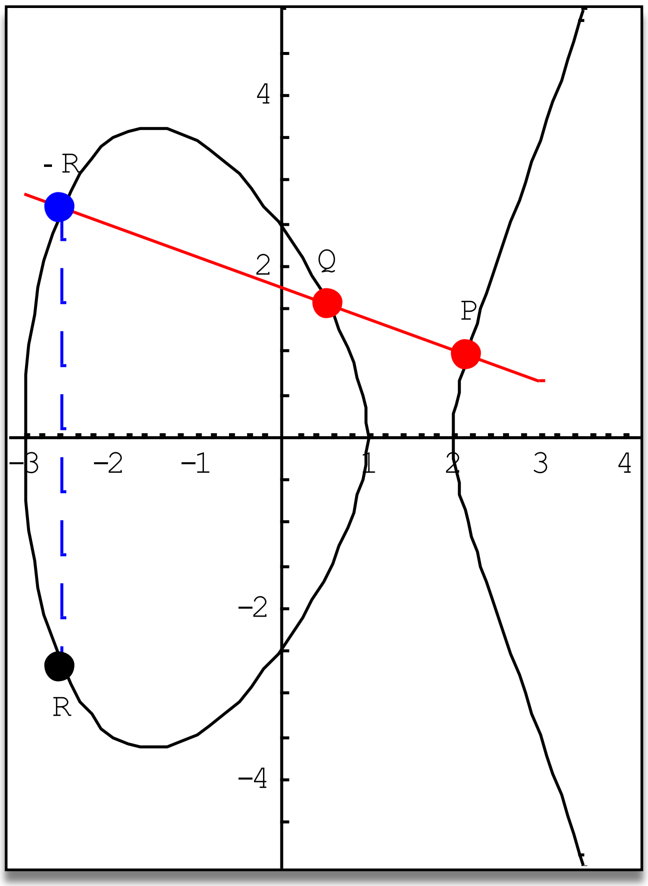


Elliptic Curve Cryptography – Amiee's Input/Insight

- After learning about elliptic curves and how to add points on these curves both graphically and algebraically, Amiee decided to use Mathematica instead of C.
- For example, the graph at the right shows how to add points on a elliptic curve algebraically.
- All graphs were constructed in Mathematica, from scratch, relying mainly on the built-in help files.



ELLIPTIC CURVE CRYPTOGRAPHY – RESULTS

- Amiee developed Mathematica notebooks that can be used to:
  - Verify an elliptic curve has a corresponding Elliptic Curve Group.
  - Construct an Elliptic Curve Group from an elliptic curve over a finite field, such as Z_p.
  - Add points on these elliptic curves.
  - Implement the ElGamel scheme for an Elliptic Curve Group.

Create an Elliptic Curve Group E

- Select an elliptic curve $y^2 = x^3 + ax + b$ over a finite field $\mathbb{Z}_n$where $n > 3$is prime. Check to make sure the curve is non-singular by making sure that $4\,a^3 + 27\,b^2 \neq 0 \pmod{n}$.

```
In[1]:= a = 1;
        b = 6;
        n = 7;

In[4]:= Mod[4 a^3 + 27 b^2, n] == 0

Out[4]= False
```

- Define elliptic curve addition algebraically

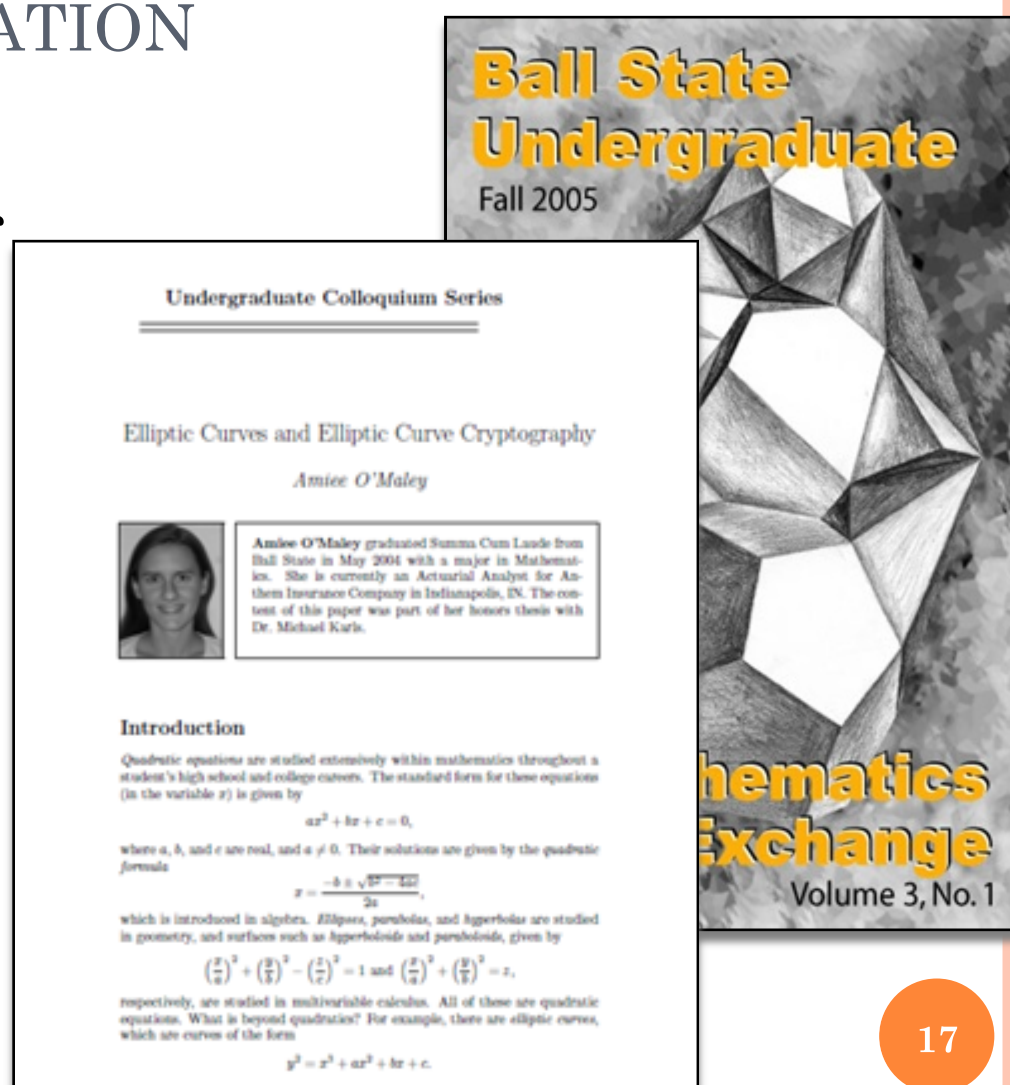
```
In[5]:= ModEllAdd[xP_, yP_, xQ_, yQ_, a_, n_] := If[xP == ∞, {xQ, yQ}, If[xQ == ∞, {xP, yP},
          If[(Mod[xP, n] == Mod[xQ, n] && Mod[yP, n] == Mod[-yQ, n]),
           {∞, ∞}, If[xP == xQ && yP == yQ,
            {Mod[(Mod[3 xP^2 + a, n] * PowerMod[2 yP, -1, n])^2 - 2 xP, n],
             Mod[-yP + (Mod[3 xP^2 + a, n] * PowerMod[2 yP, -1, n])
                (xP - (Mod[(Mod[3 xP^2 + a, n] * PowerMod[2 yP, -1, n])^2 - 2 xP, n])), n]},
            {Mod[(Mod[yP - yQ, n] * PowerMod[xP - xQ, -1, n])^2 - xP - xQ, n],
             Mod[-yP + (Mod[yP - yQ, n] * PowerMod[xP - xQ, -1, n])
                (xP - (Mod[(Mod[yP - yQ, n] * PowerMod[xP - xQ, -1, n])^2 - xP - xQ, n])), n]}
           ]]]]
```

- Find the points in E and construct the addition table for E

```
In[6]:= points = Partition[Flatten[Table[{i, j}, {i, 0, n - 1}, {j, 0, n - 1}]], 2];

In[7]:= chk = Table[PowerMod[points[[i]][[2]], 2, n] ==
          Mod[PowerMod[points[[i]][[1]], 3, n] + a * points[[i]][[1]] + b, n],
         {i, 1, Length[points]}];

In[8]:= group = Insert[Extract[points, Position[chk, True]], {∞, ∞}, 1]

Out[8]= {{∞, ∞}, {1, 1}, {1, 6}, {2, 3}, {2, 4}, {3, 1}, {3, 6}, {4, 2}, {4, 5}, {6, 2}, {6, 5}}
```

ELLIPTIC CURVE CRYPTOGRAPHY – PROJECT DISSEMINATION

- Amiee published a paper based on part of this work in the Ball State Undergraduate Mathematics Exchange.



Undergraduate Colloquium Series

Elliptic Curves and Elliptic Curve Cryptography

Amiee O'Maley

Amiee O'Maley graduated Summa Cum Laude from Ball State in May 2004 with a major in Mathematics. She is currently an Actuarial Analyst for Anthem Insurance Company in Indianapolis, IN. The content of this paper was part of her honors thesis with Dr. Michael Karls.

Introduction

Quadratic equations are studied extensively within mathematics throughout a student's high school and college careers. The standard form for these equations (in the variable x) is given by

$$ax^2 + bx + c = 0,$$

where a, b, and c are real, and $a \neq 0$. Their solutions are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which is introduced in algebra. *Ellipses*, *parabolas*, and *hyperbolas* are studied in geometry, and surfaces such as *hyperboloids* and *paraboloids*, given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1 \text{ and } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = z,$$

respectively, are studied in multivariable calculus. All of these are quadratic equations. What is beyond quadratics? For example, there are *elliptic curves*, which are curves of the form

$$y^2 = x^3 + ax^2 + bx + c.$$

Displacement of a Diving Board – Problem Statement

- Find the vertical displacement of a diving board at any point along the board at any time after the board is displaced and released.

Displacement of a Diving Board – Mathematical Background

- Cantilever Beam
- Newton's Second Law
- Bending Moment
- Shearing and Damping Forces
- Boundary Value Problem
- Beam Equation (PDE)
- Separation of Variables
- Fourier Series

$$\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4} - k\frac{\partial y}{\partial t} - g, \tag{1}$$

$$y(0,t) = 0, \quad t > 0, \tag{2}$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \quad t > 0, \tag{3}$$

$$\frac{\partial^2 y}{\partial x^2}(L,t) = 0, \quad t > 0, \tag{4}$$

$$\frac{\partial^3 y}{\partial x^3}(L,t) = 0, \quad t > 0, \tag{5}$$

$$y(x,0) = f(x), \; 0 < x < L, \tag{6}$$

$$\frac{\partial y}{\partial t}(x,0) = g(x), \; 0 < x < L. \tag{7}$$

Displacement of a Diving Board – Initial Attempts

- Brenda's first model consisted of the beam equation with a quadratic function for initial displacement.
- After deciding that an actual diving board wouldn't be feasible, Brenda contacted the Physics Department to see if they had a long ruler we could use.
- A wooden ruler of length two–meters with one end clamped to a table was chosen for our physical diving board.
- Fifteen tape-marked points along the ruler were analyzed using a video camera attached to a computer with World in Motion software.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4}} - k\frac{\partial y}{\partial t} - g, \tag{1}$$

$$y(0,t) = 0, \quad t > 0, \tag{2}$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \quad t > 0, \tag{3}$$

$$\frac{\partial^2 y}{\partial x^2}(L,t) = 0, \quad t > 0, \tag{4}$$

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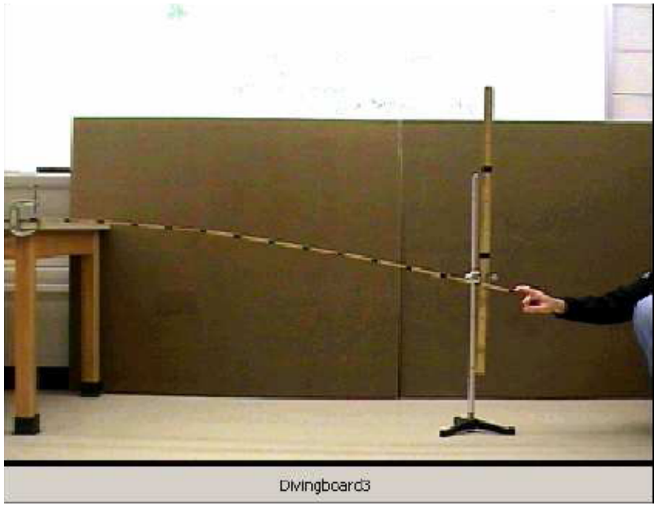
$$\frac{\partial y}{\partial t}(x,0) = g(x), \; 0 < x < L. \tag{7}$$



DISPLACEMENT OF A DIVING BOARD – BRENDA'S INPUT/INSIGHT

- Brenda was able to match model and data time period, but decided that damping needed to be accounted for in the model.
- Including a damping term in the model led to much better results.
- This is the point we got to for Brenda's Honors Thesis.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4} - k \frac{\partial y}{\partial t}} + g, \tag{1}$$

$$y(0,t) = 0, \quad t > 0, \tag{2}$$

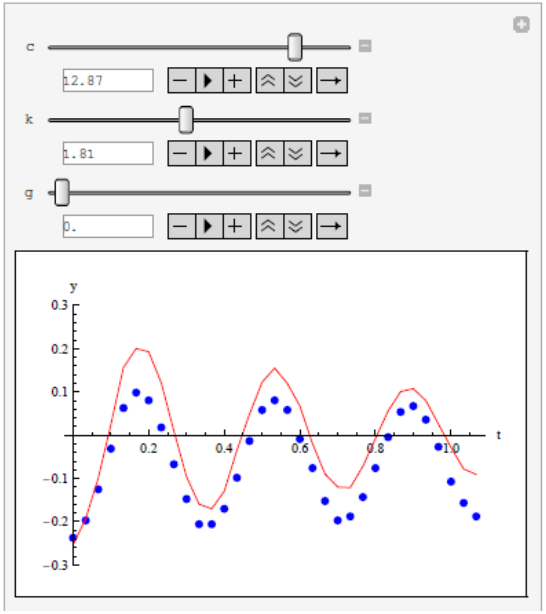
$$\frac{\partial y}{\partial x}(0,t) = 0, \quad t > 0, \tag{3}$$

$$\frac{\partial^2 y}{\partial x^2}(L,t) = 0, \quad t > 0, \tag{4}$$

$$\frac{\partial^3 y}{\partial x^3}(L,t) = 0, \quad t > 0, \tag{5}$$

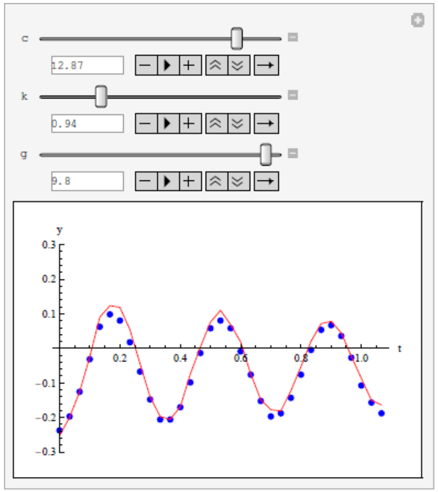
$$y(x,0) = f(x), \ 0 < x < L, \tag{6}$$

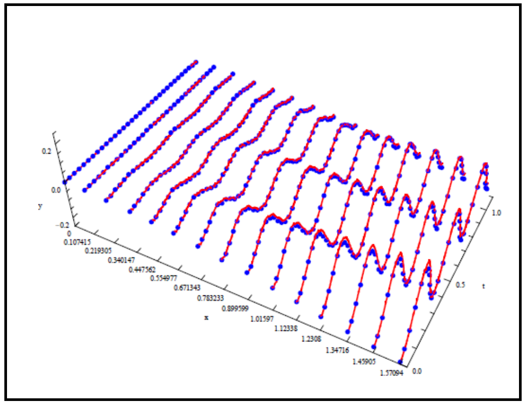
$$\frac{\partial y}{\partial t}(x,0) = g(x), \ 0 < x < L. \tag{7}$$



Displacement of a Diving Board – Results

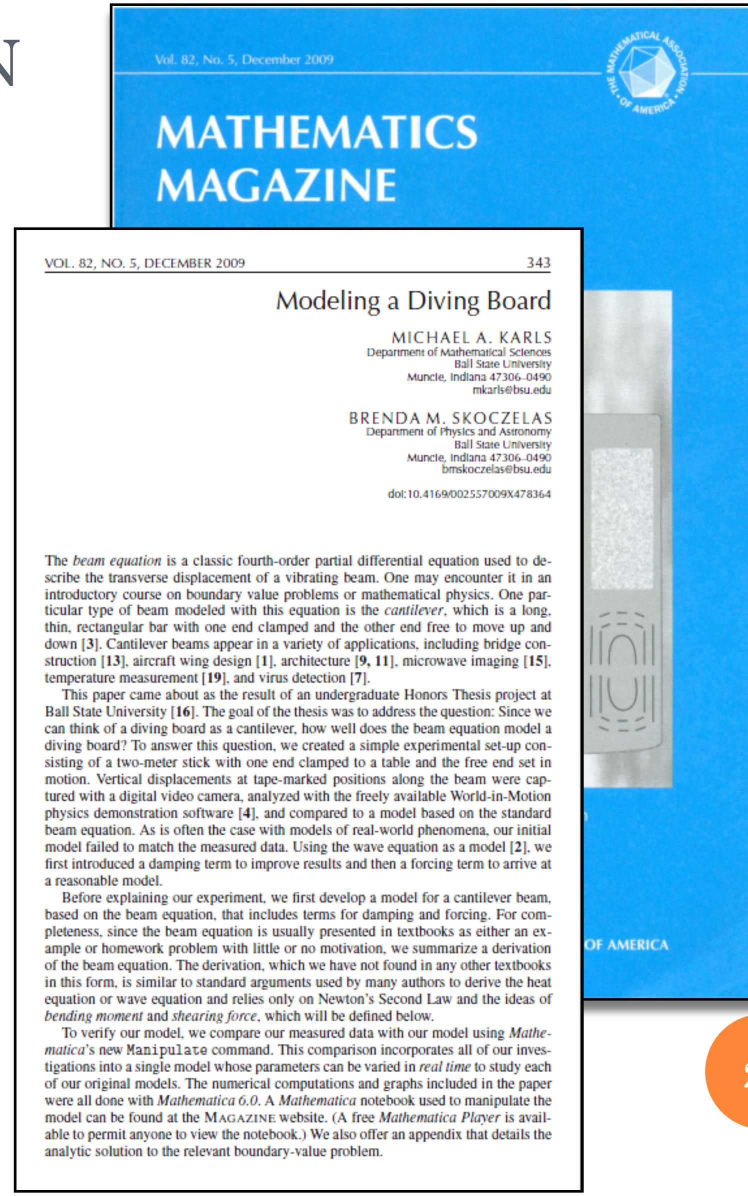
- After Brenda's thesis was completed, I continued working on the project.
- One day I had the idea to include a forcing term – it turns out that forcing due to gravity is what is needed!
- Using Mathematica's Manipulate command, one can implement all three models at once.
- One key to this approach was Brenda's choice of a quadratic as the initial displacement function!





DISPLACEMENT OF A DIVING BOARD – PROJECT DISSEMINATION

- A joint paper from this work appeared in Mathematics Magazine.



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Modeling a Diving Board

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The *beam equation* is a classic fourth-order partial differential equation used to describe the transverse displacement of a vibrating beam. One may encounter it in an introductory course on boundary value problems or mathematical physics. One particular type of beam modeled with this equation is the *cantilever*, which is a long, thin, rectangular bar with one end clamped and the other end free to move up and down [**3**]. Cantilever beams appear in a variety of applications, including bridge construction [**13**], aircraft wing design [**1**], architecture [**9, 11**], microwave imaging [**15**], temperature measurement [**19**], and virus detection [**7**].

This paper came about as the result of an undergraduate Honors Thesis project at Ball State University [**16**]. The goal of the thesis was to address the question: Since we can think of a diving board as a cantilever, how well does the beam equation model a diving board? To answer this question, we created a simple experimental set-up consisting of a two-meter stick with one end clamped to a table and the free end set in motion. Vertical displacements at tape-marked positions along the beam were captured with a digital video camera, analyzed with the freely available World-in-Motion physics demonstration software [**4**], and compared to a model based on the standard beam equation. As is often the case with models of real-world phenomena, our initial model failed to match the measured data. Using the wave equation as a model [**2**], we first introduced a damping term to improve results and then a forcing term to arrive at a reasonable model.

Before explaining our experiment, we first develop a model for a cantilever beam, based on the beam equation, that includes terms for damping and forcing. For completeness, since the beam equation is usually presented in textbooks as either an example or homework problem with little or no motivation, we summarize a derivation of the beam equation. The derivation, which we have not found in any other textbooks in this form, is similar to standard arguments used by many authors to derive the heat equation or wave equation and relies only on Newton's Second Law and the ideas of *bending moment* and *shearing force*, which will be defined below.

To verify our model, we compare our measured data with our model using *Mathematica*'s new `Manipulate` command. This comparison incorporates all of our investigations into a single model whose parameters can be varied in *real time* to study each of our original models. The numerical computations and graphs included in the paper were all done with *Mathematica 6.0*. A *Mathematica* notebook used to manipulate the model can be found at the MAGAZINE website. (A free *Mathematica Player* is available to permit anyone to view the notebook.) We also offer an appendix that details the analytic solution to the relevant boundary-value problem.

Project Sources

- Ideas for my project problems come from many places, including courses taught, research projects, or examples in books.
- The Thermos Problem was inspired by a homework question in my Boundary Value Problems course.
- The ECC Problem resulted from a sabbatical project which involved cryptography.
- The Diving Board Problem was motivated by an example on cantilever beams found in a BVP book by Nakhle Asmar.
- I am always on the lookout for more project topics!

Project Guidelines

- Choose a problem that is simple/easy to state.
- Set up regular meetings – usually once per week.
- It is ok if the problem solution is uncertain or unknown.
- Be prepared to modify the original problem.
- Let student input/insight help drive the project.
- Set high levels of expectation – students will usually rise to this level.
- Expect the unexpected – what results from a project is often much better than the initial idea.

Project Guidelines

- Of the guidelines above, the most important one is to *set up regular meetings* – this is crucial to a project's success!
- In a recent fall semester, for the first time, one of my projects failed to get off the ground.
- I attribute this in part to not being able to schedule regular meetings.
- For the following spring semester, we established a weekly meeting time, leading to successful completion of the project!

References

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