Problem C: Dog Cannot Catch: Successful Award

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Who I Am

- I am here today to advocate for additional programs that can target more high schoolers to do application-based math work.
- I am currently a sophomore in high school and am taking AP Calc BC (roughly Calculus II)
- I participated in SCUDEM with this relatively novice level of education to experience firsthand an applications-based approach

The Team’s Work:

Model 1 Assumptions

- Owner and Fritz (dog) are at the fixed location
- Fixed distance between owner and Fritz
- Two-Dimensional system
- Only force that is acting the object (food) is gravity
- The air resistance is negligible

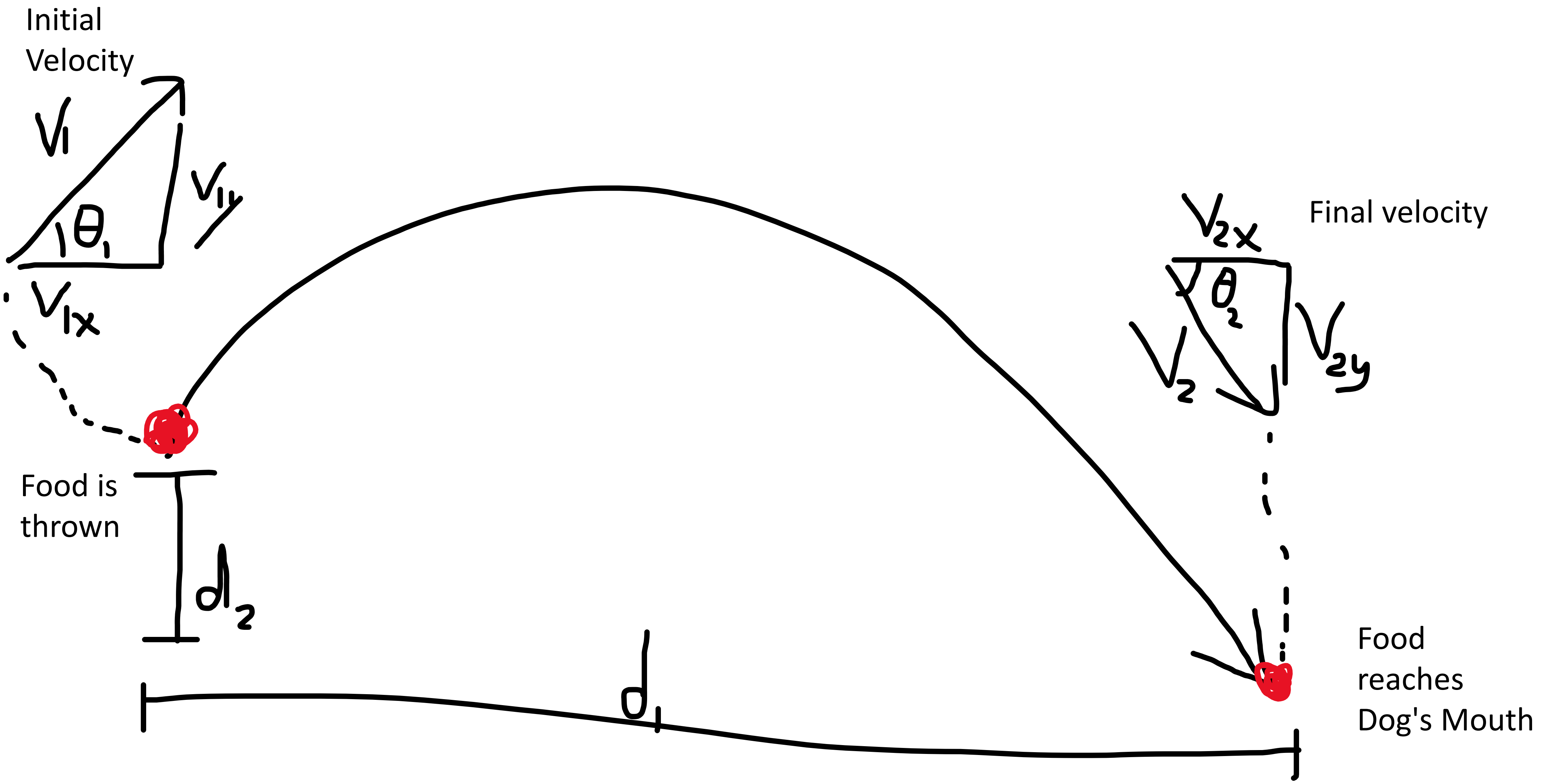
Model 1: parameters

- V1 = initial velocity
- V2 = final velocity
- T = elapsed time
- Θ1 = the angle at which the food is launched.
- Θ2 = the angle at which the food approaches Fritz
- D1 is the horizontal distance traveled (horizontal speed * time)
- D2 is the vertical distance that is traveled (vertical speed * time)

Prediction

- The final velocity can be used to predict whether the dog will be able to catch the food item or not. The larger the final velocity, the more difficult it will be for the dog to catch.

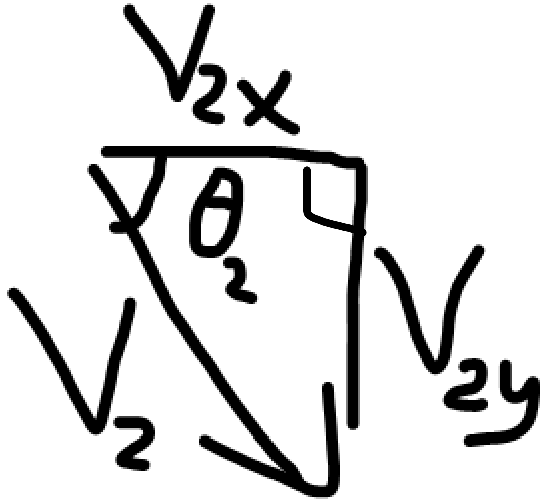
Representation

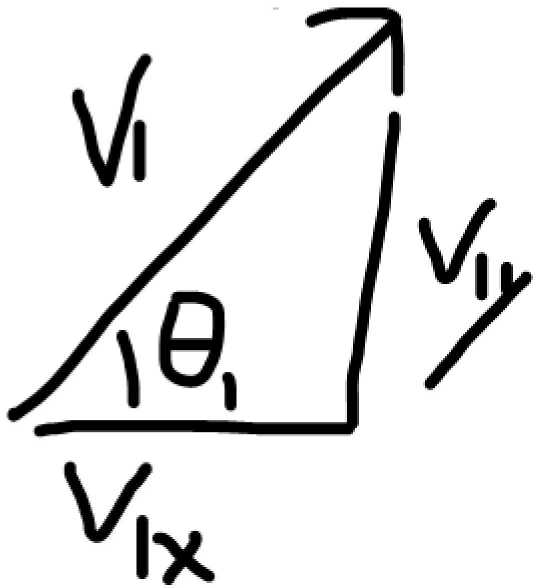


Representing V2 with initial terms

- V2 can be represented as $\sqrt{(V2X)^2 + (V2Y)^2}$
- cosΘ1 = V1X / V1
- V1X = V1 • cosΘ1
- Because we are not account for air resistance, the horizontal velocity is constant, meaning **V2X = V1X = V1 • cosΘ1**
- sinΘ1 = V1Y / V1
- V1Y = V1 • sinΘ1
- V2Y has the force of gravity that decreases the velocity over time.   V2Y = V1Y - G•T = **V1 • sinΘ1 - G•T**
- Plugging these values into the V2 equation returns

$$V2 = \sqrt{(V1 \bullet cos\Theta 1)^2 + (V1 \bullet sin\Theta 1 - G \bullet T)^2}$$





Representing Time

- Because the horizontal speed is constant and time = distance / speed, t = Horizontal Distance / Horizontal Speed
- t= D1 / V1X
- $t = \dfrac{D1}{V1 \bullet \cos\Theta_1}$

Representing Vertical Displacement (D2)

- D2 = Velocity without gravity minus the loss of vertical distance due to gravity
- D2 = V1Y • t – 0.5•g•t^2
- D2 = V1•sinΘ1•t – 0.5•g•t^2
- Plugging in $t = \frac{D1}{V1 \bullet \ cos\Theta}$ gives us

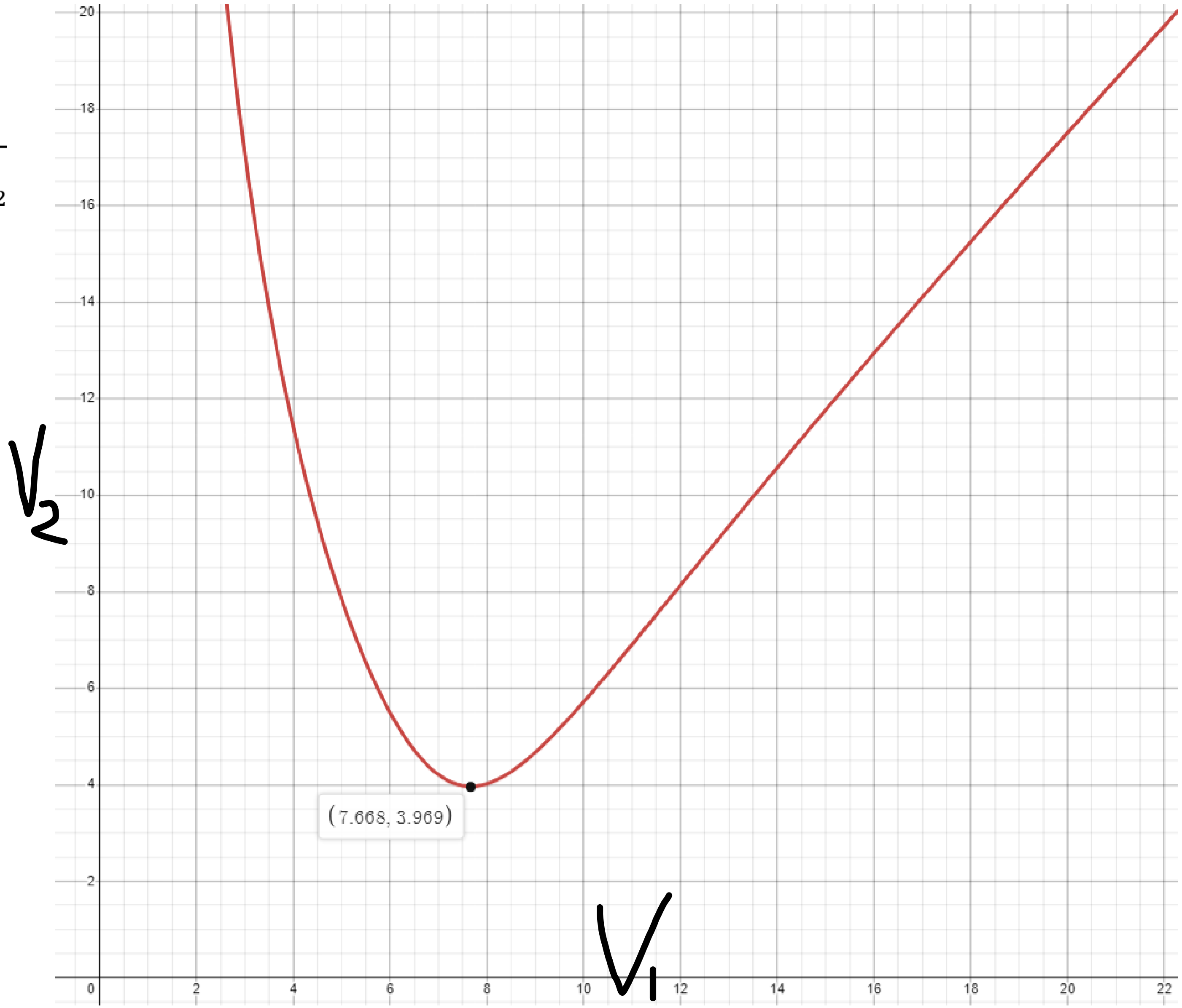
$$D2 = V1 \bullet sin\Theta 1 \bullet \frac{D1}{V1 \ \bullet \ cos\Theta 1} - 0.5 \bullet 9.8 \bullet (\frac{D1}{V1 \ \bullet \ cos\Theta 1})^2$$

Plugging in time into the V2 equation returns $V2 = \sqrt{(V1 \bullet cos\Theta1)^2 + (V1 \bullet sin\Theta1 - G \bullet \frac{D1}{V1 \bullet cos\Theta1})^2}$

- We are able to create sample conditions of Θ = 60 degrees and D1 = 3 m . Plugging in returns

$$V2 = \sqrt{(V1 \bullet cos60°)^2 + (V1 \bullet sin60° - 9.8 \bullet \frac{3}{V1 \bullet cos60°})^2}$$

- Graphing returns the graph on the right with the Y axis representing V2 and the X-axis representing V1.
- The relative minimum of the graph represents the ideal initial velocity the food should be thrown at for it to result in the **lowest final velocity** when thrown at an angle of 60° and traveling 3 m horizontally.
- The conditions that result in the smallest final velocity for this graph are V1 = 7.668 m/s and V2 = 3.969 m/s



Vertical Displacement does not matter: Solving for vertical displacement

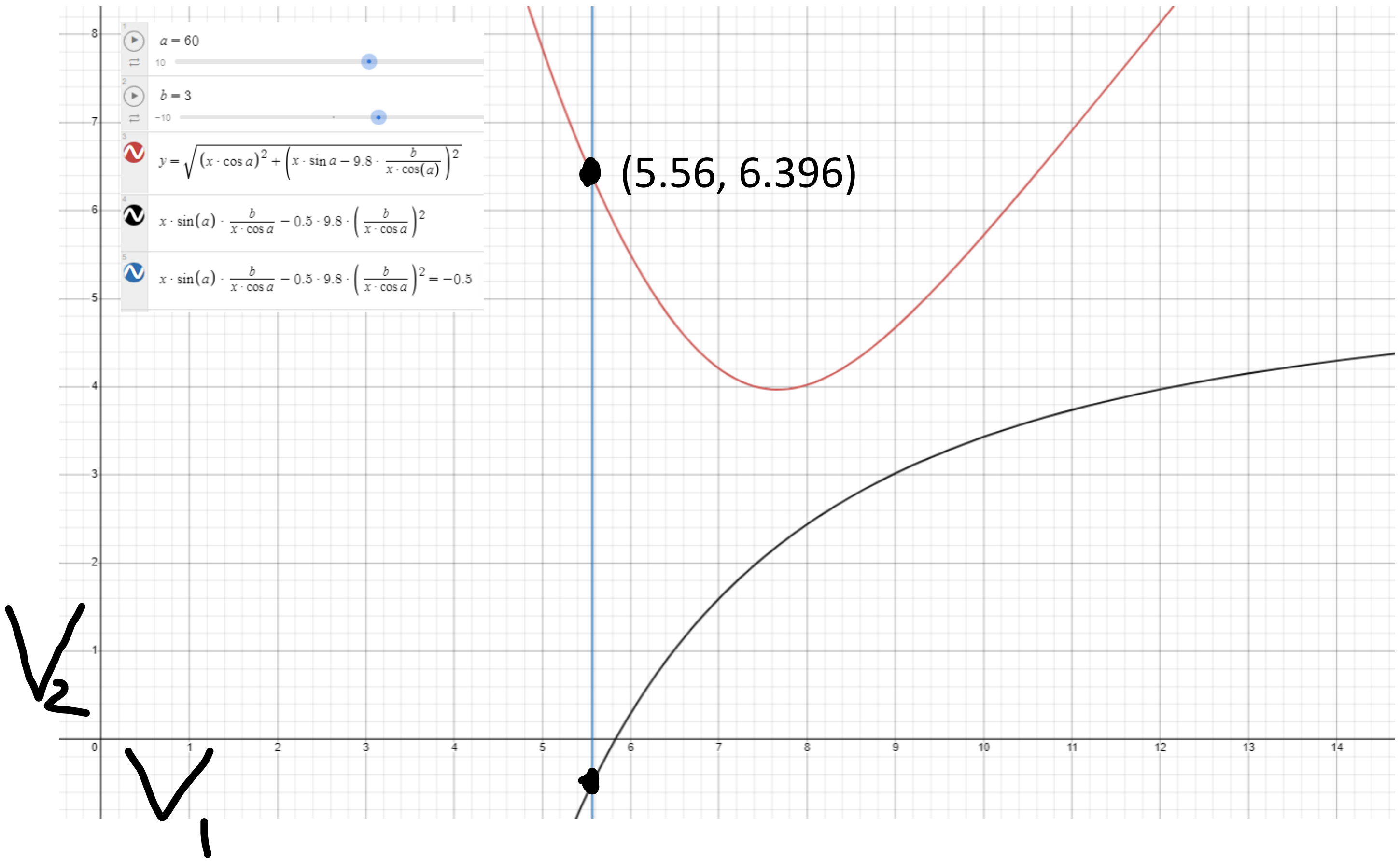
- Continuing with the values found for the conditions of Θ1=60°,D1=3 m the vertical displacement (D2) can be solved for when the V1 = 7.668 m/s because that was when the conditions resulted in the lowest final velocity in the last slide.

$$D2 = V1 \bullet sin\Theta 1 \bullet \frac{D1}{V1 \ \bullet \ cos\Theta 1} - 0.5 \bullet 9.8 \bullet \left(\frac{D1}{V1 \ \bullet \ cos\Theta 1}\right)^2$$

- Plugging in all of the parameters returns D2 = 2.196 m
- This means the food item was caught at a final height that was 2.196 m higher than the initial height in the scenario with a final velocity of V2 = 3.969 m/s.

If Vertical Displacement does matter: Solving for vertical displacement

- The dog likely receives the food item below the initial position that it is thrown. This means that we must graph the D2 equation at with the V2 equation and find the corresponding V2 value for the desired D2 displacement.
- In this example, the equation graphed represents if the dog received the food 0.5m below the initial height.
- From this, the initial and final velocities are approximately V1 = 5.56 m/s and V2 = 6.396 m/s in this case.



Question:
Does the height at which the object is thrown matter?

- The height matters because this initial height would impact the D2 value that has to be traveled in order to reach the dog. As the food object is thrown from a higher point, the value of D2 approaches negative infinity, and the value of V2 approaches infinity, meaning the final velocity increases as D2 approaches negative infinity. This can be seen in the graph of V2.

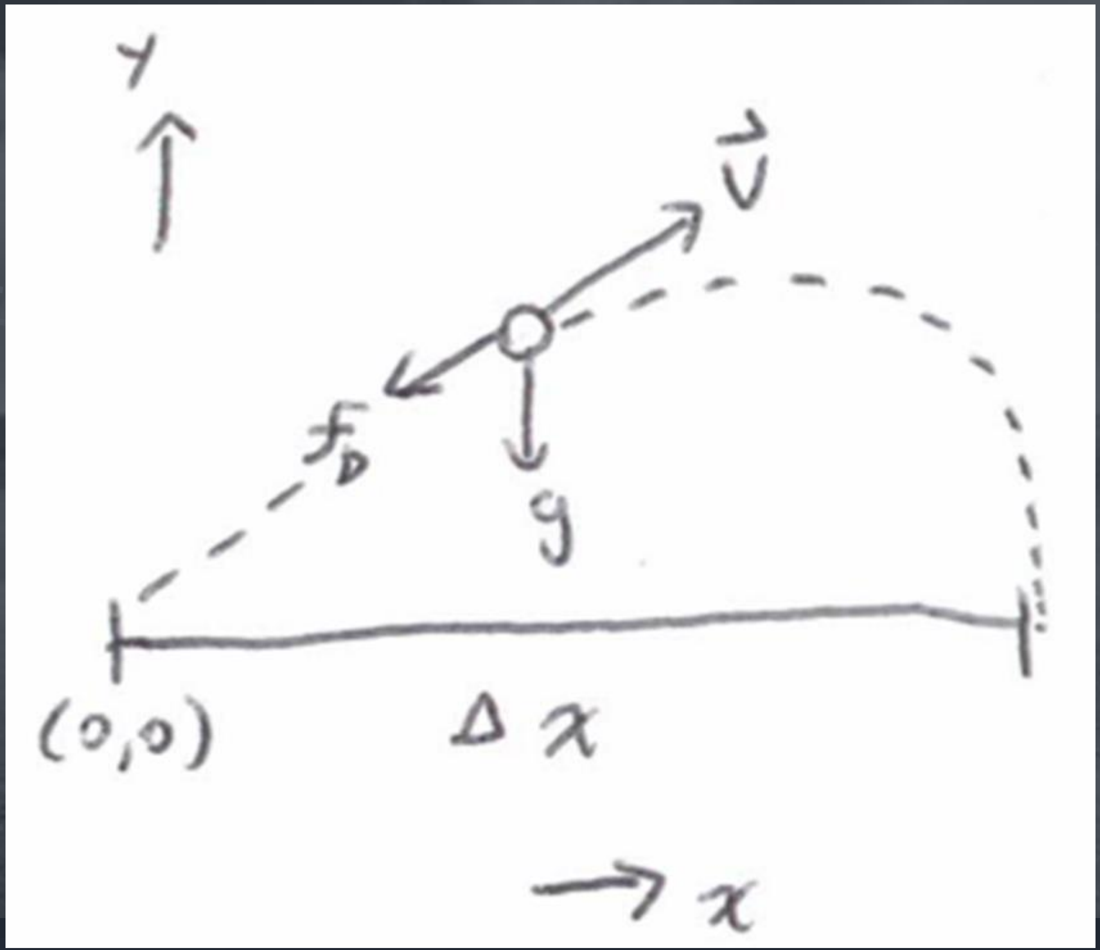
Model 2 Assumptions

- Owner’s location is fixed
- Distance between owner and Fritz changes
- No height
- Two-Dimensional system
- Only forces acting on the object (food) are gravity and air drag
- The shape and size of the object was ignored

Limitations of Model 1

Because Model 1 does not take air resistance into account, the exact velocity and distance will always be slightly inaccurately represented.

Model 2: schematic



Model 2: parameters

- F_{net}: Net force
- $a_{x,y}$: Acceleration in x and y direction
- v_{x_0,y_0}: Initial velocity in x and y direction
- $v_{x,y}$: Final velocity in x and y direction
- m: Mass of an object (food)
- t: Time of flight
- b: Linear drag coefficient
- g: Gravity of Earth

Model 2: solution

$$\Sigma F = m\vec{a}$$

$$\vec{F}_{drag} + \vec{F}_g = m\vec{a}$$

$$-mg\hat{y} - b\vec{v} = m\vec{a}$$

Model 2: solution cont.

x

$$-bv_x = m a_x$$

$$-b v_x = m \frac{dv_x}{dt}$$

y

$$-mg - b\vec{v}_y = ma_y$$

$$-b\left(\frac{mg}{b} + v_y\right) = m\frac{dv_y}{dt}$$

Model 2: solution cont.

$$v_x = C_2 e^{-bt/m}$$

At $t = 0$

$$v_{x0} = C_2 e^0 = C_2$$

$$\therefore v_x = v_{x0} e^{-bt/m}$$

$$\frac{mg}{b} + v_y = C_4 e^{-bt/m}$$

At $t = 0$

$$\frac{mg}{b} + v_{y0} = C_4 e^0$$

$$\therefore \frac{mg}{b} + v_y = \left(\frac{mg}{b} + v_{y0}\right) e^{-bt/m}$$

Model 2: solution cont.

$$\therefore v_x = v_{x_0} e^{-bt/m}$$

$$\therefore \frac{mg}{b} + v_y = \left(\frac{mg}{b} + v_{y_0}\right) e^{-bt/m}$$

$$\frac{dx}{dt} = v_{x_0} e^{-bt/m}$$

$$\frac{dy}{dt} = v_{y_0} e^{-bt/m} - \frac{mg}{b}\left(1 - e^{-bt/m}\right)$$

$$\Delta x = \frac{m v_{x_0}}{b}\left(1 - e^{-bt/m}\right)$$

$$\Delta y = \frac{m}{b}\left(v_{y_0} + \frac{mg}{b}\right)\left(1 - e^{-bt/m}\right) - \frac{mgt}{b}$$

As $t \to \infty$, $e^{-bt/m} \to 0$

Thus $\Delta x = \dfrac{m v_0}{b}$

Model 2: solution cont.

$$x = \frac{mv_{x_0}}{b}\left(1 - e^{-bt/m}\right)$$

$$\frac{bx}{mv_{x_0}} = 1 - e^{-bt/m}$$

$$1 - \frac{bx}{mv_{x_0}} = e^{-bt/m}$$

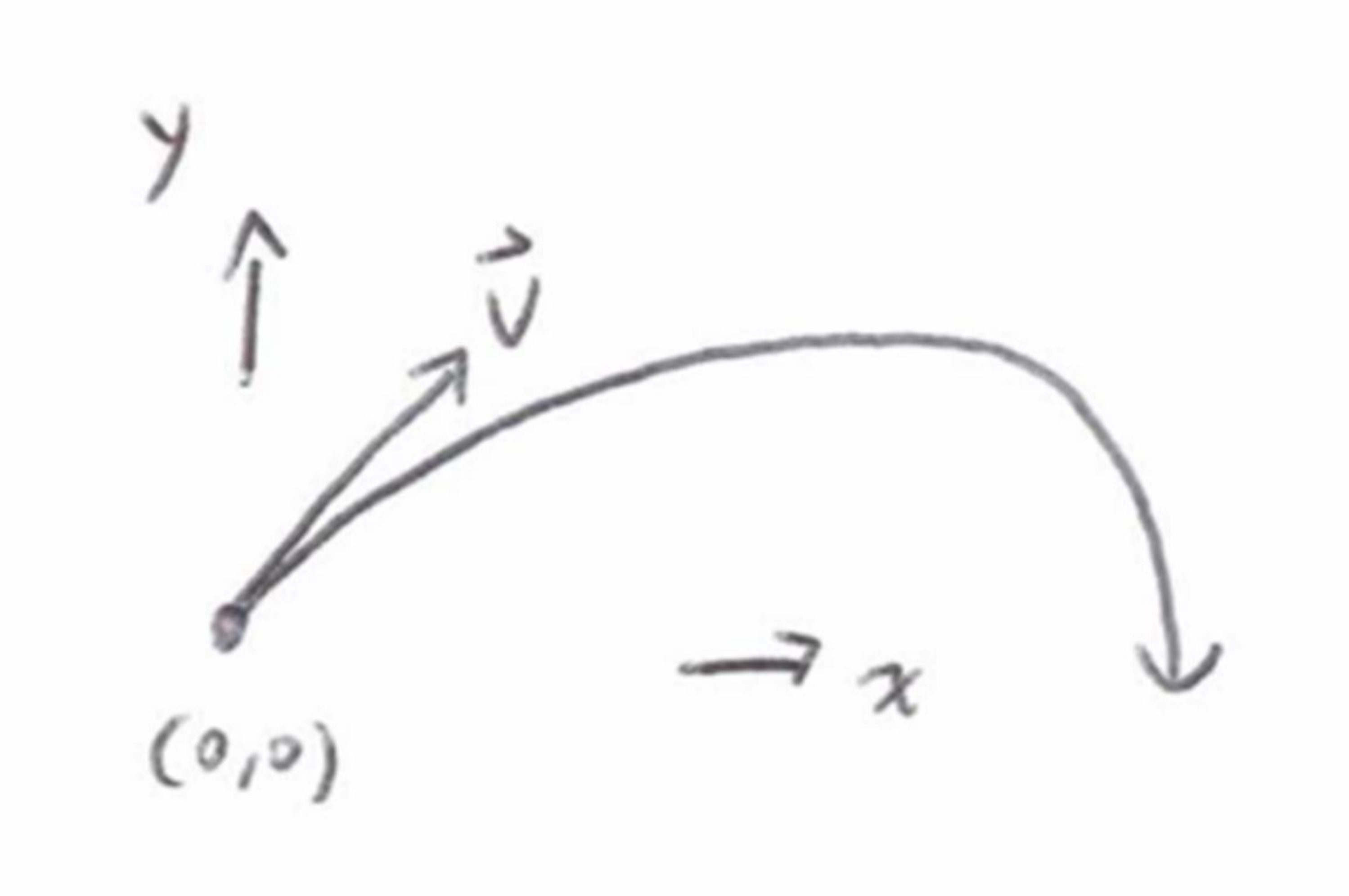
$$t = -\frac{m}{b}\ln\left(1 - \frac{bx}{mv_{x_0}}\right)$$

$$y = \frac{m}{b}\left(v_{y_0} + \frac{mg}{b}\right)\left(\frac{bx}{mv_{x_0}}\right) - \frac{mg}{b}\left(-\frac{m}{b}\ln\left(1 - \frac{bx}{mv_{x_0}}\right)\right)$$

$$y = \frac{v_{y_0} + \frac{mg}{b}}{v_{x_0}}x + \frac{m^2 g}{b^2}\ln\left(1 - \frac{bx}{mv_{x_0}}\right)$$

$$y = \frac{bv_{y_0} + mg}{bv_{x_0}}x + \frac{m^2 g}{b^2}\ln\left(1 - \frac{bx}{mv_{x_0}}\right)$$

Model 2: solution cont.



Model 2: limitations

- The size and shape of an object were not accounted for
- Linear drag equation assumes a constant drag coefficient
- Ignores other forces e.g. lift force

What I Gained

- Despite receiving only a successful award, I have learned much more about mathematical modeling from the rest of the community
- More high schoolers should be exposed to higher levels of math and additional application-focused problems, even if they do not fully understand

Acknowledgements: All of you

Thank you to:

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- Our coach Dr. Sujin Kim for encouragements
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