Network Diagrams (e.g., Flow Graphs and Block Diagrams) in Understanding the Structure and Behavior of Differential Equations

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Session: Special Topics within Differential Equations Modelling Courses
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Dr. Carey Witkov
Assistant Professor of the Practice
Worldwide campus, Dept of Math, Science & Technology
Embry-Riddle Aeronautical University

INTRODUCTION

Visual methods for Differential Equations

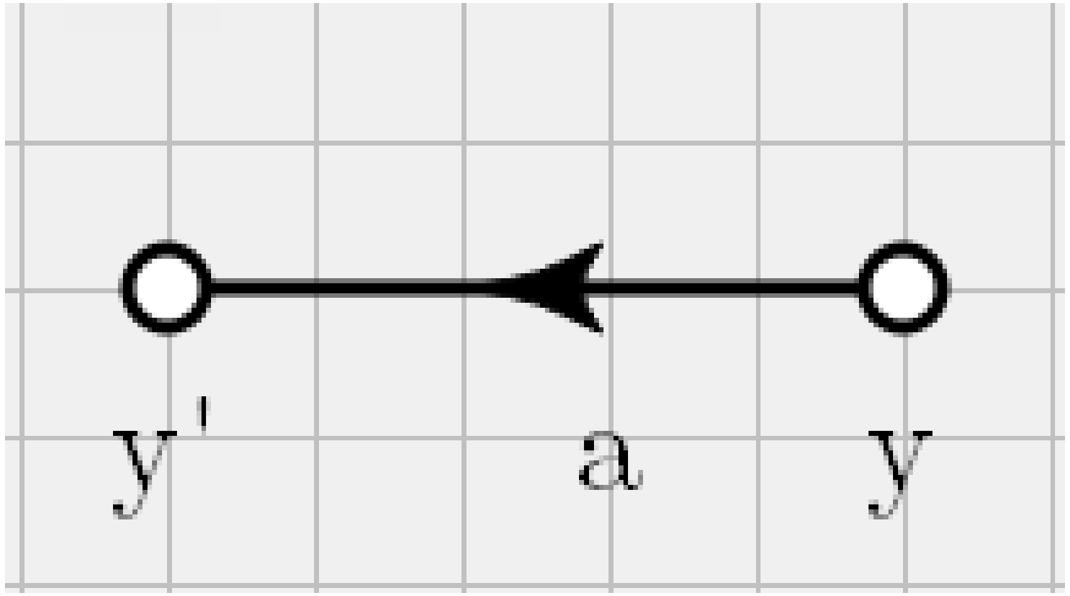
- Coordinate-based graphical methods (e.g., slope fields, phase plane plots).
- Coordinate-free network diagrams (e.g., flow graphs and block diagrams):
  - Topological (connectedness) and causal (directional).
  - Reveal structure and interconnections, e.g., feedback loops.
- "A differential equation is like a machine that is capable of movement and ready for action."
  - J. M. A. Danby, North Carolina State University, Computer Applications to Differential Equations, Reston Pub., 1985.
- **The differential equation becomes the object of study.**

Example 1
Signal Flow Graph
A simple ODE

The simplest differential equation

$$y' = ay$$

Nodes: variables

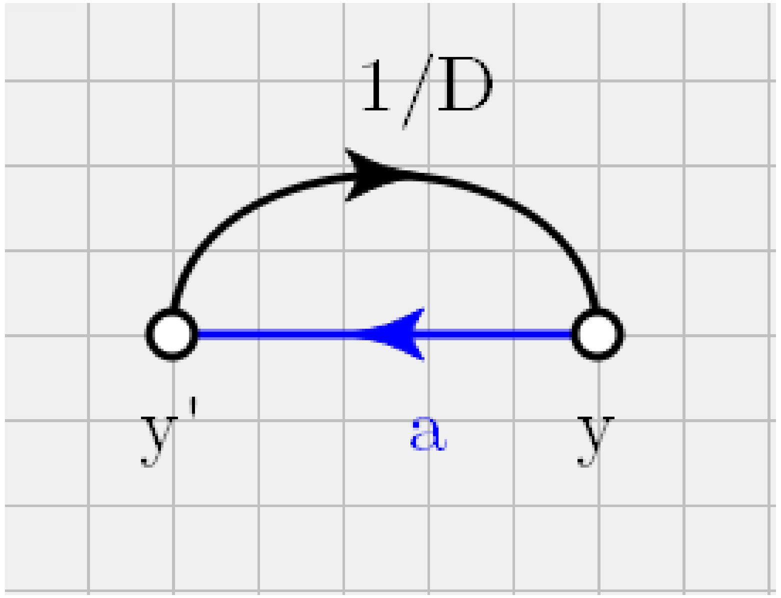


Edges: multiplier

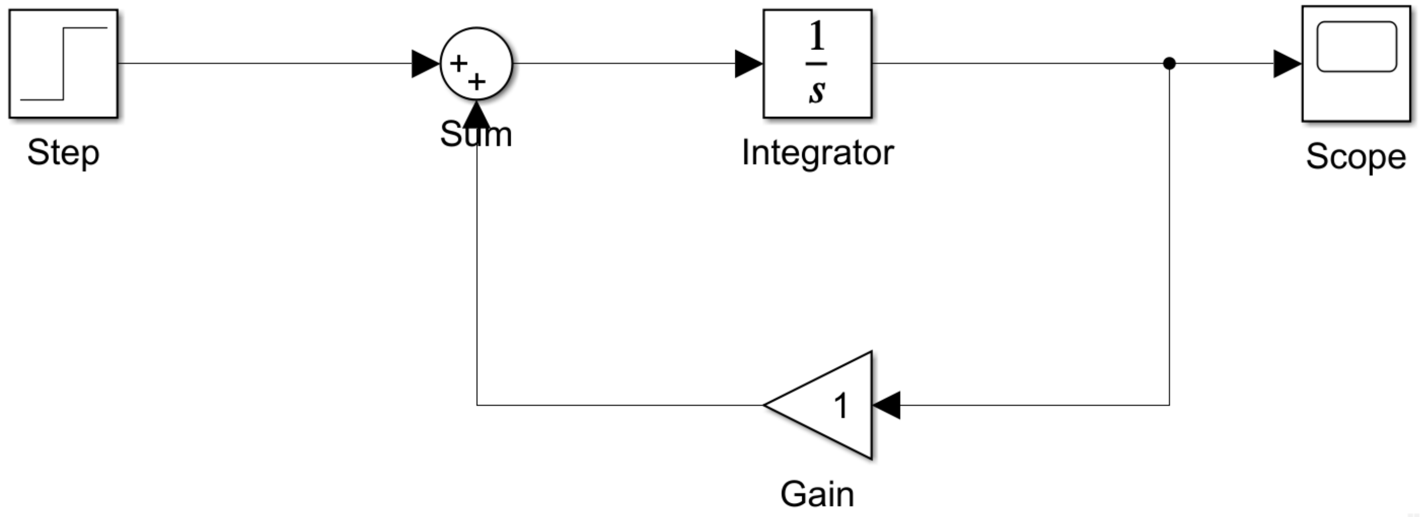
A 2nd (implied) equation

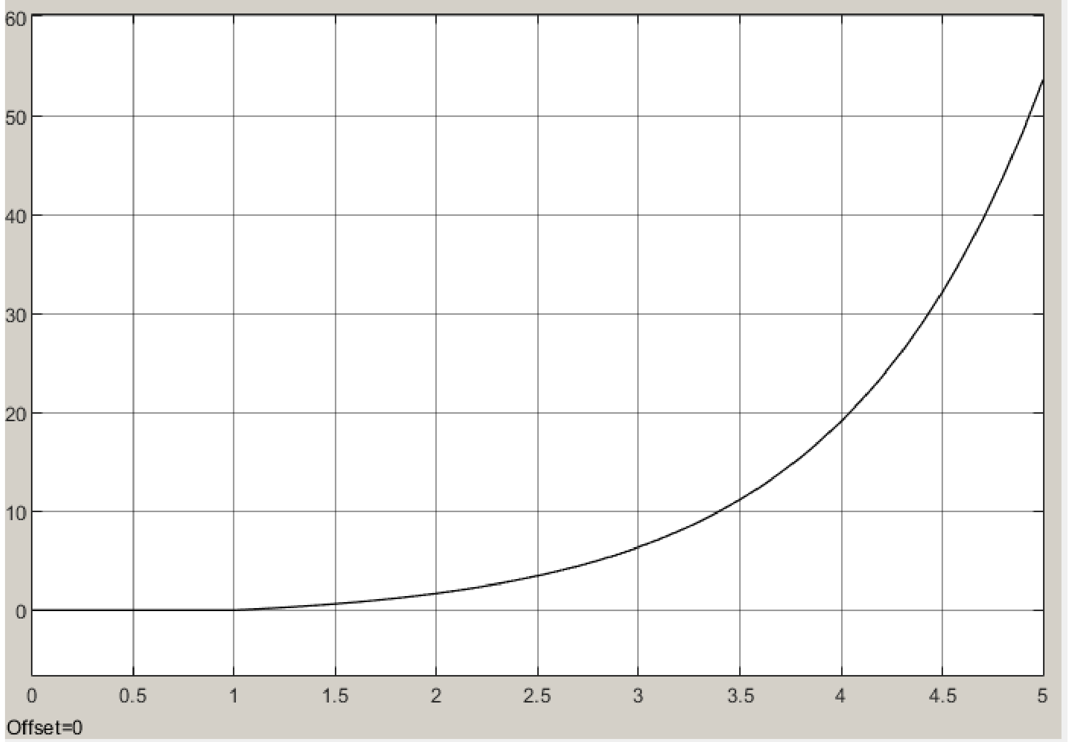
Differentiation: D operator $$y' = Dy$$

Integration: Inverse D operator $$y = \frac{1}{D}y'$$



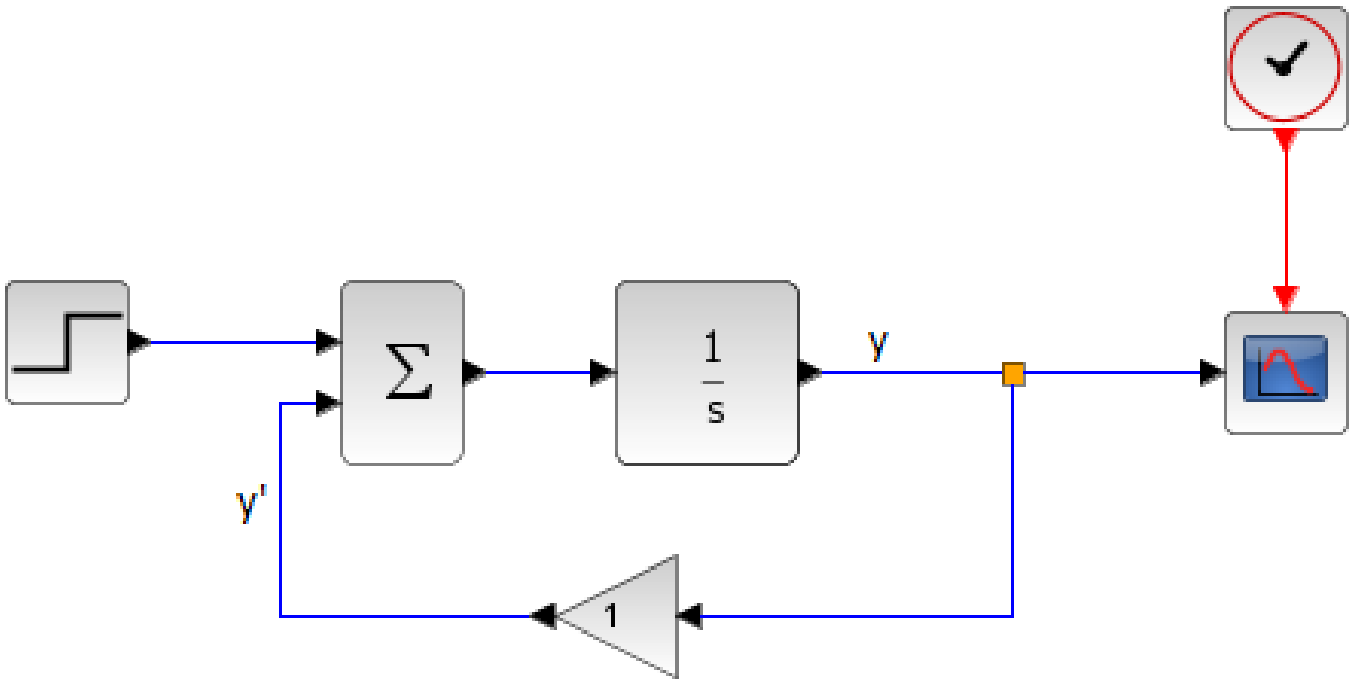
From SFG to executable block diagram in Simulink

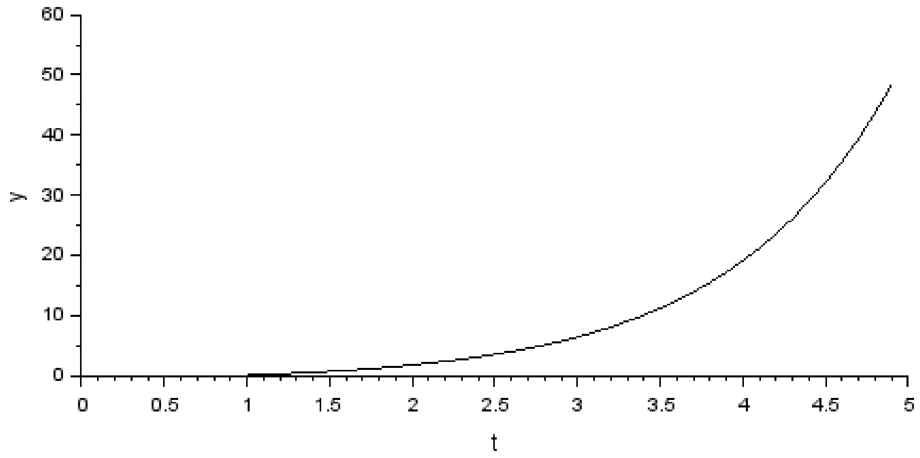




From SFG to executable block diagram in Scilab Xcos

https://www.scilab.org





Example 2
Signal Flow Graph
2^{nd} order ODE

Signal Flow Graph

Step 1: Start with an ODE.

$$y''(t) + 3y'(t) + 2y(t) = 5x(t)$$

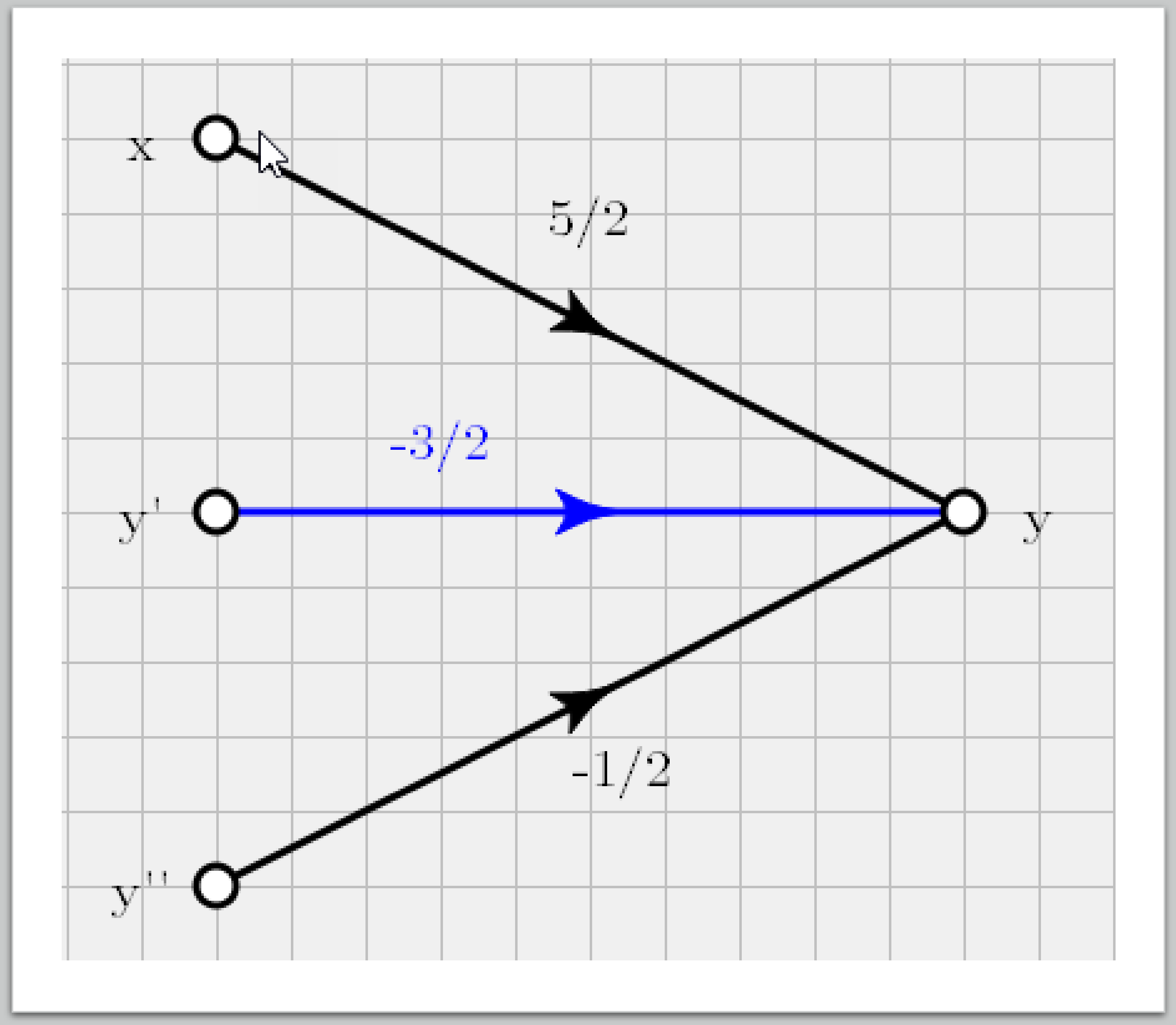
Step 2: Rearrange to solve for y(t).

$$y(t) = \frac{5}{2}x(t) - \frac{3}{2}y'(t) - \frac{1}{2}y''(t)$$

Signal Flow Graph

Step 3: Create a **partial signal flow graph** (nodes: variables; edges: transmissions).

(Incorrectly) assume all variables are independent except y.



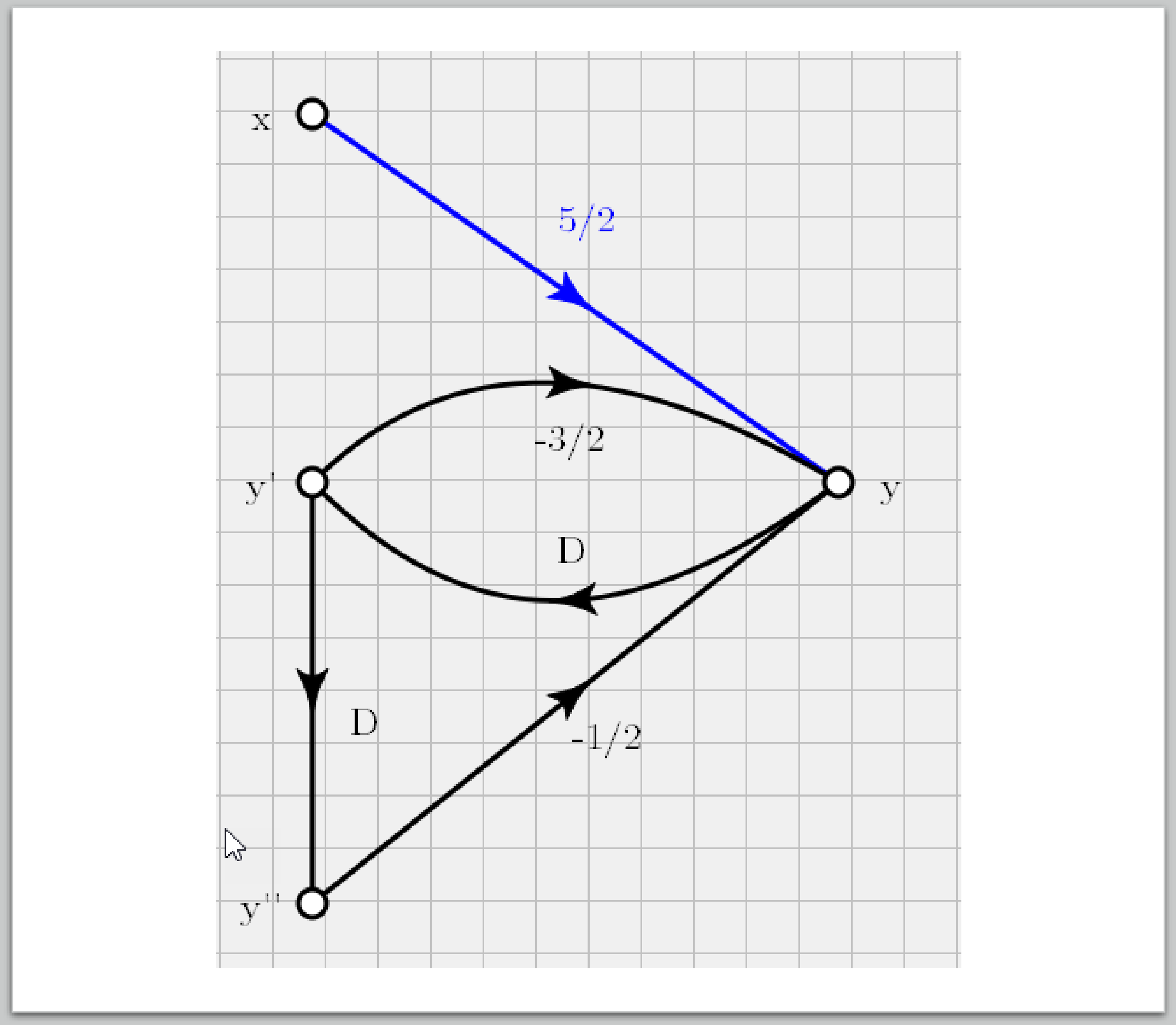
Step 4: Write down the implied equations

$$y' = Dy$$

$$y'' = Dy' = D^2 y$$

Signal Flow Graph

Step 5: Draw a **signal flow graph** using differential operator D, relating y to its derivatives.

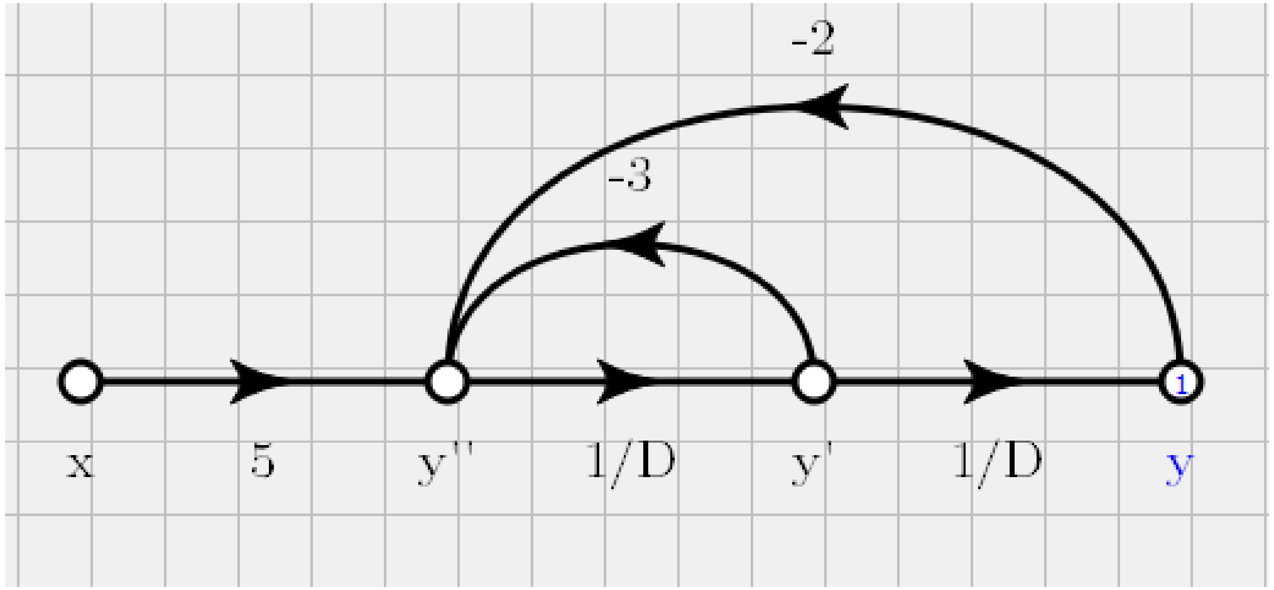


Signal Flow Graph

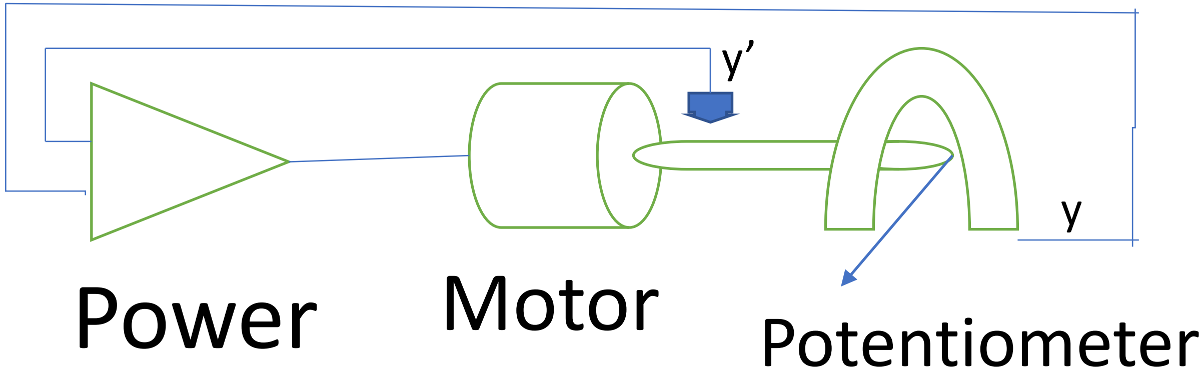
Step 6: Solve for the highest derivative:

$y'' = 5x - 3y' - 2y$

Draw SFG from source to response.

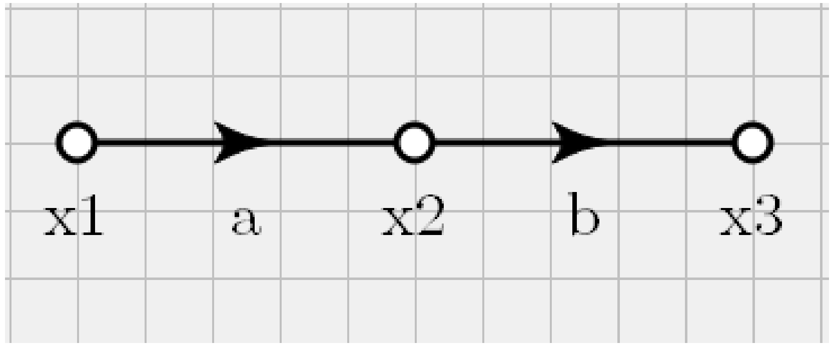


Motor position controller

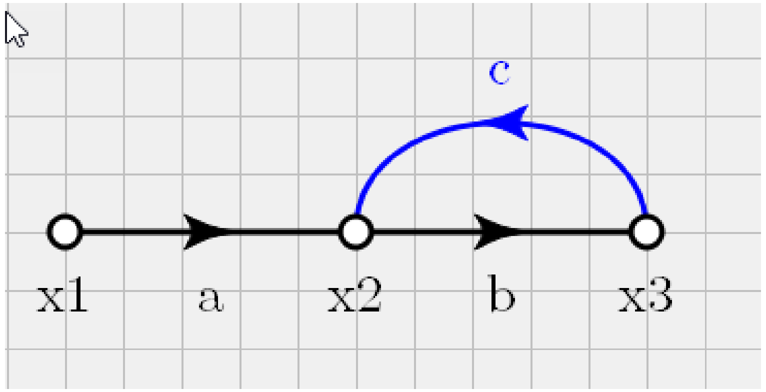


Example 3
Signal Flow Graph
2^{nd} order ODE
Sensitivity analysis

Open-loop vs closed-loop



Open-loop

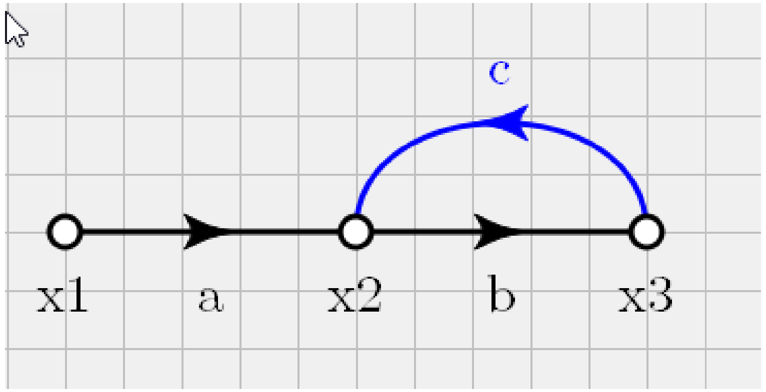


Closed-loop

Sensitivity analysis 1

Knowledge of transmittance between nodes can be used to determine sensitivity of overall gain on parameters.

Multiplier



$$\frac{x_2}{x_1} = a$$

$$x_2 = ax_1 + cx_3$$

$$x_3 = bx_2$$

$$\frac{x_3}{b} = ax_1 + cx_3$$

$$\frac{x_3}{x_1} = G = \frac{ab}{1-bc}$$

Open-loop $\frac{x_3}{x_1} = ab$

Multiplier effect of loop: $\frac{1}{1-bc}$ $\frac{x_3}{x_2} = \frac{1}{1-bc}$

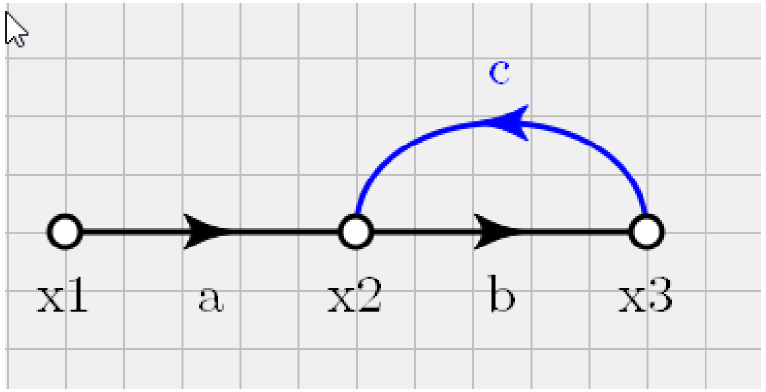
$$\frac{x_3}{x_1} = ab[1 + bc + (bc)^2 + (bc)^3 + \ldots + (bc)^n]$$

Lynch, W., Truxal, J.G. Introductory System Analysis, NY: McGraw-Hill, 1961.

Sensitivity analysis 2

Knowledge of transmittance between nodes can be used to determine sensitivity of overall gain on parameters.

Multiplier



$$\frac{x_2}{x_1} = a$$

$$\frac{x_3}{x_2} = \frac{1}{1-bc}$$

$$\frac{x_3}{x_1} = G = \frac{ab}{1-bc}$$

$$S_{g_k}^{G} = \frac{\frac{\partial G}{G}}{\frac{\partial g}{g_k}} = \frac{g_k}{G}\frac{\partial G}{\partial g_k}$$

$$S_a^G = 1$$

$$S_b^G = \frac{1}{1-bc}$$

$$S_c^G = \frac{bc}{1-bc}$$

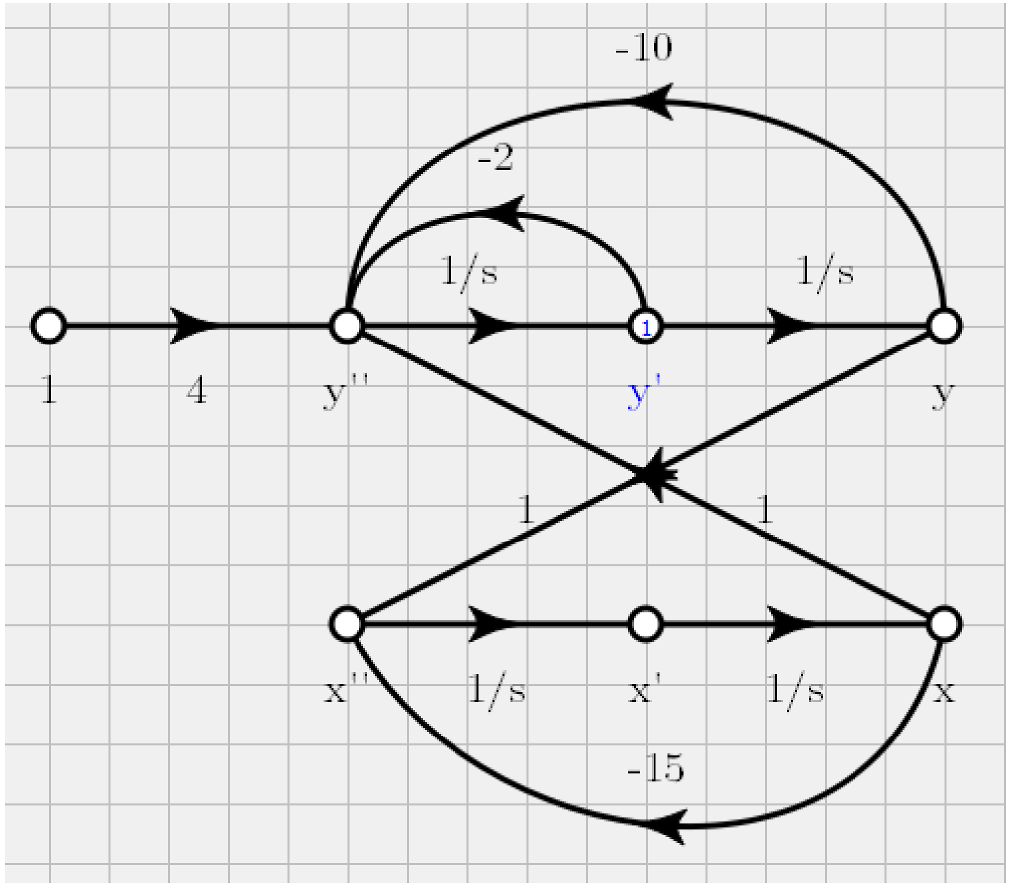
Lynch, W., Linear Control Systems – A Signal-Flow Graph Viewpoint (Ch 2 in Mishkin, E. , Braun, L., Adaptive Control Systems), NY: McGraw-Hill, 1961

Truxal, John G., Introductory System Engineering, NY: McGraw-Hill, 1972.

Systems of differential equations

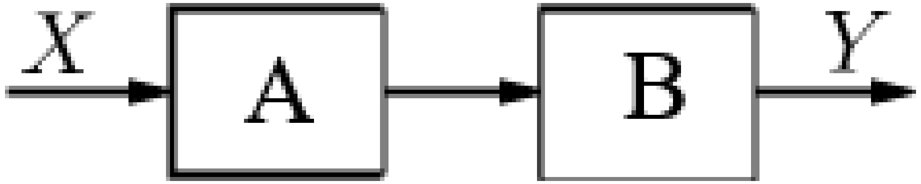
$$y'' = 4 + x - 2y' - 10y$$

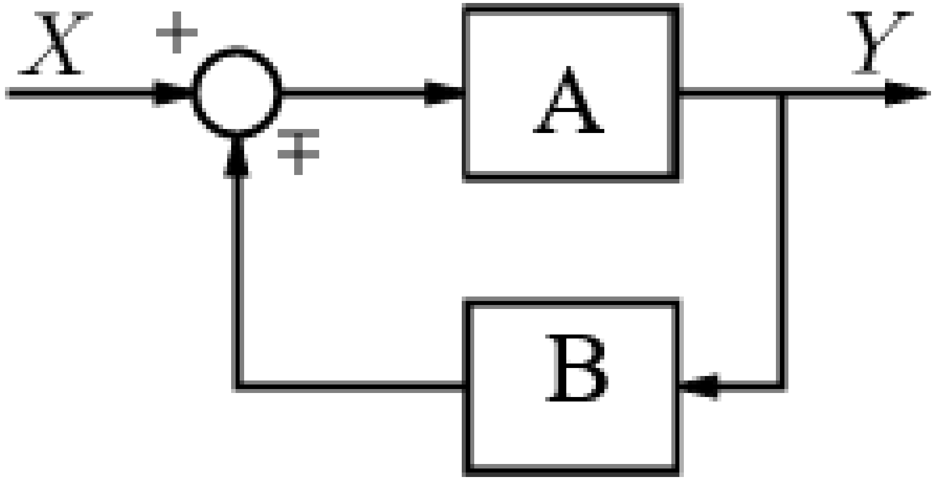
$$x'' + 15x - y = 0$$



Example 4
Block diagram
Nonlinear oscillator (Duffing)
Feedback view of nonlinearity

Block Diagrams





Forced undamped Duffing equation

$$y''(t) + \omega_o^2 y(t) \pm b[y(t)]^3 = A\cos(\omega_d t)$$

A special case of the pendulum

$$\theta'' + b\sin(\theta) = A\cos(\omega_d t)$$

$$\theta'' + b(\theta - \frac{\theta^3}{3}) = A\cos(\omega_d t)$$

$$\theta'' + b\theta - b\frac{\theta^3}{3} = A\cos(\omega_d t)$$

Transfer function for the undamped harmonic oscillator

$$y''(t) + \omega_o^2 y(t) = x(t)$$

$$D^2 y(t) + \omega_o^2 y(t) = x(t)$$

$$s^2 Y(s) + \omega_o^2 Y(s) = X(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{1}{s^2 + \omega_o^2}$$

If $\omega_o = 1$ then $H(s) = \frac{1}{s^2+1}$

D = s (for linear systems w/o initial conditions)
The Substitution Rule

(using properties of the derivatives of exponentials)

Let P(D) = a polynomial differential operator

Prove $P(D)e^{st} = P(s)e^{st}$

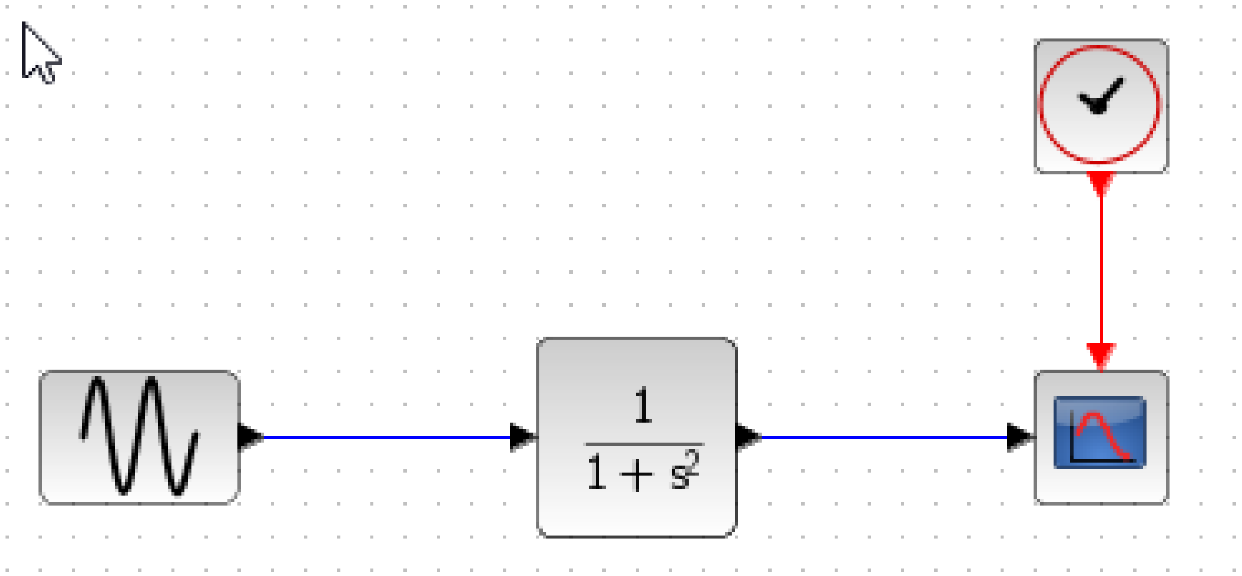
By example, $P(D) = D^2 + 8D + 7$

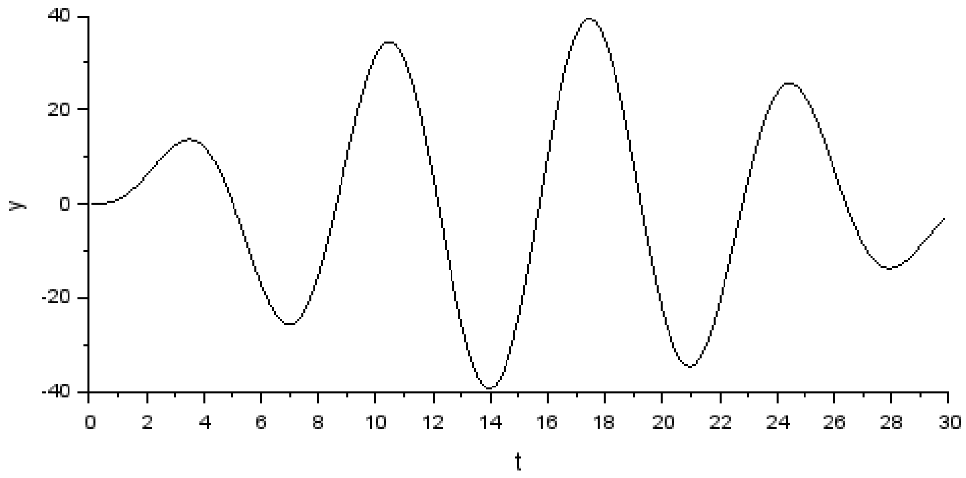
$$P(D)e^{st} = s^2e^{st} + 8se^{st} + 7e^{st}$$

$$P(D)e^{st} = (s^2 + 8s + 7)e^{st} = P(s)e^{st}$$

Adapted from Jeremy Orloff, Complex Variables with Applications, LibreText.
https://math.libretexts.org/@go/pages/51237

Executable block diagram using Scilab Xcos





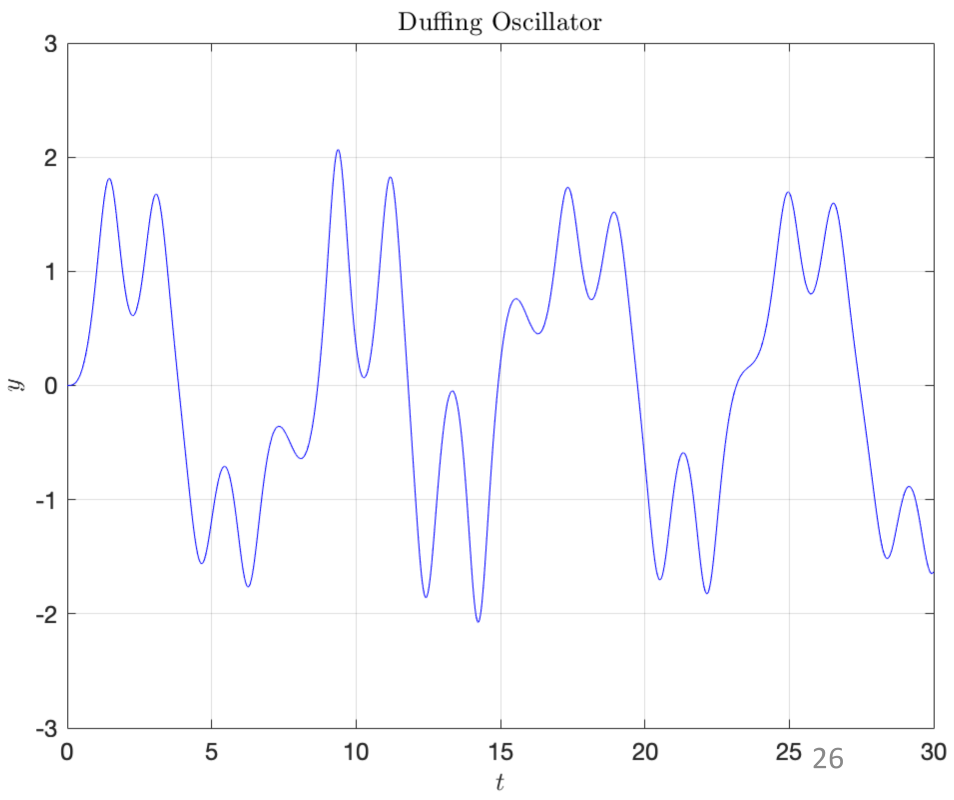
Code-based simulation of Duffing equation

File 1: Function m-file

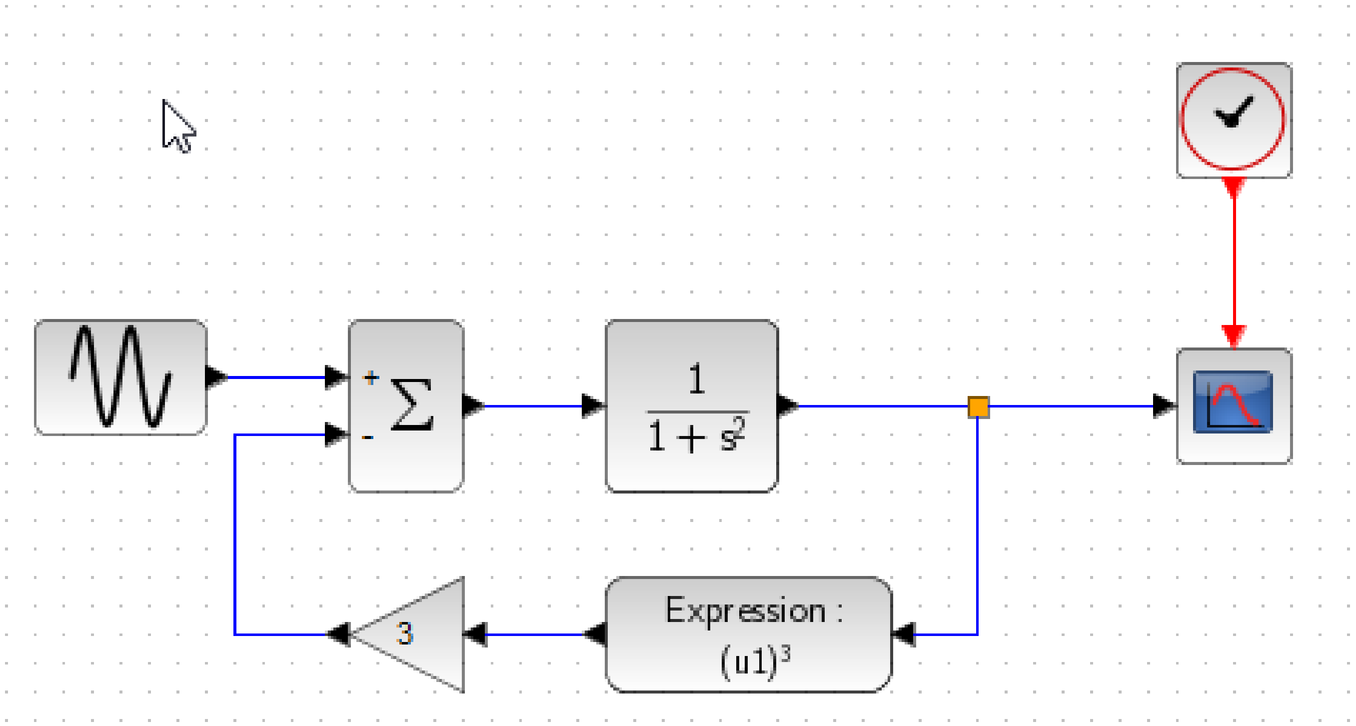
```
function yp = duffing(t,y,p)
a = p(1); b = p(2); c = p(3); w = p(4);
yp1 = y(2);
yp2 = -a*y(1) - b*y(1)^3 + c*sin(w*t);
yp = [yp1;yp2];
end
```

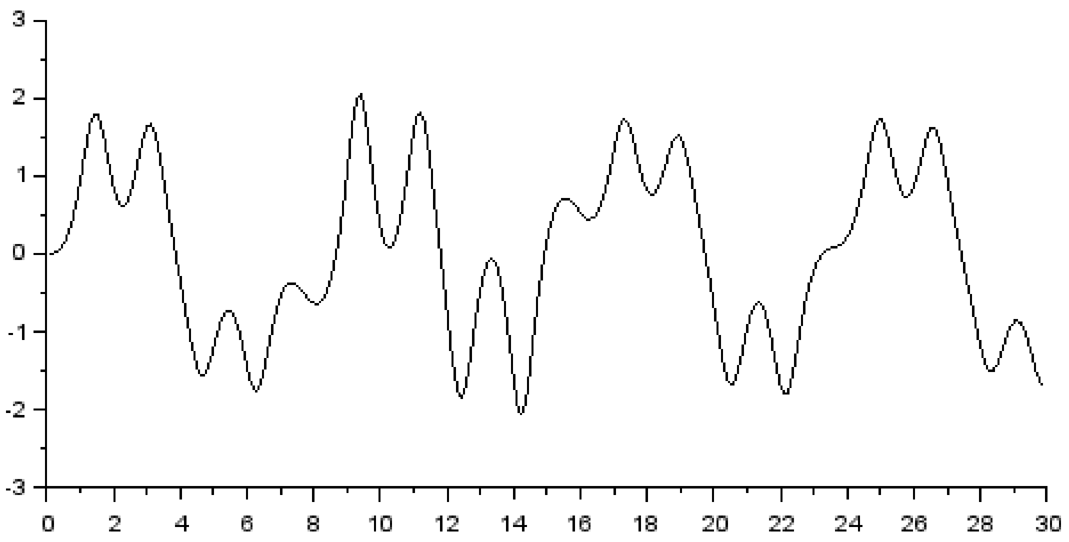
File 2: Calling and plotting script

```
clear all
figure(1)
title1 = 'Duffing Oscillator';
xlab = '$t$';
ylab = '$y$';
t1 = [0:0.002:30];
p1 = [1,3,8,0.8]; % y_coeff=1; nonlin_coeff=3; Amp=8; forcing_freq=0.8 (system_freq=1)
[t,y] = ode45(@duffing2,t1,[0;0],[],p1);
plot(t,y(:,1),'b-')
grid;
xlim([0 30]);
ylim([-3 3]);
font1= 'Times New Roman';
xlabel(xlab,'FontSize',14,'FontName',font1,'interpreter','latex');
ylabel(ylab,'FontSize',14,'FontName',font1,'interpreter','latex');
title(title1,'FontSize',18,'FontName',font1,'interpreter','latex');
set(gca,'FontSize',12);
print -depsc duffing.eps
```



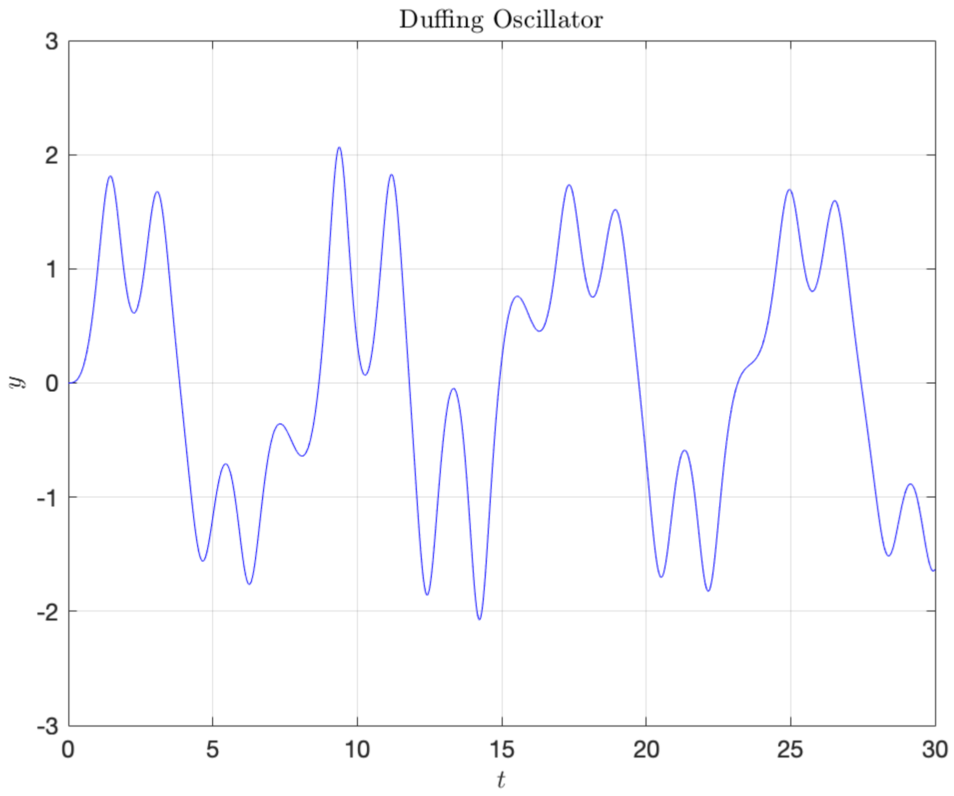
Block Diagram simulation of Duffing oscillator



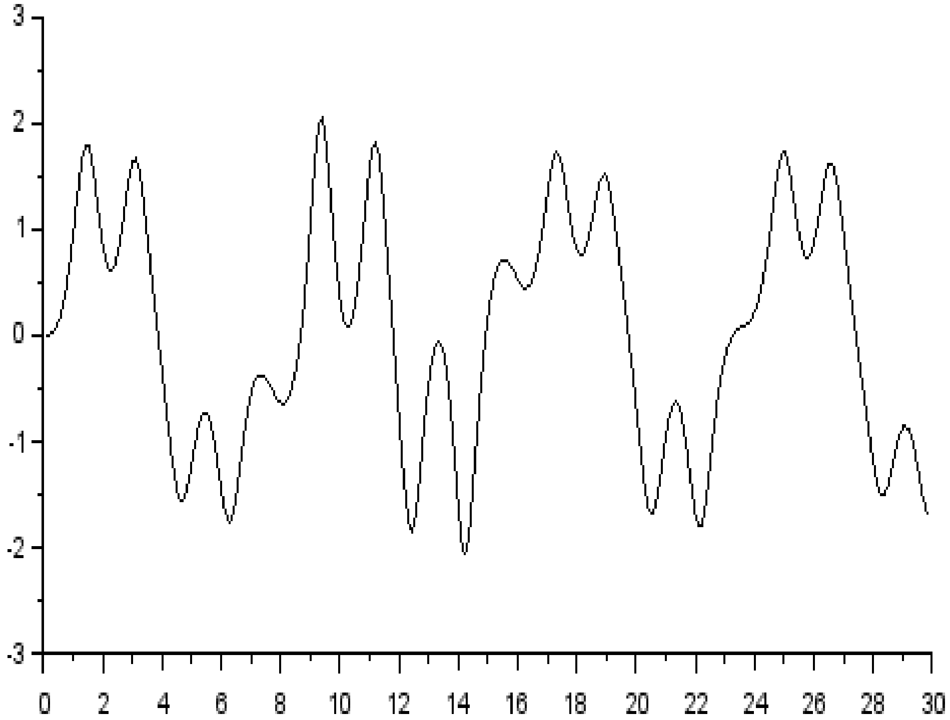


Jordan, D W, and Peter Smith. *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers*. Oxford Univ Press, New York, 2007.

Simulation results comparison



Script file simulation



Block diagram simulation

Example 5
Block diagram
Optimal feedback structure & parameters

Optimal control – electromechanical example - calculus of variations

$$x_1' = -x_1 + u$$
$$x_2 = x_1$$

u = control (e.g., force)

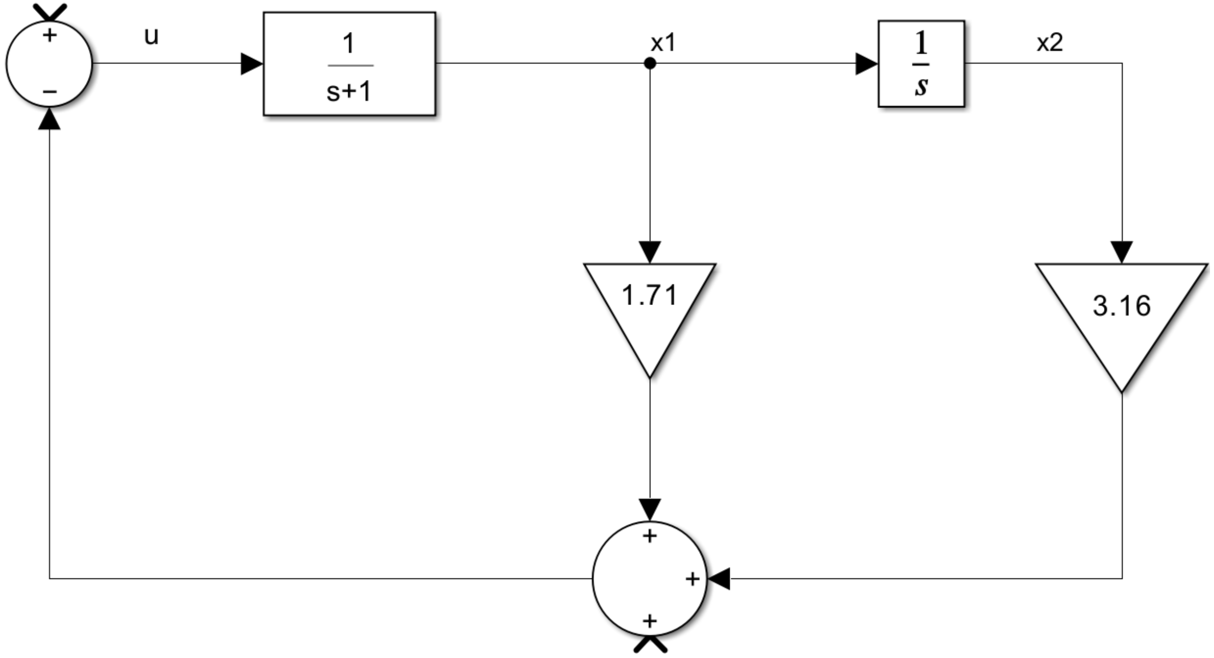
Minimize $\int_0^T x_2^2 dt$

Noton, A.R.M., Introduction to Variational Methods in Control Engineering, Oxford: Pergamon Press, 1965.

ODE control law vs block diagram control law

Control law

$$u(0) = -1.71x_1(0) - 3.16x_2(0)$$



Proportional and integral control with negative feedback.

Summary of Network (diagrammatic) methods

Potential benefits

1. An alternate visual perspective.
2. Insight into feedback loops within ODEs and systems.
3. Easy conversion to executable block diagrams.
4. Answers design-oriented system questions, e.g., sensitivity to parameters (performance improvement, robustness, structural stability).