Simulation and Modeling Blue Whale and Krill Populations

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SIMIODE Modeling Scenario 6-025

Introduction

Blue whales and krills are commonly used in teaching predator-prey (Lotka-Volterra) model. We present how we engage students in modeling and simulation activities to observe the population changes in both species and their life cycles numerically and graphically. A few facts of blue whales and krills make this model more intriguing.

- Blue whales are listed as endangered species. There are approximately between 10,000 and 25,000 blue whale left in the world (about 10% of what they once were). The blue whale is the largest animal on earth.

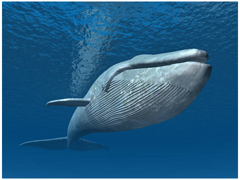

Figure 1: Blue Whale

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Figure 2: Krill

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- Krill eat phytoplankton (the free-floating single-celled variety are responsible for absorbing a massive amount of carbon from the atmosphere). Blue whale extinction would result in a marked acceleration in climate change which could have a cataclysmic impact on global habitability.

Assumptions

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- In the absence of predators and uncrowded conditions, the krill population density grows at a rate $a = 100\%$ per year.
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- In the absence of predators and uncrowded conditions, the krill population density grows at a rate $a = 100\%$ per year.
- The presence of blue whales decreases the krill population density. The rate of decrease is proportional to the product of B and K where the proportionality constant is $b = 0.02$.
- The initial population of blue whales is $B_0 = 20$ thousand and the initial krill density is $K_0 = 5$ tons/acre.

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4. From the last two assumptions, how can we express $\frac{dK}{dt}$?
5. What are the equilibrium points of the system (let the coordinates be $x = K$ and $y = B$)?

Use Euler Method to Simulate the Solutions

The students should have the following model after answering the previous five questions.

$$\frac{dB}{dt} = -m * B + n * B * K$$
$$\frac{dK}{dt} = a * K - b * B * K$$

The equilibrium points are $(0, 0)$ and $(\frac{m}{n}, \frac{a}{b})$. From the assumptions, $m = 0.8, n = 0.1, a = 1, b = 0.02, B_0 = 20$ and $K_0 = 5$, so equilibrium points are $(0, 0)$ and $(8, 50)$.

We can estimate $B(t)$ and $K(t)$ using Euler method (i.e. using difference equations to approximate differential equations) where $\frac{dB}{dt} \approx \frac{\triangle B}{\triangle t}$ and $\frac{dK}{dt} \approx \frac{\triangle K}{\triangle t}$, and let $\triangle t = 1$.

Demonstrate One Iteration of Euler Method

To estimate B_1 and K_1, do the following:

$$\begin{aligned} B_1 &= B_0 - m * B_0 + n * B_0 * K_0 \\ &= 20 - 0.8 * 20 + 0.1 * 20 * 5 \\ &= 14 \end{aligned}$$

and

$$\begin{aligned} K_1 &= K_0 + a * K_0 - b * B_0 * K_0 \\ &= 5 + 1 * 5 - 0.02 * 20 * 5 \\ &= 8 \end{aligned}$$

If time allows, an implicit solution can be obtained before the simulation demonstration: $\frac{B^a}{e^{b*B}} = c(\frac{e^{n*K}}{K^m})$ where c is a constant that can be determined by the initial conditions.

Student Modeling Activity Using the Simple Simulation Method

The next modeling activity is to ask the students to complete four additional iterations and fill in the following table.

Time (in years)	Krill Density (in tons/acre)	Blue Whale Population (in thousands)
0	5	20
1	8	14
2		
3		
4		
5		

Note that the approximation errors become very large after a few iterations. Other methods such as Runge-Kutta 4 method should be used for simulation if more than a few iterations are required. Otherwise, the students may draw incorrect conclusions due to numerical errors from the simple simulation method.

Student Activity for Using Excel to Do the Simulation

This part is better done in a lab. Students are encouraged to use a technology (such as Excel or MATLAB) to answer the questions and provide a chart. If some students have trouble with technology, provide them a template prepared before the class.

1. Estimate $B(6)$ and $K(6)$ where $B_0 = 20$, $K_0 = 5$ and $\triangle t = 0.1$ using the simulation method, and make a chart that plots the krill population density (horizontal axis) versus the blue whale population (vertical axis) over those six years.

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2. Provide a chart that plot time t versus $B(t)$ and $K(t)$, respectively.

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2. Provide a chart that plot time t versus $B(t)$ and $K(t)$, respectively.
3. Using the simulation results in your Excel spreadsheet, what do you notice about the population density of the krill and the population of blue whales over the six years? If one population or the other becomes approaches extinction, give a plausible explanation. If one or both populations approaches extinction, give an estimate of how long it takes for this to happen.

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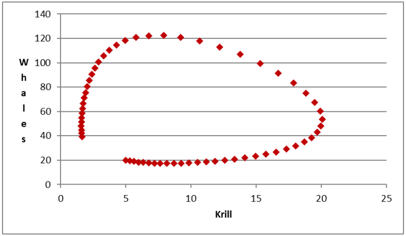
4. Using the observations and results above, classify the equilibrium points as asymptotically stable, or stable, or unstable, respectively.

Charts from the Simulations



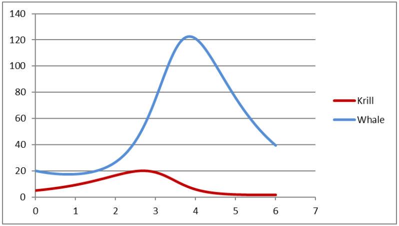


Figure 3: Blue Whales and Krill (K(t) vs B(t), t vs K(t) and t vs B(t)

Student Activity for Sensitivity Analysis

During this activity, students are asked to perform sensitivity analysis by changing the initial conditions B_0 and K_0 and observe the long term behavior of $B(t)$ and $K(t)$.

1. For each scenario (each row) in the following table, using simulation to answer the four questions in the previous activity.

Scenario	$K(0)$	$B(0)$
1	2	25
2	2	50
3	50	20
4	50	10
5	2	10
6	8	100

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2. State what this model seems to say about the blue whales population and the population density of krill.

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- The equilibrium point $(0, 0)$ is unstable.
- The equilibrium point $\left(\frac{m}{n}, \frac{a}{b}\right)$ is stable but not asymptotically stable. The trajectories are periodic.

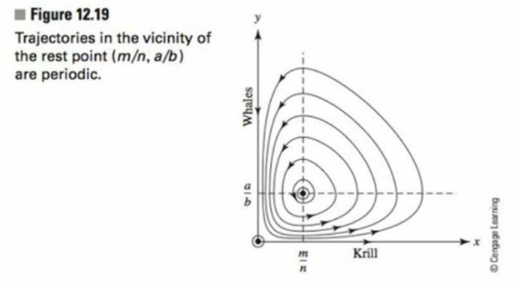

Figure 4: Trajectories and Equilibrium Points

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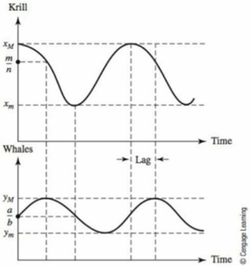
- Along a single trajectory, both population fluctuate between their maximum and minimum values.
- When the krill are plentiful, the whale population has its maximum rate of increase but that the whale population reaches its maximum value after the krill population is on the decline. The predator lags behind the prey in a cyclic fashion.

Figure 12.20
The whale population lags behind the krill population as both populations fluctuate cyclically between their maximum and minimum values.



Activities on Model Improvements and Additional Sensitivity Analysis

Students are asked to revisit the assumptions of the model and make an improvement of the model. Also, additional sensitivity can be done if one or more of the values of the coefficients m, n, a and b are changed (but be aware the approximation errors may lead to incorrect conclusions).

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2. Make an improvement (in both the written description of the model and add its mathematical counterpart to the model) in the model to overcome a simplification.

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4. Extra credit: modify the model to include another source of food for the blue whales such as copepods or fish.

References

- SIMIODE Modeling Scenario 6-025
- *Mathematical Modeling* (4th edition) by Meerschaert, Mark M, Elsevier Inc.
- *A First Course in Mathematical Modeling* (5th edition) by F. R. Giordano, W. P. Fox, and S. B. Horton, Cengage Learning.