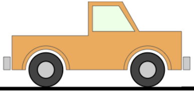

Pull-back Cars and Pop Cycles:

Vehicles for a First Modeling Experience

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Abstract

This presentation will outline recently developed modeling scenarios that examine the velocity and distance traveled of specific toys. Instructors have the option to use included data or guide their students through a hands-on modeling experience in which data is extracted from cellphone videos of a toy in motion.

- 3-099-PullBack-ModelingScenario, *Modeling the Velocity of a Pull-Back Toy*, (2021).
  `http://dx.doi.org/10.25334/4C59-Q293`
- 3-100-Ripcord-Toys-ModelingScenario, *Ripcord-Powered Toys: Modeling Rolling with Slipping*, (2022).
  `http://dx.doi.org/10.25334/KPHH-KC97`
- 3-103-PullBackCars-ModelingScenario, *Hands-On Modeling with Pull-Back Cars*, to appear, (2023).

Common Elements

Each modeling scenario examines a first-order differential equation describing the velocity of a mechanically powered toy. There are several other common elements:

- The models are elementary and can be used early in a first course on differential equations.
- Students use real data for parameter estimation.
- Students have the option to conduct hands-on experiments or use provided data.
- Experimental data is extracted from cell phone videos through frame-by-frame analysis.
- Experiments can be conducted during class or outside of class.

Part 1:

Modeling the Velocity of a Pull-Back Toy

Pull-Back Toys

Pull-back toys come from many sources and in many shapes, sizes, and quantities.

Cereal Box (2004)

Grocery Store (2022)

Amazon (2021)

The clockwork motor winds up over a short distance in reverse, but winds down over a longer distance as the car rolls forward.

The Mathematical Model

Equations of Motion:

The motion of a pull-back toy can be modeled using two first-order differential equations in terms of the position $x(t)$ and velocity $v(t)$:

$$\frac{dx}{dt} = v \qquad \text{and} \qquad \frac{dv}{dt} = M(x) - kv,$$

where $M(x)$ represents the locomotive force provided by the pull-back motor and k quantifies frictional losses.

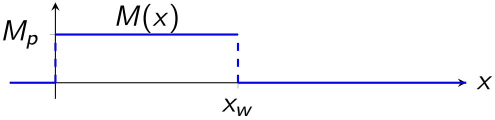
Note: The force provided by the motor depends explicitly on the position because the internal spring winds down over a fixed distance.

Overview

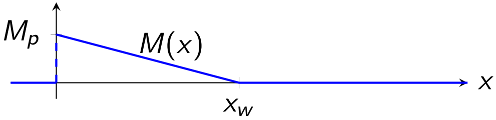
The goal of this modeling scenario is to predict the velocity and distance traveled of a fully wound pull-back toy as functions of time.

Key Features:

- Students have the option to collect data or use provided data.
- Students use numerical methods to make model predictions based on unknown parameter values: M_p, x_w, and k.
- Students perform parameter estimation by minimizing squared error.
- Students work with two proposed models.

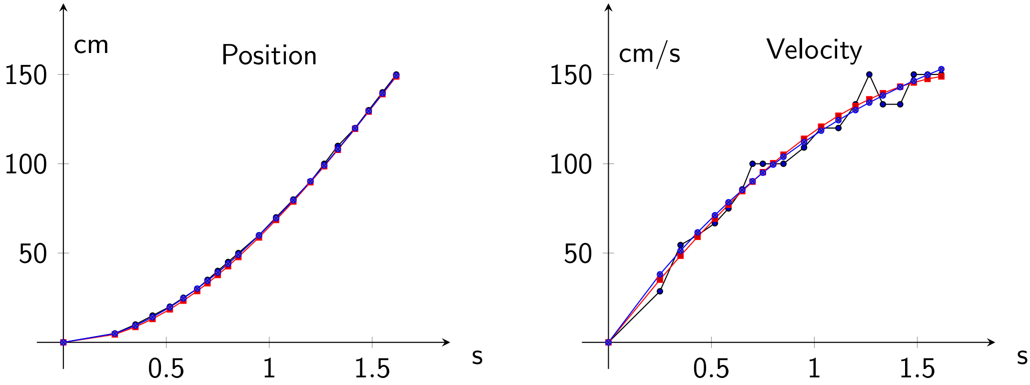


Constant Torque Model



Linear Torque Model

Results with the Provided Data

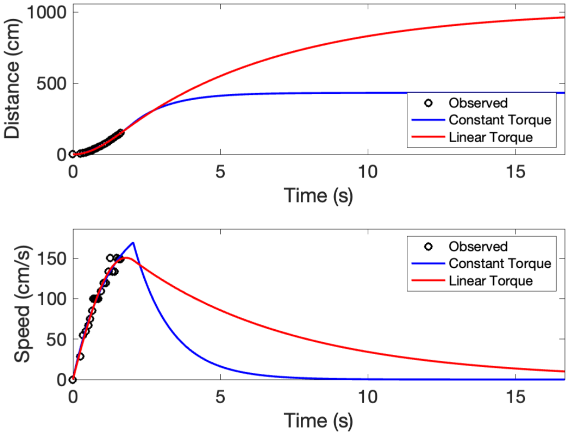


Both the linear torque (red) and constant torque (blue) models provide a good fit with the observed data (black) during acceleration.

Long-Term Predictions

Observations:

- The constant k representing frictional losses is much smaller for the linear torque model.
- The linear torque model predicts a much greater total distance traveled.
- The prediction of the constant torque model provides the better fit with observed values for the total distance traveled.



Classroom Implementations

Formats we've used (in Calculus classes):

- In-class experiments (1 period) and data extraction (1 period), pre-programmed Excel worksheet provided for analysis, out-of-class group report writing.
- Group project with experiments and Excel implementation of Euler's method done outside of class, followed by in-class student presentations (1 period).
- Pre-lab reading, in-class experiments (1 period), and post-lab analysis.

Students said:

> This made it feel like we were in the real world solving real-world problems.

> This project reminded me of related rates and how those are actually reasonable real-world problems.

Student Reflections: Perspective on Applied Math

Looking back over the entire process of **collecting data**, performing calculations, and reflecting upon both the information found and steps taken has really allowed me to see the importance of applied math and its usefulness to **find certain values that are not easily measurable**.

This project opened my eyes to the intricacy and vast extent of math behind even such a simple movement as that of a little toy car. This project shows how **math is quite amazing** and just how much of it is used or explains things without us even knowing.

I honestly would recommend this project to future classes because of how **realistic** it felt. We were given a **problem that needed an answer**, and through applied math we were able to **experiment and calculate** a solution with prior knowledge we learned in class.

Part 2:

Hands-On Modeling with Pull-Back Cars

Overview

The goal of this modeling scenario is to explore the effect of pull-back distance on the maximum velocity and distance traveled for a pull-back car.

Key Features:

- Students have the option to collect data or use provided data.
- This modeling scenario adopts the constant-torque model.
  - Both the linear-torque and constant-torque models perform well during acceleration.
  - The constant torque model is easier to solve analytically.
- Separate frictional constants are adopted during wind-down and after wind-down.
- Students derive and implement an analytical solution of the model for a fixed pull-back distance.
- Students adapt the model to account for a variable pull-back distance.

The Mathematical Model

Acceleration Phase: $0 \leq x \leq x_w$

$$x''(t) + k_1 x'(t) = M_p \quad \Longrightarrow \quad x(t) = \frac{M_p}{k_1} t + \frac{M_p}{k_1^2}\left(e^{-k_1 t} - 1\right), \quad 0 \leq t \leq t_w,$$

where t_w is defined by $x(t_w) = x_w$.

Deceleration Phase: $x \geq x_w$

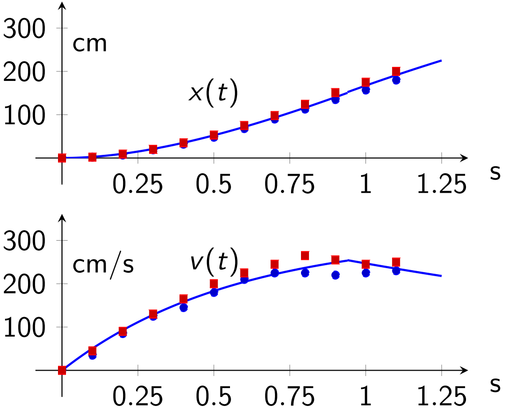
$$x''(t) + k_2 x'(t) = 0 \quad \Longrightarrow \quad x(t) = a_0 + a_1 e^{-k_2 t}, \quad t \geq t_w,$$

where a_0 and a_1 are determined from $x(t_w)$ and $x'(t_w)$.

The Sample Data (25cm pull-back distance)

The model was applied to data generated with the Darda pull-back car shown below.

Darda HP-Castrol (2015)



The Effect of Pull-Back Distance

Assumptions:

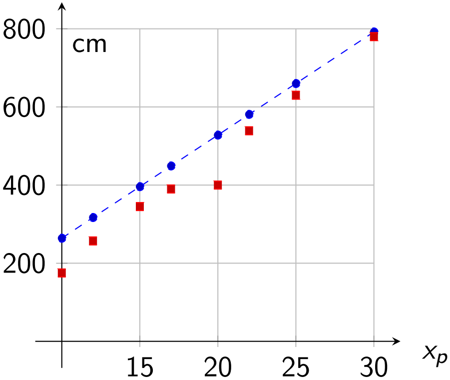
- The energy stored in the spring is proportional to the square of the pull-back distance. (Hooke's Law)
- The constants k_1 and k_2 are unaffected by the pull-back distance.

Adapting the Model

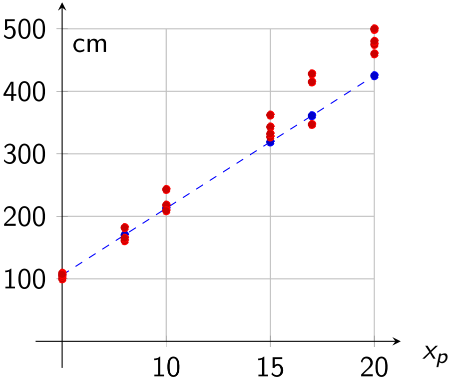
1. The base model is determined experimentally using a fixed pull-back distance x_p and leads to values for x_w, M_p, k_1, and k_2.
2. The alternate model for pull-back distance $\tilde{x}_p$ then uses

$$\tilde{x}_w = x_w \frac{\tilde{x}_p}{x_p} \qquad \text{and} \qquad \tilde{M}_p = M_p \frac{\tilde{x}_p}{x_p}.$$

Model Validation (Total Distance Traveled)



Darda Pull-Back Car



Tow-Mater Pull-Back Truck

Note: Observed distances (red) vs. Predicted distances (blue)

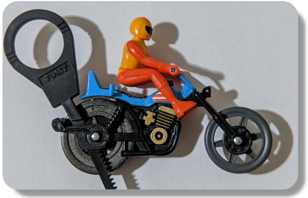
Part 3:

Ripcord-Powered Toys: Modeling Rolling with Slipping

Ripcord-Powered Toys (Tomy Pop Cycles)

A super fun class of ripcord-powered toys are the Tomy Pop Cycles of the 1980s. They are somewhat scarce, but can frequently be found for sale on eBay.

Part of the Tomy Pop Cycle Lineup



Pop Cycle with Ripcord

A quick pull on the ripcord is used to spin the rear wheel. The high rotational speed of the rear wheel stabilizes the toy while also providing a locomotive force as the wheel slips on the floor.

The Mechanics of Ripcord-Powered Toys

Three States of a Rolling Wheel

1. Pure Rolling: $v = \omega R$

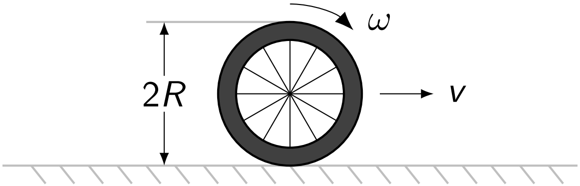
   The wheel rolls at a constant speed v.

2. Rolling with Slipping: $v < \omega R$

   The frictional force at the point of contact acts to reduce ω and increase v.

3. Rolling with Skidding: $v > \omega R$

   The frictional force at the point of contact acts to increase ω and decrease v.



Overview

The goal of this modeling scenario is to build a mathematical model for the position and velocity of a ripcord-powered toy.

Key Features:

- Students have the option to collect data or use provided data.
- Students derive and implement an analytical solution of the model.
- Students perform parameter estimation by minimizing the total squared error between the model and their experimental data.
- The acceleration rate should be [nearly] constant until pure rolling is achieved.
    - The acceleration rate is influenced by the friction provided by the test surface.
    - The maximum velocity will depend on the initial rotational speed of the wheel.

The Mathematical Model

Acceleration Phase: $0 \le t \le t_0$

$$x''(t) + k_1 x'(t) = M \quad \Longrightarrow \quad x(t) = \frac{M}{k_1} t + \frac{M}{k_1^2}\left(e^{-k_1 t} - 1\right), \quad 0 \le t \le t_0,$$

where t_0 is defined as the time when pure rolling is achieved.

Deceleration Phase: $t \ge t_0$

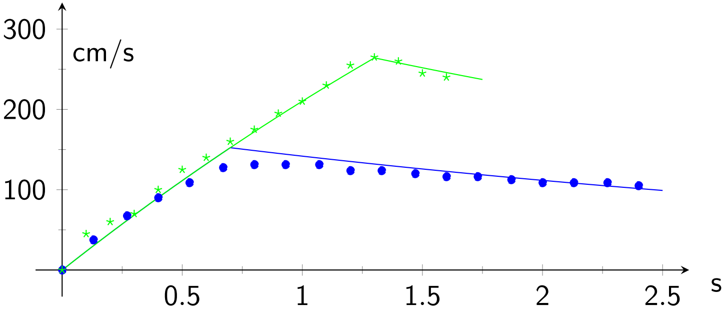
$$x''(t) + k_2 x'(t) = 0 \quad \Longrightarrow \quad x(t) = a_0 + a_1 e^{-k_2 t}, \quad t \ge t_0,$$

where a_0 and a_1 are determined from $x(t_0)$ and $x'(t_0)$.

Note: The model is very similar to that of the pull-back cars, but the key difference is that the time at which pure-rolling is achieved depends only on the initial rotational speed of the wheel rather than the distance traveled.

Model Validation

The graph below includes velocity data for a strong pull ($t_0 = 1.3$) and a soft pull ($t_0 = 0.7$) along with the corresponding model predictions for these values of t_0.



The model predicts that a Pop Cycle can travel more than ten meters on a strong pull!

Other Ripcord-Powered Toys

Another class of ripcord-powered toys consists of the Kenner Super Sonic Power (SSP) Racers of the 1970s. Used examples are readily available on eBay.

Kenner SSP Bonnie Bike

Do you have any ideas for a new modeling scenario?

The use of cellphones for data acquisition through video recordings seems like it could lead to lots of interesting modeling projects.

Video recordings could provide data for...

- the position of an object along a linear scale;
- the level of a fluid in a vessel;
- the position of a needle on a mechanical measuring device, e.g., a scale or voltmeter;
- the angle of a swinging pendulum (with appropriate reference background);
- the reading on a digital/analog thermometer.