Discovering Bifurcations

Bifurcation, UMAP journal, 43 (2) (2022) 97-107.

SIMIODE Expo 2023

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What are some problems facing the nation and the world that you are concerned about?

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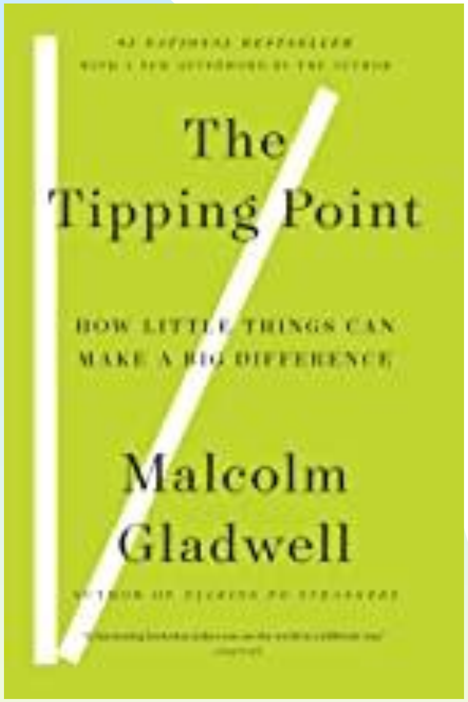
One common answer:

Climate Change

A goal of our course is to see how the mathematics we learn can connect to these problems you are concerned about

How many of you include the topic of bifurcations in your differential equations course?

The Importance of Bifurcations = Tipping Points



The Tipping Point:

How Little Things Can Make a Big Difference

Malcolm Gladwell

The Importance of Bifurcations = Tipping Points

Climate Emergency

- Temperature rises so that Greenland ice sheet melts
- Permafrost melts releasing stored carbon
- Burning too many Amazon trees, the forest will convert to grasslands

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Climate Emergency

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Students Assignment: what are other tipping points in the earth climate system?

The Importance of Bifurcations = Tipping Points

Counter-intuitive:

A small change in a system should have a small impact.

The Importance of Bifurcations = Tipping Points

Counter-intuitive:

A small change in a system should have a small impact.

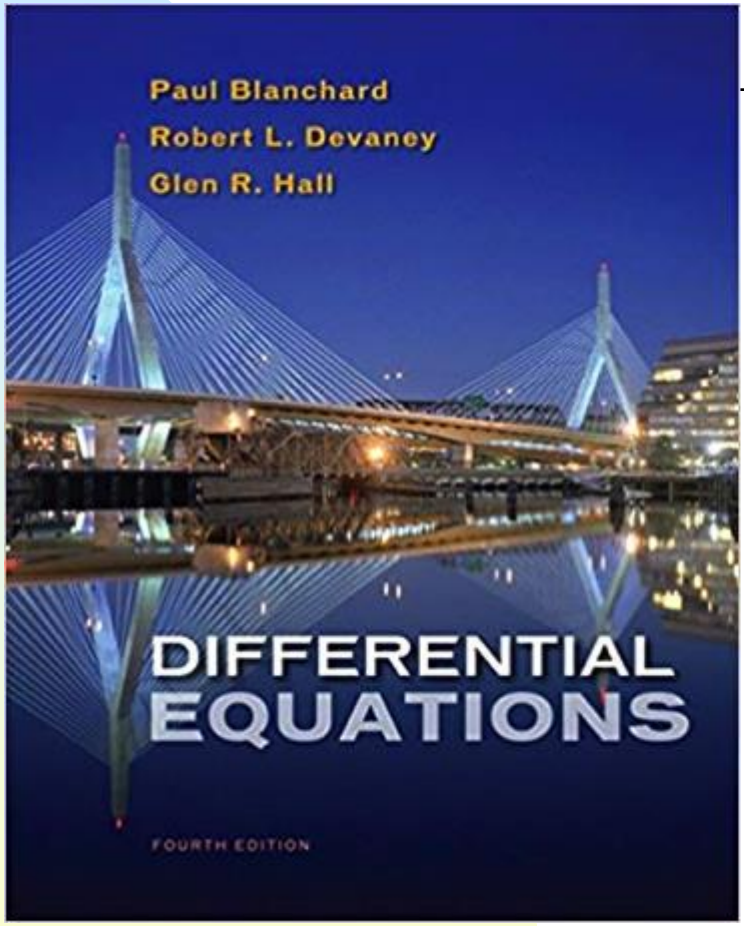
NOT!!

TED Ed

http://ed.ted.com/lessons/is-our-climate-headed-for-mathematical-chaos-victor-j-donnay#review

Is our climate headed for a mathematical tipping point?

Differential Equations



Population growth:

Exponential Model

$$\frac{dP}{dt} = kP \Rightarrow \quad P(t) = P_0 e^{kt}$$

Unlimited growth

Population growth: Logistic model

Limits to growth - carrying capacity N

$$\frac{dP}{dt} = kP(1 - P/N)$$

Population growth: Logistic model

Simplification: Change variables, can assume

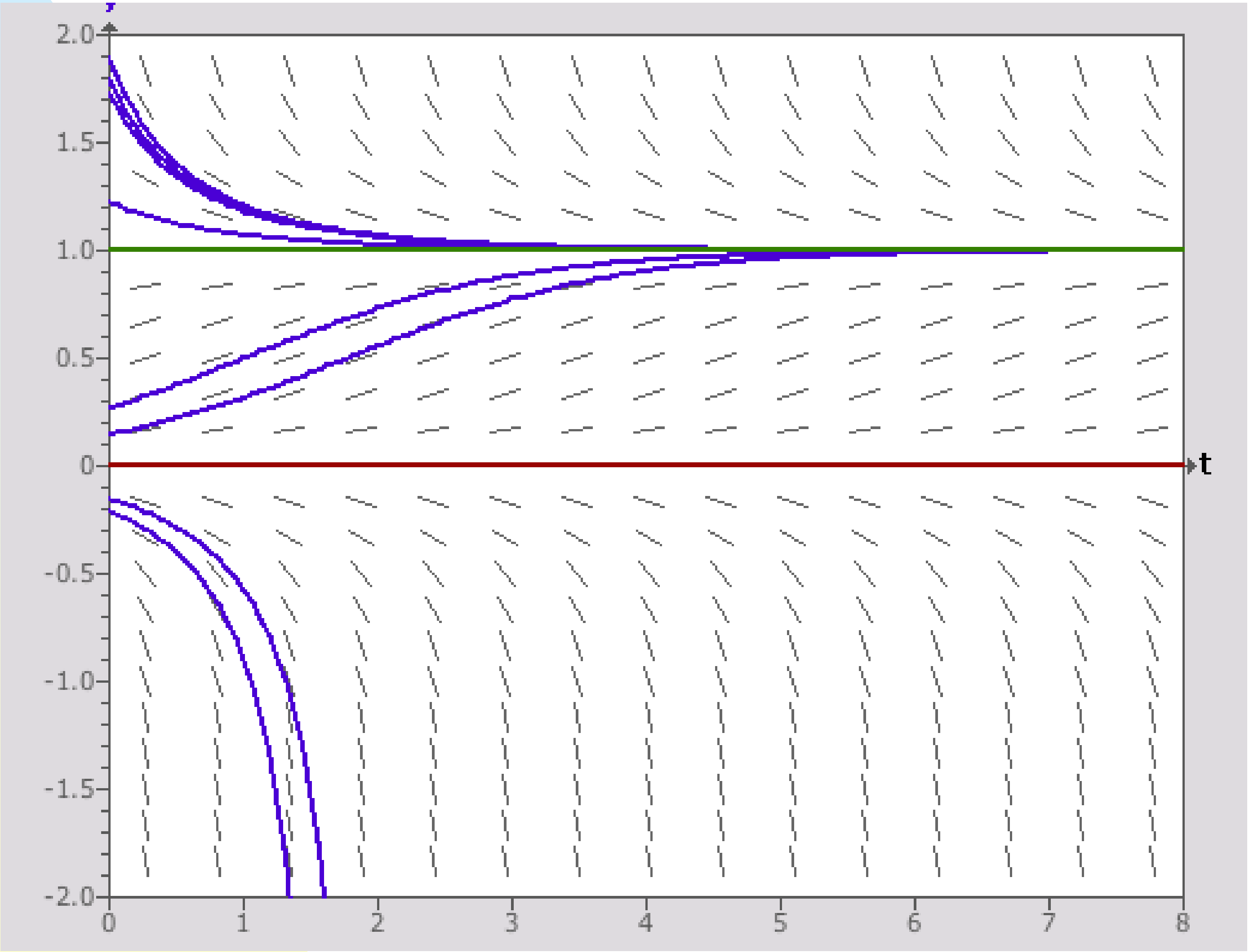
k = 1, N = 1 .

$$\frac{dP}{dt} = P(1 - P)$$

Slope Field Analysis:

$$\frac{dP}{dt} = P(1 - P) = \text{ slope of sol'n curves}$$

P axis



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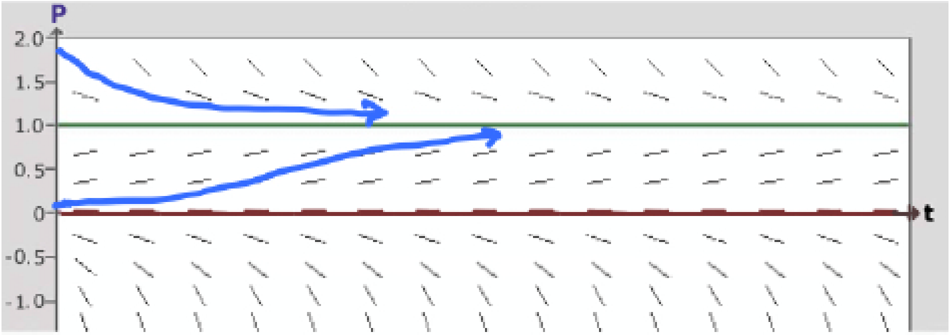
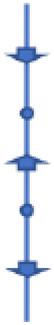


(a) Slope field diagram with solutions from two initial conditions.



(b) Phase line diagram.

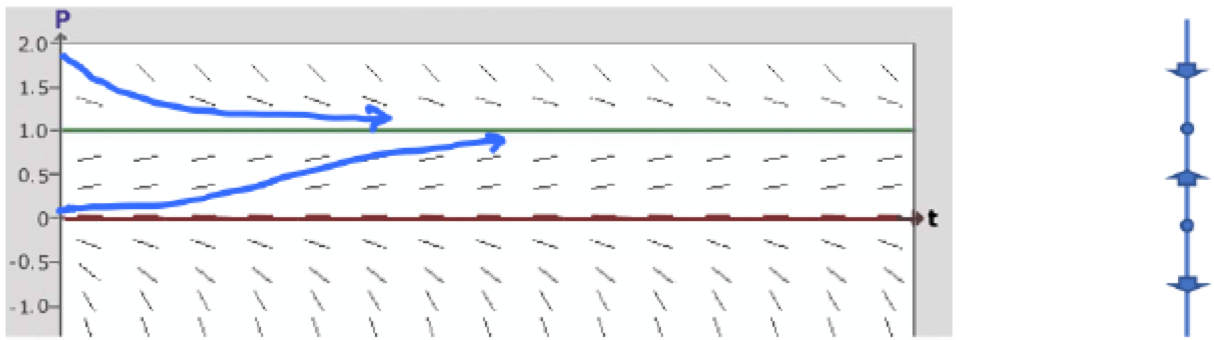
Equilibrium Sol'ns:

P = 1 attracting/sink/stable

P = 0 repelling/source/unstable

Slope Field Analysis:

$$\frac{dP}{dt} = P(1 - P) = \text{f(P)} = \text{slope of sol'n curves}$$



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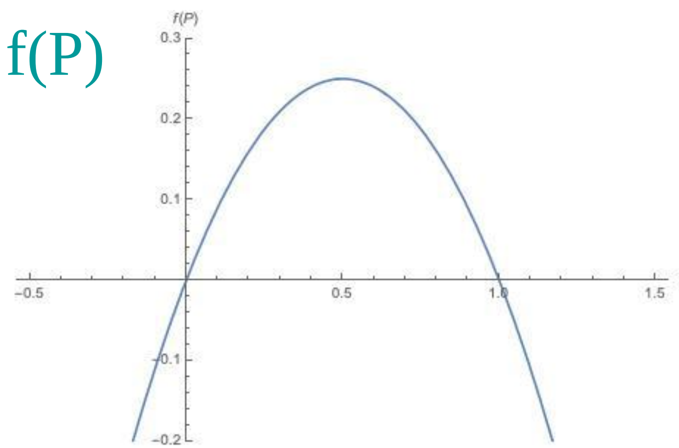
f(P)



Equilibrium Sol'ns:

P = 1 attracting/sink

P = 0 repelling/source

The graph of f(P), whose zeros are equilibrium solutions and whose sign indicates whether the derivative dP/dt is positive or negative

Logistic growth with harvesting

$$\frac{dP}{dt} = P(1 - P) - C$$

Harvesting rate "C"

Fish caught per year

Trees harvested per year

Logistic growth with harvesting

$$\frac{dP}{dt} = P(1 - P) - C$$

Harvesting rate "C"

How does changing the fishing level C impact population levels?
To maximize our profits, how many fish should we catch?

Fish caught per year

Trees harvested per year

Discover the Bifurcation Diagram

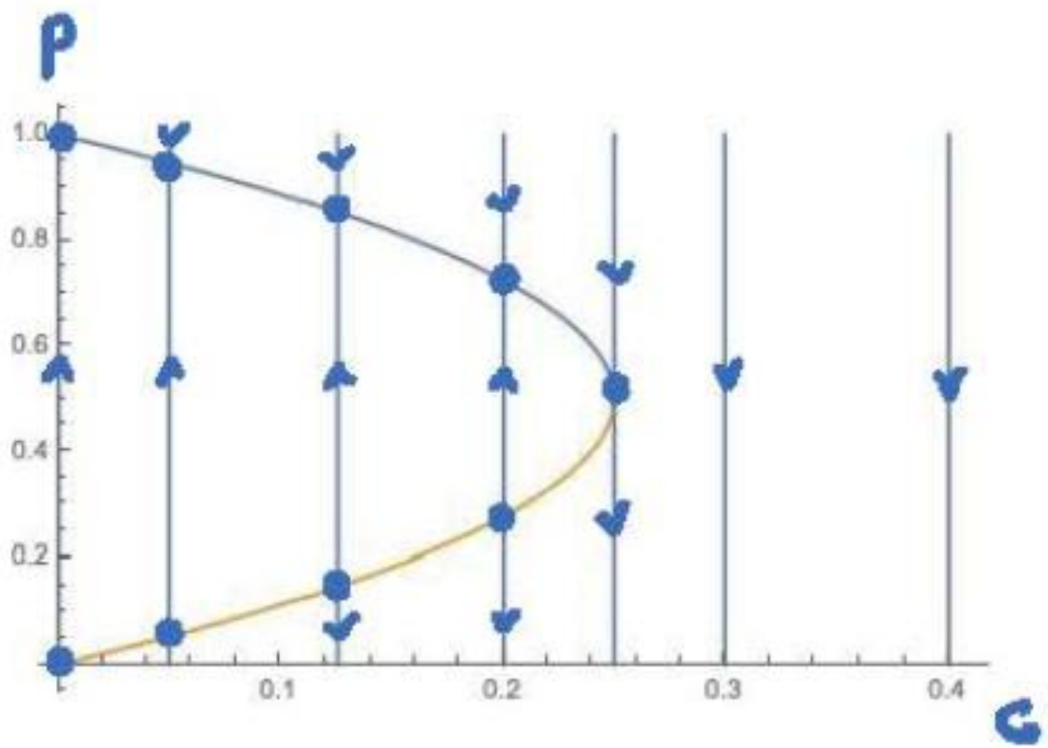


Figure 4. Bifurcation diagram for (3).

Discover the Bifurcation Diagram

- Carry out Phase Line Diagram analysis for various c values.

Group work: Jigsaw approach

- Divide students into groups (of 4)
- Students in the group are assigned (2) different c values
- Go off and work with students from other groups who have the same c values
- Return to original group, share your results, combine findings.

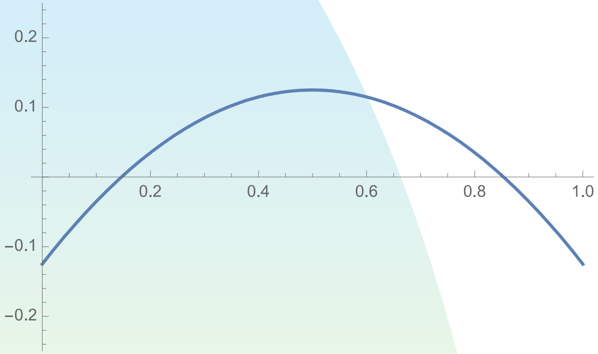
For Each c Value

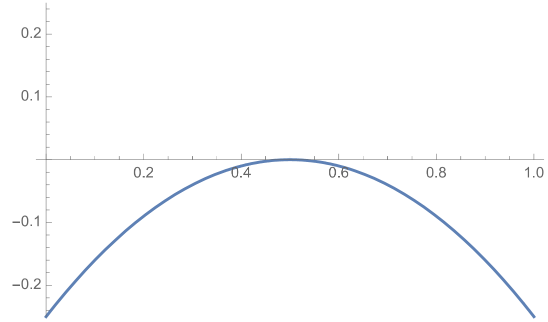
- Draw the graph of the slope function $f(P) = -P^2 + P - c$ as a function of P.
- Determine the equilibrium points.
- Draw the corresponding phase line diagram.
- Indicate whether an equilibrium point is attracting or repelling.

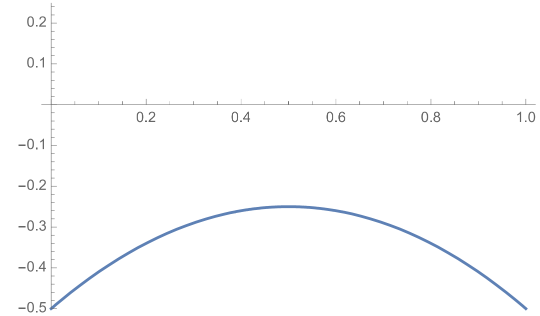
For Each c Value

c = .125 c = . 25 c = .5

f(P) = slope function







Phase Line: equilibrium points, up/down arrows

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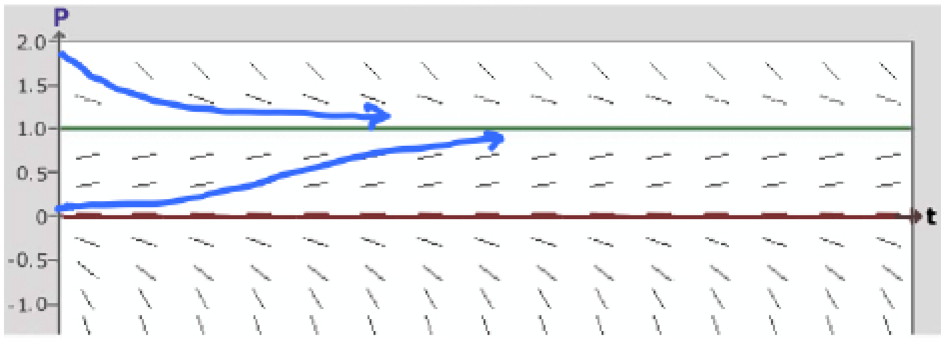

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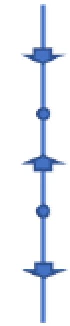
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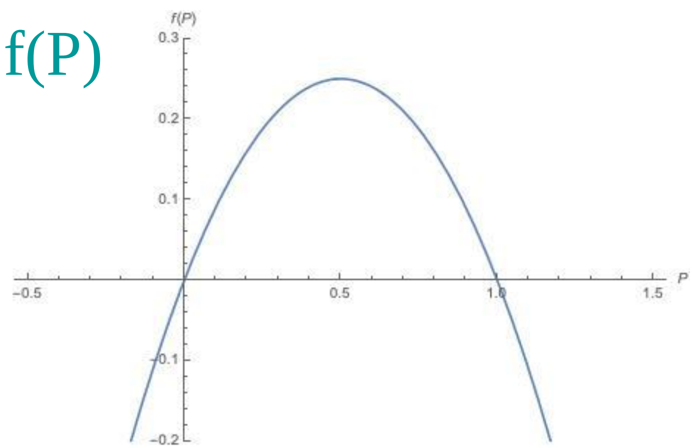
f(P)



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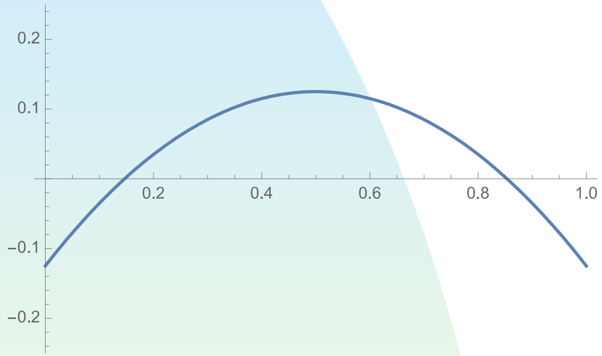

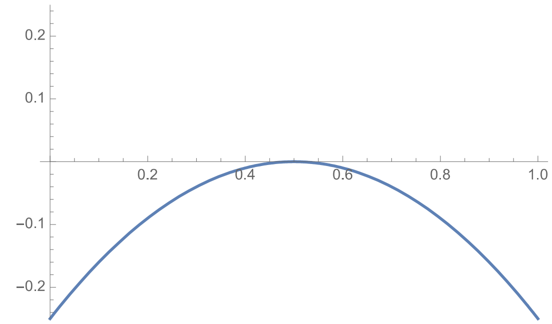

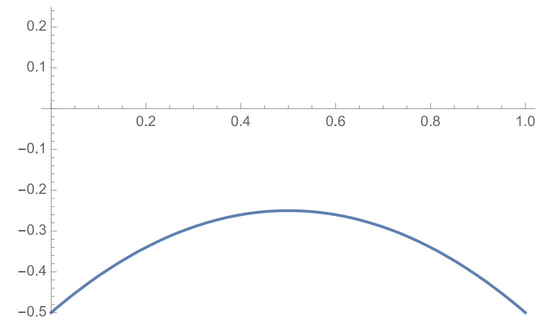

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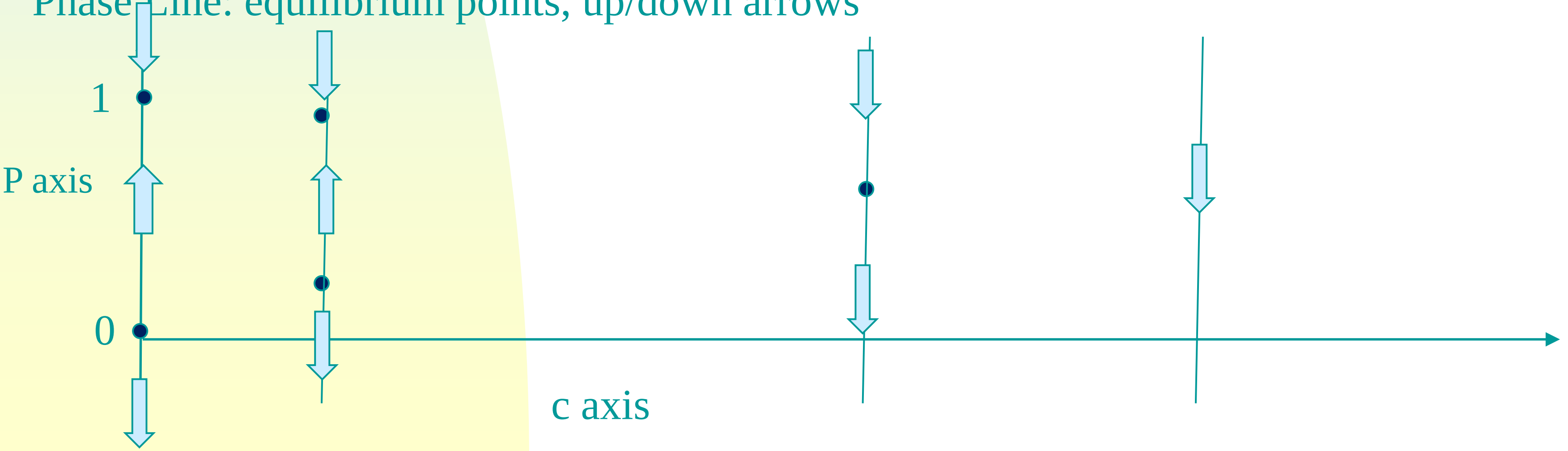


Bifurcation Diagram – Tipping Point

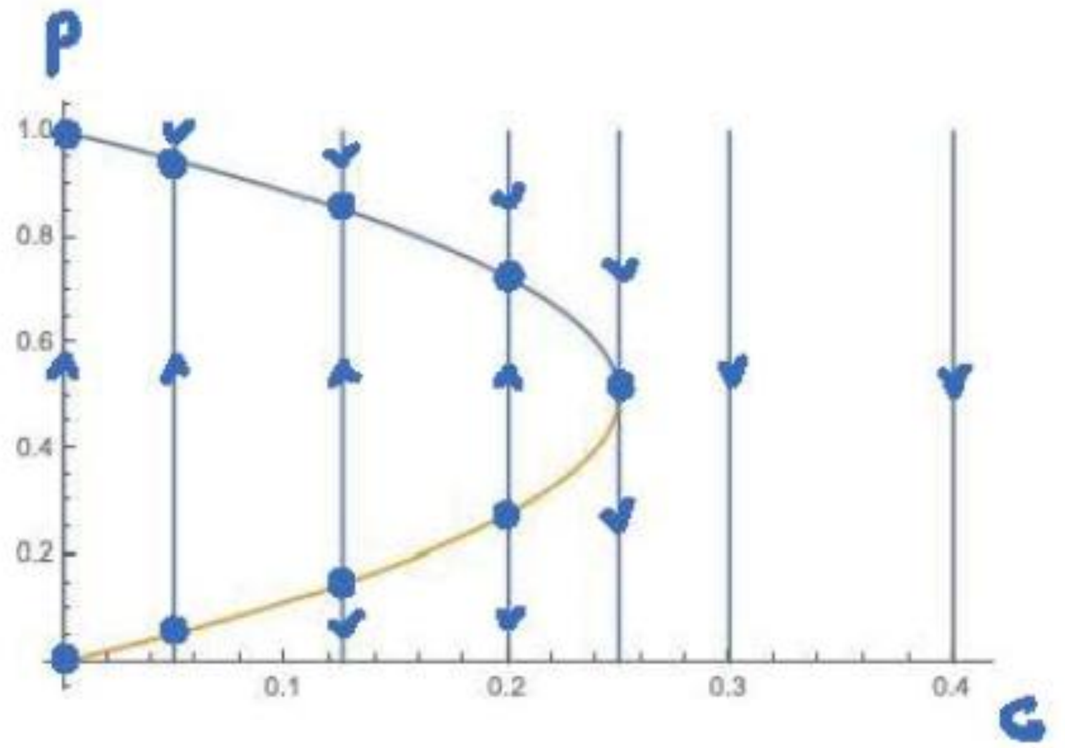
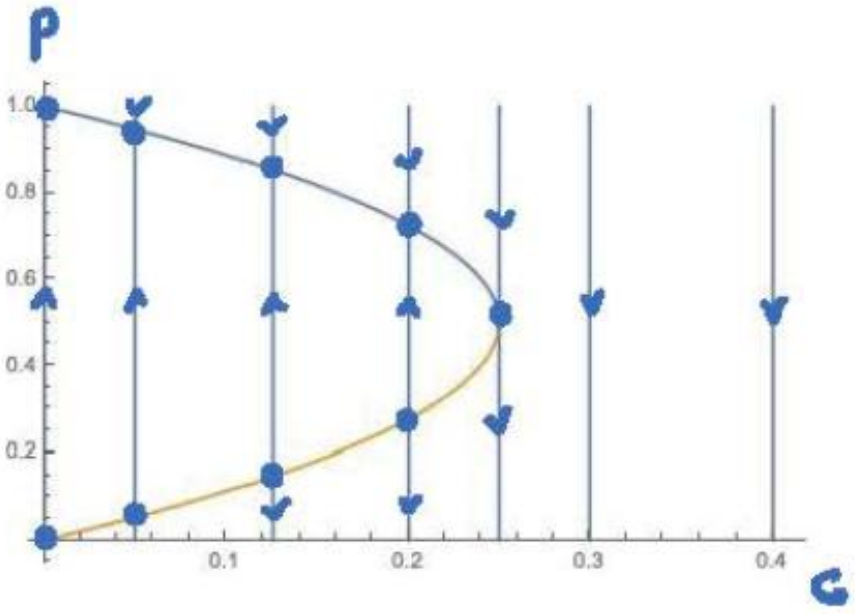


Figure 4. Bifurcation diagram for (3).

Add more c values and phase lines. Connect the dots.

Bifurcation Diagram – Tipping Point



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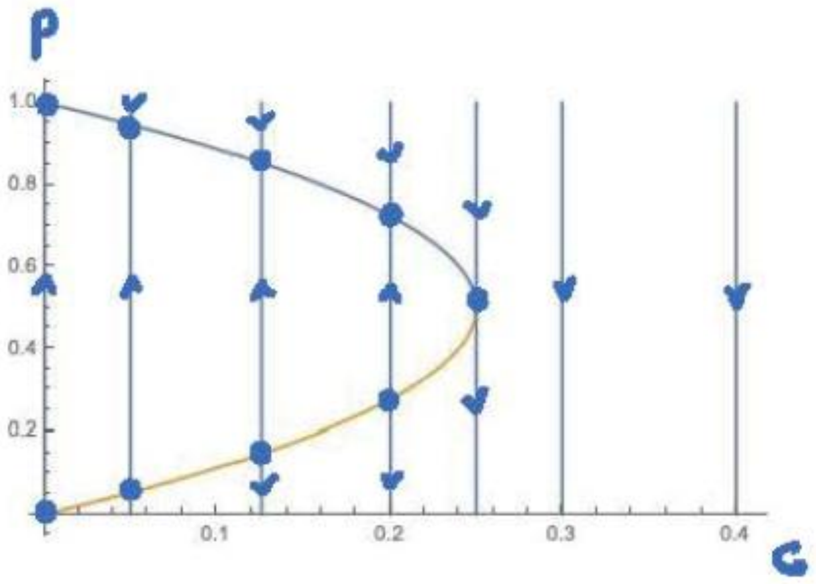
Q: What happens to the fish population (long term) if the fishing level c is high?
If the fishing level c is low to moderate?

Bifurcation Diagram – Tipping Point



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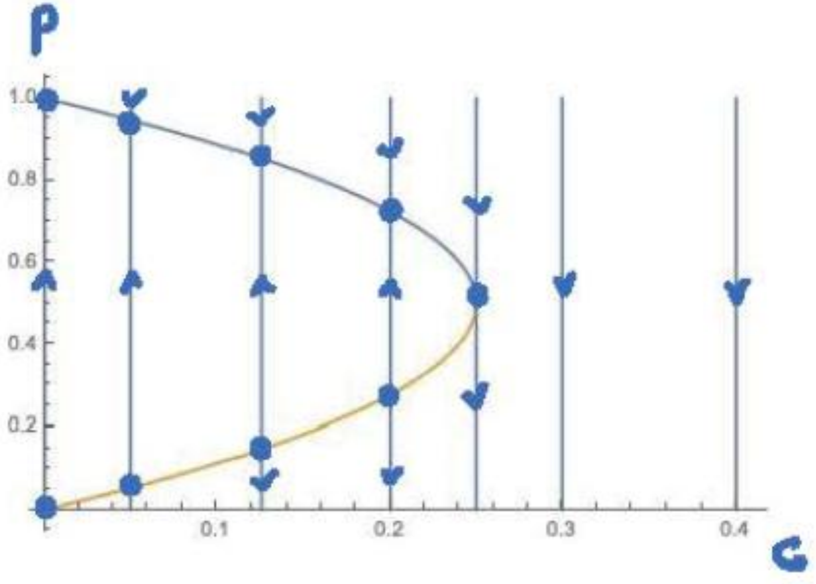
Recommendation: The government Department of Fisheries in partnership with business and environmental groups has set up a Fisheries Commission. The goal of the Commission is to set a quota for how many fish can be caught each year. As an expert on the mathematics of fish populations, you are being called to testify before the Fisheries Commission.

Based on your mathematical analysis, what recommendation will you give to the Commission?

Bifurcation Diagram – Tipping Point



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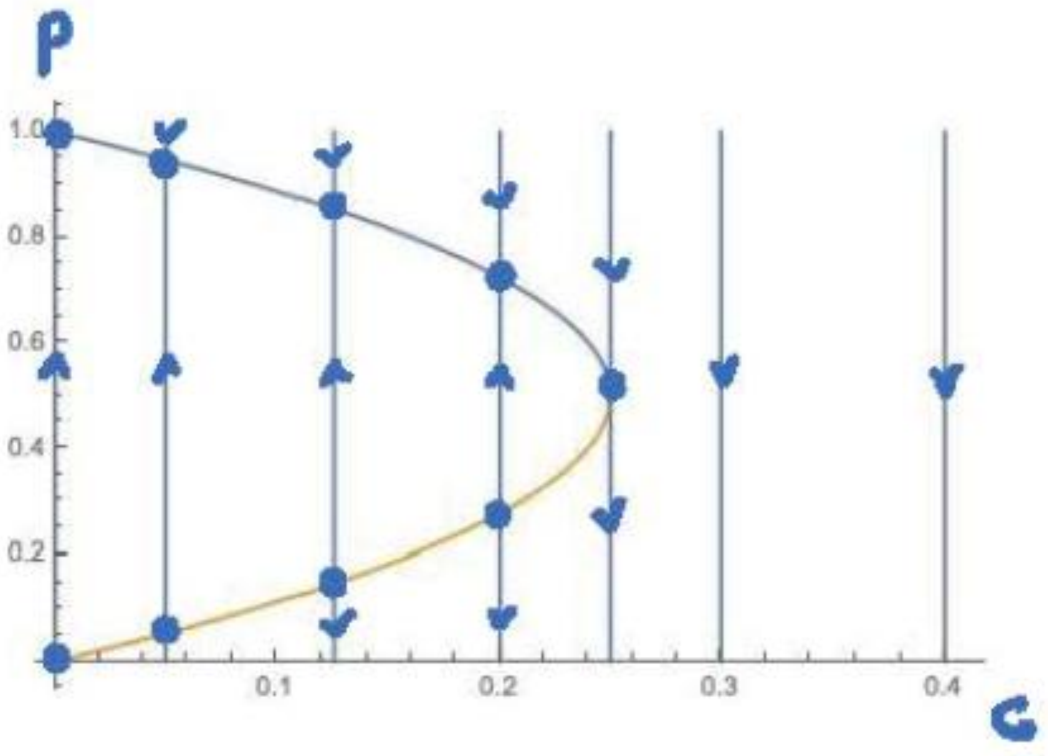
Q: What is the critical fishing level below which the fish population will survive and above which the fish population will die out?

This value is called a tipping point or in mathematical terminology a bifurcation point

Bifurcation Diagram – Tipping Point



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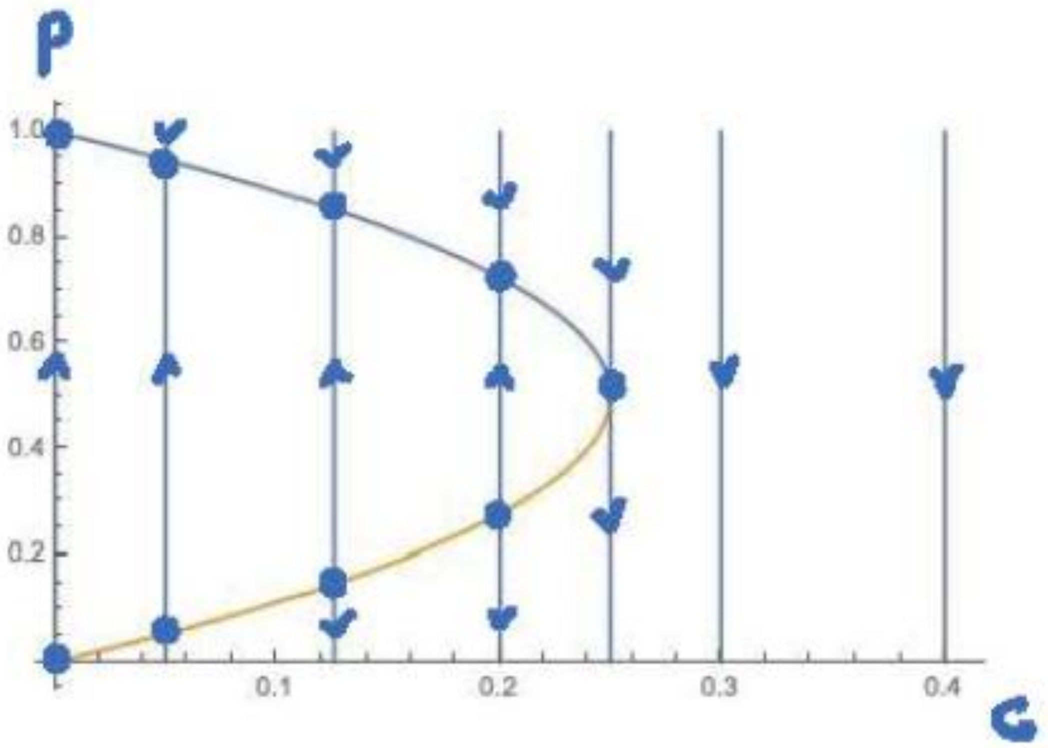
Bifurcation value c = .25 = tipping point

c < .25, stable steady state fishing population

c > .25, fish population crashes, extinction

Bifurcation Diagram – Tipping Point



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Small change in c causes dramatic in behaviour.

Bifurcation value c = .25
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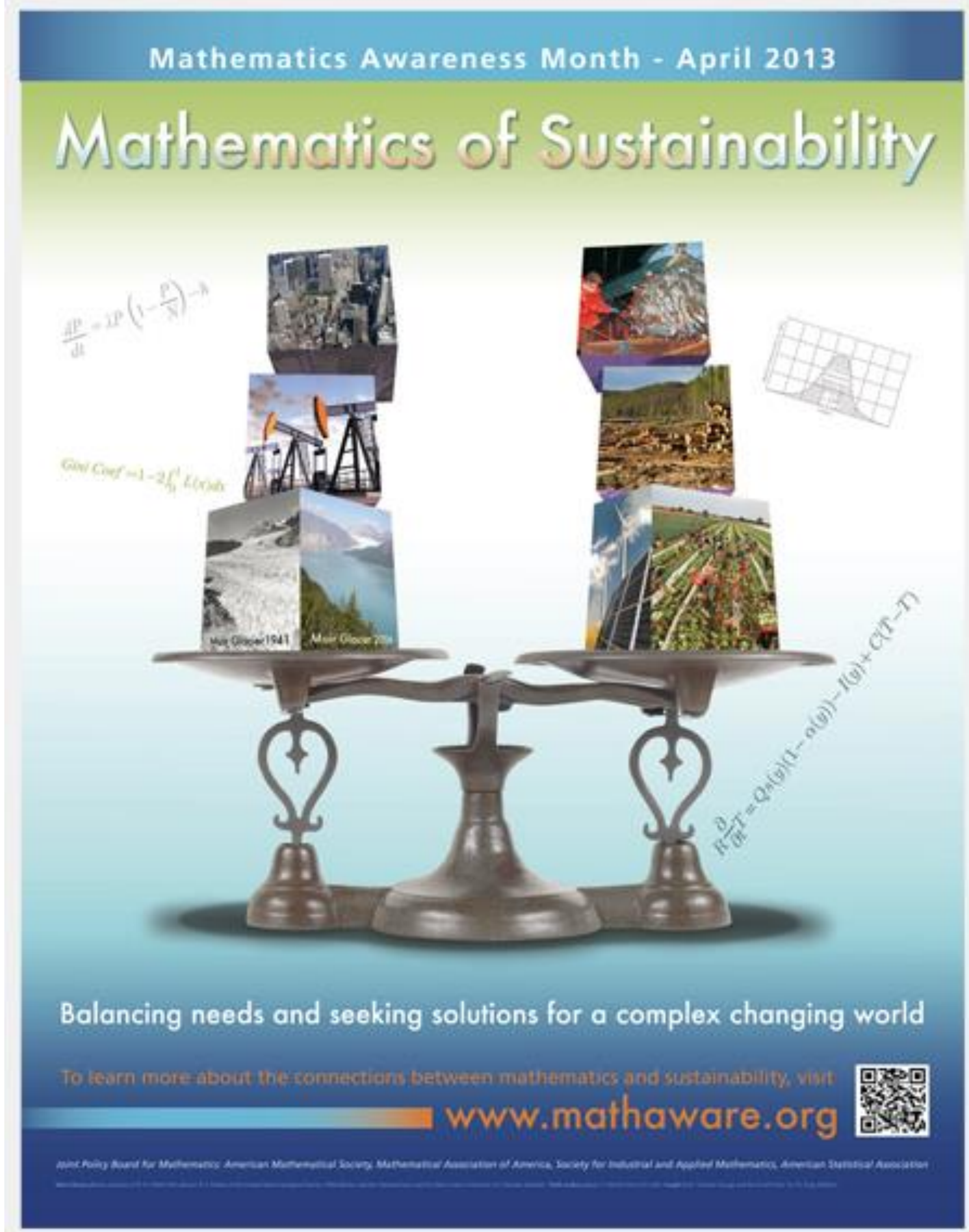
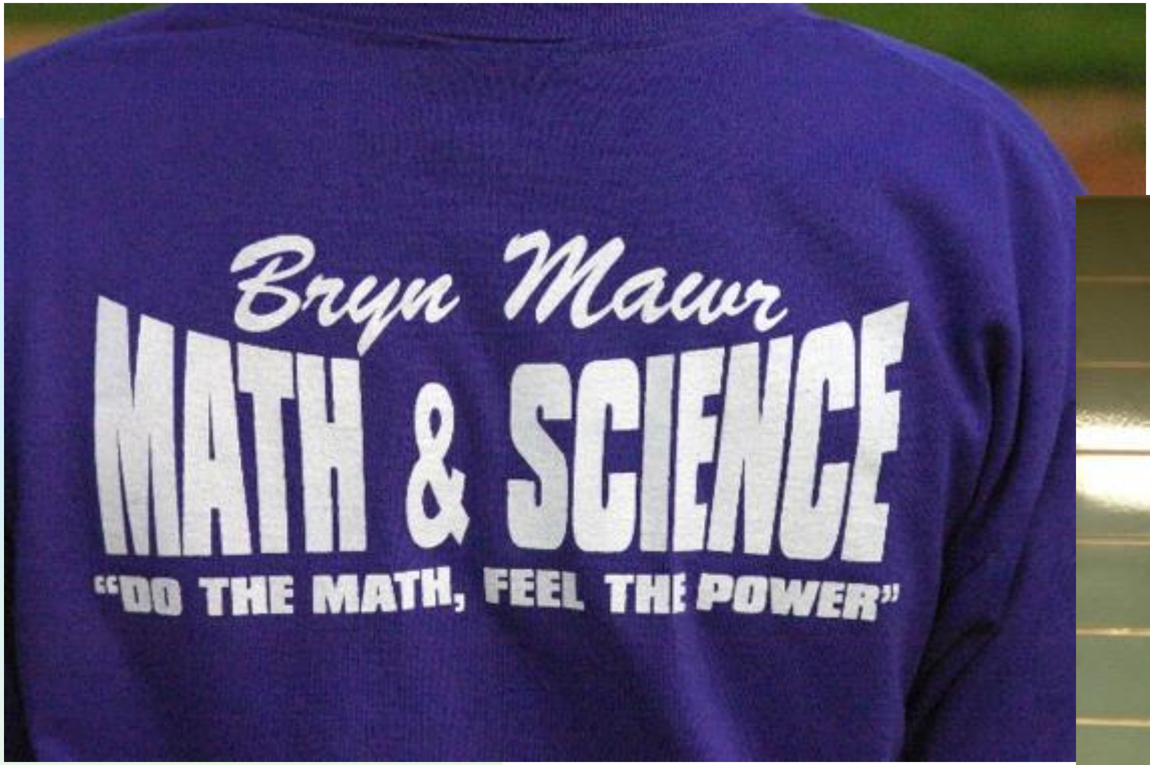
ww2.amstat.org/mam/2013/

Theme essays





2012 AMS Exemplary Department Award

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