Inquiry Oriented Differential Equations: A Student-Centered Curriculum

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Outline

- Who Are We?
- History of Inquiry Oriented Differential Equations (IODE)
- What is Inquiry?
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- What is the value of IODE?
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Who Are We?

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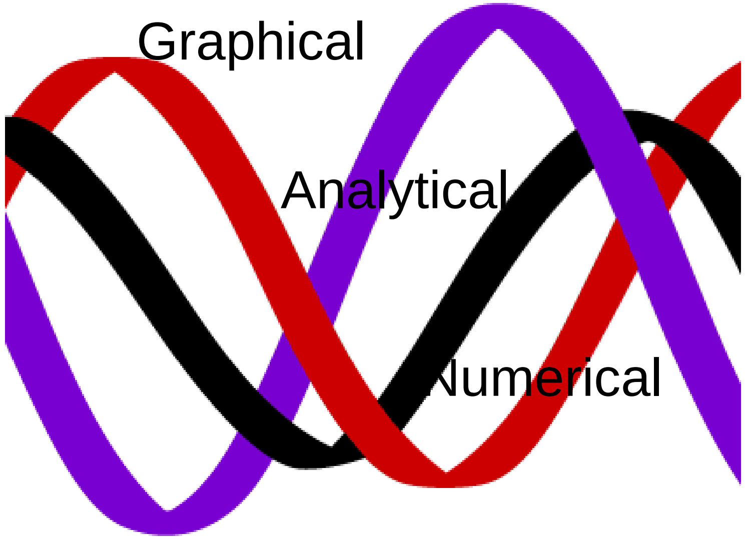
History of IODE

- **Instructional design theory**
  - Realistic Mathematics Education (RME) (Freudenthal, 1991)
    - Challenge learners to organize key subject matter at one level in order to produce new understanding at a higher level
- **Research-based curriculum**
  - "Framework for interpreting students' understandings of and difficulties with ideas central to new directions in differential equations" (Rasmussen, 2001, p. 55)
    - Function-as-solution dilemma
    - Students' intuitions and images
    - Students were learning analytic, graphical, and numerical methods in a compartmentalized manner

What is Inquiry?

- **Our use of the term inquiry aspires to aligns with the four pillars of Inquiry Based Mathematics Education (IBME) (Lauren & Rasmussen, 2019):**
  - students engage deeply with coherent and meaningful mathematical tasks,
  - students collaboratively process mathematical ideas,
  - instructors inquire into student thinking, and
  - instructors foster equity in their design and facilitation choices.

Philosophy of IODE



- **Using Realistic Contexts**
- **Developing DEs as models in three ways**
  - Using Laws of Science
  - Using Data
  - Modifying Existing Models
- **Model-of and Model-for** (Rasmussen et al., 2019)
- **Integrating Analytical, Graphical, Numerical Concepts**
- **Qualitative Approaches to Autonomous DEs**
- **Guided Reinvention of Concepts (Realistic Mathematics Education)**
- **First, Understanding, then Procedure**

What is the value of IODE?

- Utilizes qualitative, quantitative, and numerical methods integrated together
- Students take more ownership for learning
- Involves more than standard analytical techniques that only solve some DEs
- Context driven to engage and motivate students, while bringing out important mathematical ideas
- Modeling and Prediction are central themes
- “Capstone experience”- brings in calculus, etc.

Course Outline

- **First Order DEs**
  - Qualitative and Graphical Approaches
    - Modeling (Population)
    - What exactly is a differential equation and its solution?
    - Slope fields
  - A Numerical Approach
    - Euler's method
  - An Analytic Approach
    - Understanding solutions as continually changing slopes in Euler's method
    - Separation of variables
- Linear Differential Equations
  - Modeling (Salty (brine) tanks)
  - Integrating Factors (developed as the "Reverse Product Rule")
- Uniqueness of Solutions
  - The Uniqueness Theorem
- Autonomous Differential Equations
  - Phase Lines
- Modeling with Autonomous Differential Equations
  - Modeling (Newton's Law of Cooling)
  - Building phase lines from graphs of dx/dt vs x
- The Effect of Varying a Parameter in Autonomous Differential Equations
  - Modeling (Population and harvesting)
  - Bifurcation theory
  - Climate Change

Course Outline Continued

- **Systems, Second Order, & Nonlinear DEs**
  - Introduction to Systems
    - Modeling (Nonlinear Systems)
    - Euler's method for systems
    - 3D solution space
    - Phase planes, vector fields, nulliclines
  - Spring Mass System and Linear System
    - Modeling (Springs)
    - Straight Line Solutions
- Damped and Undamped Linear System
  - Complex eigenvalues
- Eigentheory Applied to Linear Systems
  - Using eigenvalues to find solutions to linear systems
- Second Order Linear Differential Equations
  - Method of undetermined coefficients
- Nonlinear Systems
  - Modeling (Pendulum)
- Laplace Transforms

Example Unit

- **Fish harvesting leading to the reinvention of the bifurcation diagram which then leads to a climate change problem**

IODE Unit 8: The Effect of Varying a Parameter in Autonomous Differential Equations

Fish Harvesting

A mathematician at a fish hatchery has been using the differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ as a model for predicting the number of fish that a hatchery can expect to find in their pond.

1. Use an autonomous derivative graph, a phase line, and a slope field to analyze what this differential equation predicts for future fish populations for a range of initial conditions. Present all three of these representations and describe in a few sentences how to interpret them.

2. Recently, the hatchery was bought out by fish.net and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). This means that the previous differential equation used to predict future fish populations needs to be modified to reflect this new plan. For the sake of simplicity, assume that this new plan can be taken into consideration by including a constant, annual harvesting rate k into the previous differential equation. Below are two modifications to the differential equation that may account for the new plan, as well as an option to create your own modification. Do you agree with (a) or (b)? If yes, explain why. If no, create your own modification and explain your reasoning.

(a) $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P}{25}\right) - kP$ (b) $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P-k}{25}\right)$ (c) Create Your Own

IODE Unit 8: The Effect of Varying a Parameter in Autonomous Differential Equations

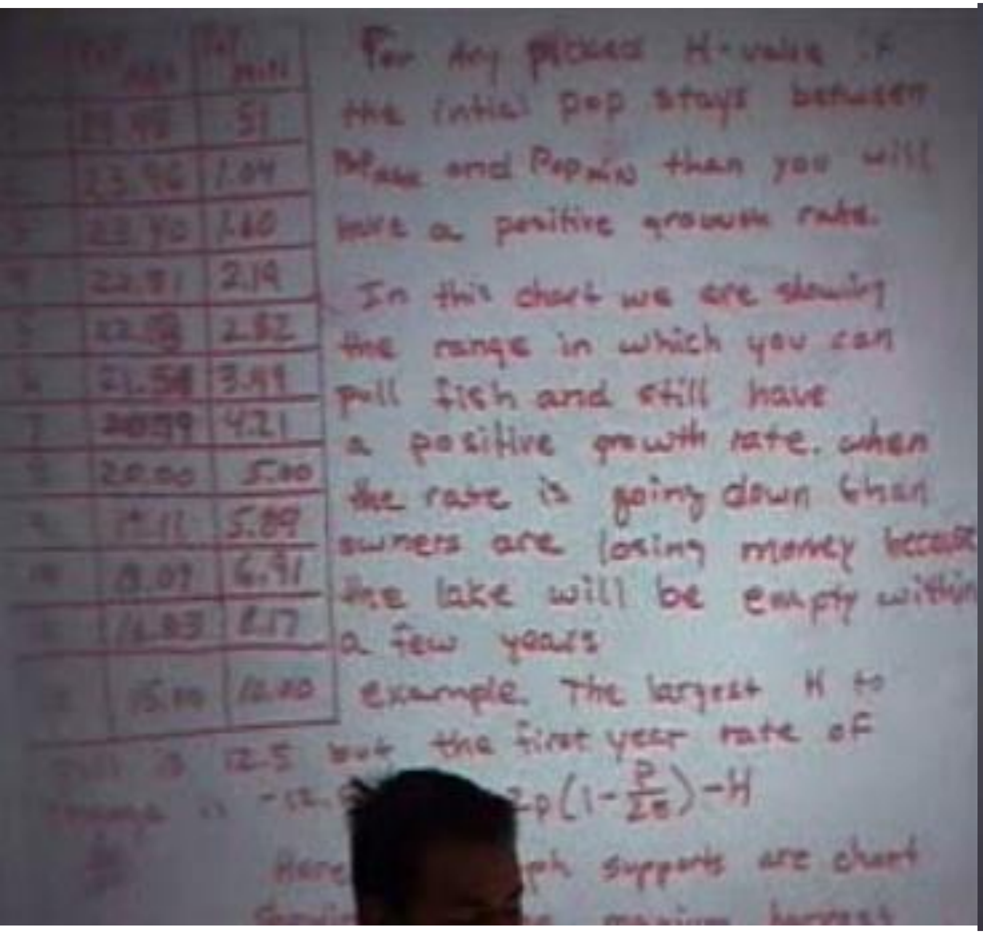
3. Your team of consultants settled on $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P}{25}\right) - k$ to model the new fishing plan. Analyze the effect of different choices for the value of k on the fish population. Synthesize your analysis in a **one page** report for the new owners that illustrates the implications that various choices of k will have on future fish populations. Your report may include one or more graphical representations but must communicate the effect of different k values in a concise way.

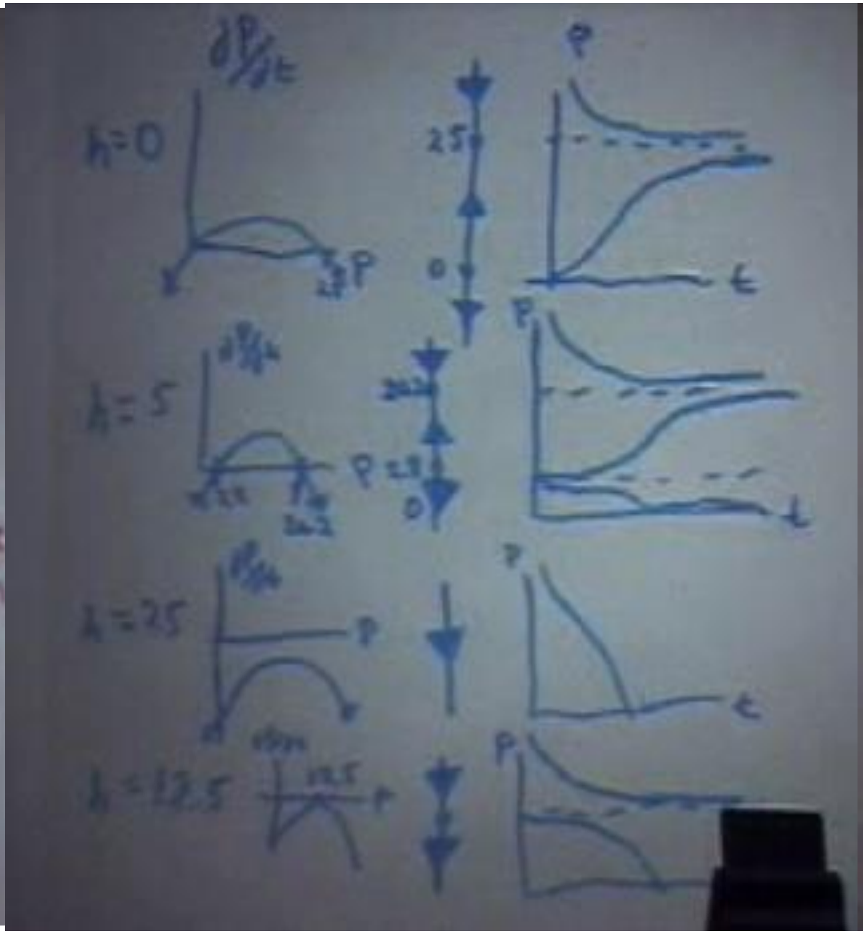
Modeling & Symbolizing → Reinvention of a bifurcation diagram

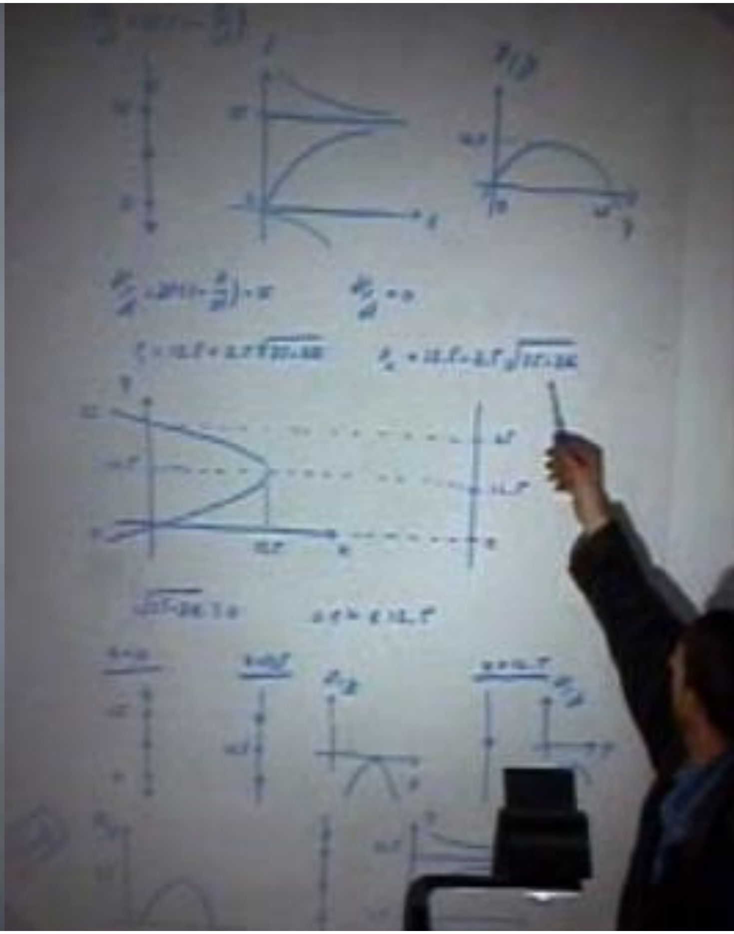
Important feature:

Did not ask students to find the best harvesting rate

Student Work







4. In studying climate, scientists are often concerned about positive feedback loops: two or more processes that amplify each other, creating a system of amplification that leads to a vicious cycle. One example is the interaction of water vapor with global temperature. If global temperature increases, the capacity of the atmosphere to contain evaporated water vapor also increases. If water resources are available, this would result in an increased amount of water vapor in the atmosphere. Water vapor is a greenhouse gas, thus if a climate system has more water vapor in the atmosphere, the global temperature will increase due to the increased insulation of the atmosphere. This positive feedback loop will eventually equilibrate at a higher temperature. Some scientists predict that a global increase in average temperature of just two degrees would be enough to kick off a system of positive feedback loops that would equilibrate at a temperature at least 6 degrees higher than we have now. This 6-degree increase would be enough to turn rainforests into deserts and melt ice caps. It may even redistribute the areas of the world that can support human life, i.e. making previously uninhabitable places, like the northern reaches of Siberia and Canada, inhabitable (though they may not support agriculture) and previously inhabitable places, like coastal cities, uninhabitable.

   (a) A modern pre-industrial average temperature at the equator is about 20 degrees Celsius. Assuming that our current global climate system has not undergone this vicious cycle, model this system with a phase line. What are the essential features of that phase line?

   (b) What is a simple differential equation that corresponds to your above phase line?

For more see Dunmyre, Fortune, Bogart, Rasmussen, & Keene (2019)

(c) A group of scientists came up with the following model for this global climate system:

$$\frac{dC}{dt} = \frac{1}{10}\left(C - 20\right)\left(22 - C\right)\left(C - 26\right) - k$$

where C is the temperature, in Celsius, and k is a parameter that represents governmental regulation of greenhouse gas emissions. Assume the baseline regulation corresponds to $k = 0$, increasing regulation corresponds to increasing k, and the current equatorial temperature is around 20 degrees. To what equatorial temperature will the global climate equilibrate?

(d) Sketch a bifurcation diagram and use it to describe what happens to the global temperature for various values of k.

In the context of climate science and government regulation, we specifically desired a negative k value to correspond to less regulation, that is, deregulation. Doing so necessitated the differential equation contain a “−k” so that a negative k results in a positive shift of the average equatorial temperature. While this differential equation does not capture the complexity of climate change science, it captures the long-term behavior described in the exposition to the extended problem.

(e) Suppose at the start of a new governmental administration, the temperature at the equator is about 20 degrees Celsius, and $k = 0$. Based on the model and other economic concerns, a government decides to deregulate emissions so that $k = -0.5$. Later, the Smokestack Association successfully lobbied for a 5% change, resulting in $k = -0.525$. Subsequently, a new administration undid that change, reverting to $k = -0.5$, and eventually back to $k = 0$. What is the equilibrium temperature at the equator after all of these changes?

(f) Use your bifurcation diagram to propose a plan that will return the temperature at the equator to 20 degrees Celsius.

Student Work

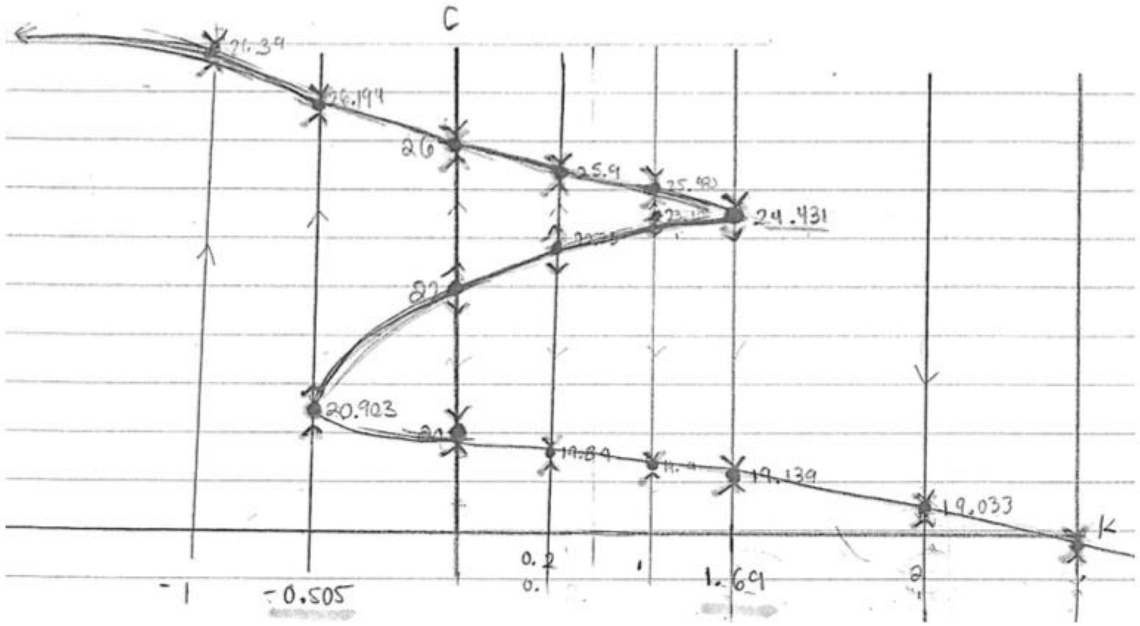


Figure 2: Example of student work.

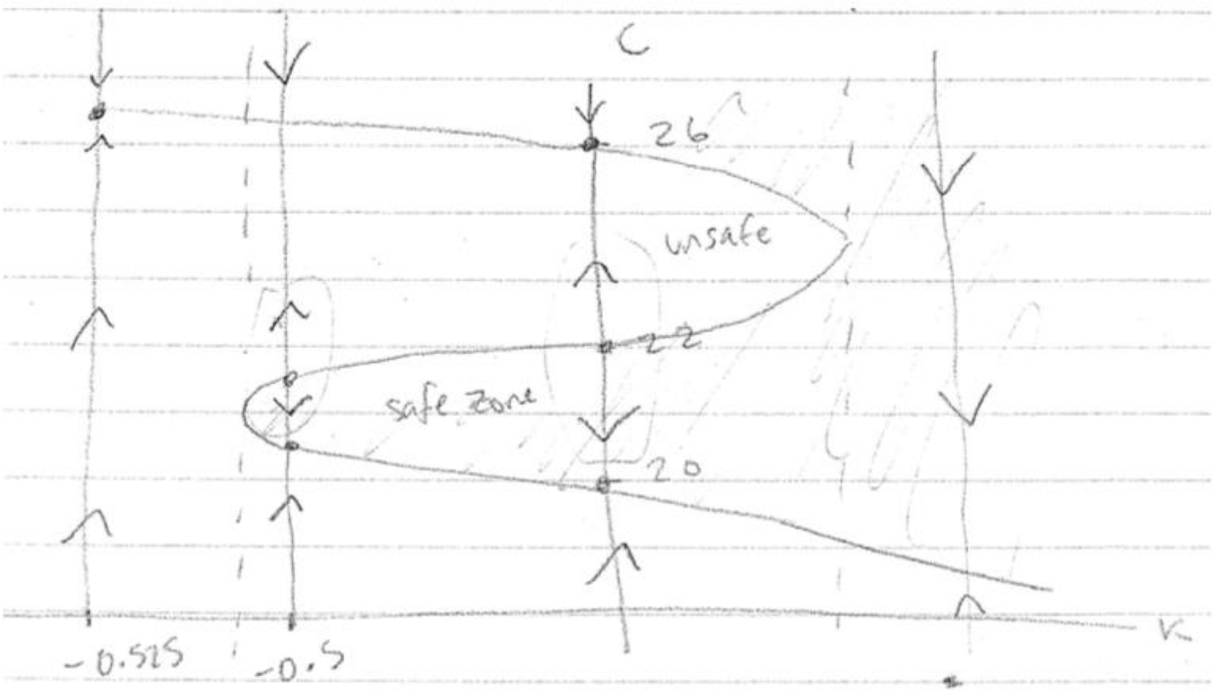


Figure 3: Example of student work.

After we get $k = -0.525$, we lose two equilibrium points and are forced to jump to the highest equilibrium point $y = 26$. Even after going back to $k = 0$ we still have to settle at $y = 26$ because we already jumped passed the repeller $y = 22$, which is sort of the "over-the-hill" point.

How to Access

- Materials are available for free at https://iode.sdsu.edu
- On the main page you will fill out a form to get a password to get access to the materials
- Materials in the form of PDFs but we also supply the LaTeX files
- Additional online resources are on the website

Thank You & Questions

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