Fostering Mathematical Curiosity: Using Real-Life Modeling to Teach Differential Equations

Major Brittany Oletti
United States Military Academy

UNITED STATES MILITARY ACADEMY
WEST POINT®

Agenda

1. CURRENT LANDSCAPE
2. FRAMEWORK
3. APPLICATION DETAILS
4. ADVANTAGES & CHALLENGES
5. MOTIVATION

Current Landscape

- Average college freshman will be placed into one of the following courses:
  - Precalculus
  - Calculus I
  - Calculus II
  - Calculus III
- Students might not take a modeling course
  - Math majors
  - STEM majors
  - Non-STEM majors

“While less common, a first-year course in mathematical modeling that incorporates just-in-time techniques from calculus, discrete mathematics, and/or statistics could serve as a pathway to recruit new majors and also provide "existing” first-year math majors with a richer sense of purpose in their undergraduate studies.”

- MAA

Population Demographics

- College freshmen (first semester) who have BOTH the calculus background and have an aptitude for math
- All have already taken AP/IB Calculus
- Some have completed through Vector Calculus
- SAT Math: 765
- AP Exam: 4.2
- Total Enrollment: ~325 students

Our Approach

IN-CLASS BOARD PROBLEMS

- Interactive lecture & daily practice and feedback

PROGRESSIVE MADS

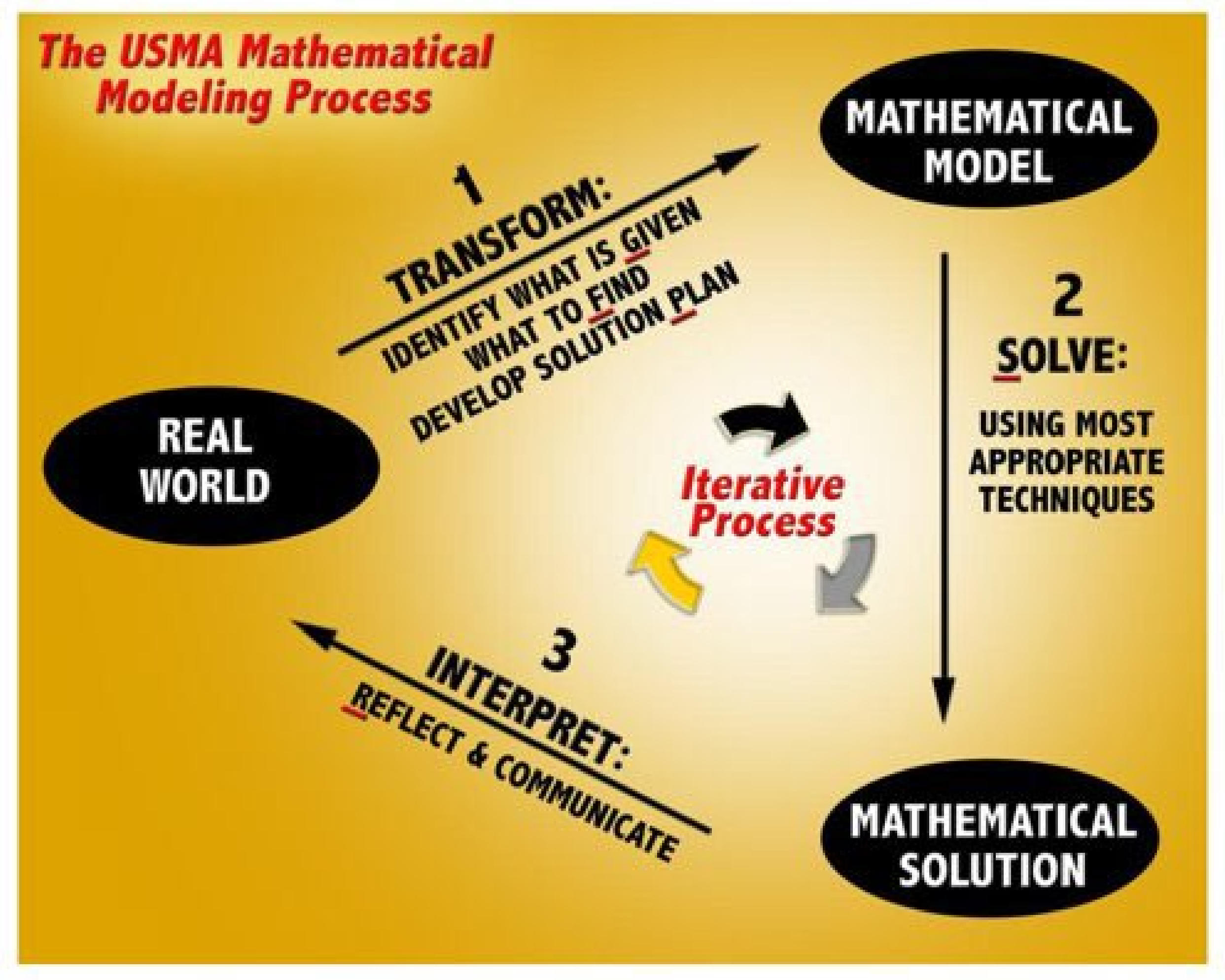
- Modeling Application Days (MAD) that bridge the gap between in class mechanics and a developed model.

END OF TERM PROJECT

- Project selected by the cadets to investigate further and build models out

Framework

Block 1	• In-Class Board Problems • Introduce fundamentals of Modeling and ODEs • First MAD assigned	4 weeks
Block 2	• In-Class Board Problems • Develop Modeling further & intro systems of ODEs • Second MAD assigned/presented	6 weeks
Block 3	• In-Class Board Problems • Refine Modeling process further & intro higher order • Project Introduced	4 weeks
Block 4	• Present Final Projects • Final Exam	4 weeks



Modeling Basics

USMA Mathematical Modeling Triangle

Modeling Application Day – First Order

MA153 Modeling Applications Day 1: Ebola Modeling [1]

Instructions. This is a group (2-3 cadets) assignment worth 120 points, consisting of your group's written report in a Mathematica notebook. Your group will submit a single copy of your report with a group cover page and associated works cited by 2359 on Thursday, 15 September, 2022. You may attach any additional work as an appendix. You do not need to cite your textbook, Mathematica, this document, or your group partners.

Ebola virus disease or EVD is a potentially fatal disease affecting humans. The virus is thought to be carried by animals such as fruit bats and was discovered in 1976 when the first human outbreaks occurred [1]. The World Health Organization (WHO) recognizes March 2014 to be the start of the worst Ebola outbreak in history. The outbreak was primarily contained to West Africa, beginning in Guinea and spreading to Liberia and Sierra Leone [1].

The outbreak and swift spread of Ebola in West Africa sparked fears worldwide for its potential to reach other regions. In September 2014, the Center for Disease Control (CDC) cited model predictions for the number of Ebola cases that reached as high as 550,000 cases by January 20, 2015 in Liberia and Sierra Leone alone [2]. When the model was corrected for under reporting, the number of cases jumped to 1.4 million. At the time of the prediction, Liberia and Sierra Leone combined for 2,407 cases while the model fairly accurately estimated 2,618 cases [2].

Date	t (months)	Total Cases	Cases in Guinea	Cases in Liberia	Cases in Sierra Leone
3/22/2014	0	49	49	-	-
4/14/2014	1	194	197	27	-
5/12/2014	2	260	248	12	-
6/16/2014	3	528	398	33	97
7/14/2014	4	982	411	174	397
8/13/2014	5	2,115	519	786	810
9/14/2014	6	5,335	942	2,710	1,673
10/14/2014	7	9,191	1,472	4,249	3,252
11/11/2014	8	14,383	1,919	6,878	5,586
12/14/2014	9	18,569	2,416	7,797	8,356
1/11/2015	10	21,261	2,806	8,331	10,124

$$\frac{dN}{dt} = kN(1 - \frac{N}{M}). \qquad (1)$$

Modeling Application Day – Systems of ODEs

MA153 Modeling Applications Day 2: Combat Modeling

Instructions. This is a group (2-3 cadets) assignment worth 120 points, consisting of your group's submission and **4-6 minute presentation**. Each group will submit, at a minimum: a cover page, works cited and acknowledgment of assistance, a copy of their presentation, and anything additional required by your instructor (e.g., supporting Mathematica work). You will have approximately one week to complete this assignment, and your **submission is due at the start of Lesson 31** (06 OCT 2023). You do not need to cite the textbook, Mathematica (MMA), this document, your instructor, or your group partners. If you use generative Artificial Intelligence (e.g., ChatGPT or WolframAlpha) you need to acknowledge its assistance.

The foundational relationship in the LEs says that the rate of change of the strength of each side in a conflict is proportional to the size of the opponent's force. If we let $r(t)$ be the enemy (red) force size, $b(t)$ be the friendly (blue) force size, and c_r and c_b represent the respective relative combat effectiveness of the two fighting forces we have the following system of equations (which we will refer to as System 1).

$$\begin{pmatrix} r'(t) \\ b'(t) \end{pmatrix} = \begin{pmatrix} 0 & -c_b \\ -c_r & 0 \end{pmatrix} \begin{pmatrix} r(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} -c_b b(t) \\ -c_r r(t) \end{pmatrix} \tag{1}$$

4 Exploration

The following components for additional investigation will be assigned to each group based on instructor directions.

1. *Groups 1 and 4*: How would you expand your system of equations to account for the use of U.S. Army aviation elements in a portion of the fight (i.e., an additional Blue force)? (*hint*: add a term for how b_h effects $r'(t)$ in System 2, and add an equation for b_h' where you model for an aviation element which starts and 0, increases over the fight, and then decreases back to 0 at the end of the fight?) How does this expansion change the battle?
2. *Groups 2 and 5*: How would the system have changed if we considered a nonhomogeneous system? Modify two components to explore this:
   (a) Modify the ICs to consider a smaller initial U.S. Force size of 400 Soldiers.
   (b) Consider how to model the impact of U.S. artillery on the rate of change of red forces, assuming that the artillery effects only the Taliban forces. Given these two modifications to our model, how would a smaller force with artillery perform?
3. *Groups 3 and 6*: How could we add a component to the system where the size of the Red forces might cause reinforcements? (i.e., where the rate of change of $r(t)$ was influenced positively by a value of $r(t)$?) What effect would this have on the fight, and give some context to your analysis.

Deliverable: One slide on the solution to your modified system of ODEs. Explain your modification to the model, discuss how you developed it, and provide a visualization of the new solution.

Modeling Application Day – Higher Order

GENERAL MATHEMATICAL MODEL: The mass in consideration will be only the mass of the satellite that is added onto the trailer. To simplify the analysis of a suspension system, we assume that the mass and any forcing function on the suspension are evenly distributed; in other words, the suspension system can be treated as one spring. Thus, this simplified suspension system behaves according to the differential equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t),$$

Large Scale Project

- Student Selected
- Observe a phenomena and attempt to model
- Explore canonical models such as SIR, Lanchester Equations, and Predator-prey models
- Project Topics this year:
  - India's Urban Infrastructure and Environment
  - Uber vs. Lyft Competition Revenue
  - E. Coli in Crandall Pool
  - August Complex Fire in California
  - The Pacific Garbage Patch

Project Submission 1

UNITED STATES MILITARY ACADEMY WEST POINT.

Introduction

Overview

- Using the Lotka-Volterra equations and population data of these two species, we created a visualization of their population over time

Figure 1: Ngorongoro Crater, Tanzania

So What?

- This modeling process can be applied to other species that fall within a predator-prey relationship, and predict their future populations

UNITED STATES MILITARY ACADEMY WEST POINT.

Transform – Variables

<u>Variables</u>:

t = time since the start of the observation that started in 1964 (in years).

L(t) = The population of lions as a function of time.

Z(t) = The population of zebras as a function of time.

<u>Assumptions:</u>

- Zebras only die to lions, and lions only eat zebras. Lions only die to starvation.
- The rate of change of both populations is proportional to the number of interactions between the two populations.
- No lions or zebras are being added or subtracted from their population after t(0).
- Populations are 50% male, 50% female.

UNITED STATES MILITARY ACADEMY WEST POINT.

Transform – Equations

<u>Lotka-Volterra Base Model</u>

$$Z'(t) = az - bZl$$
$$L'(t) = cZl - dl$$

<u>Our Model</u>

$$Z'(t) = (0.02)z - (0.0001)Zl$$
$$L'(t) = (0.001)Zl - (4)l$$

<u>Defining Constants</u>

a = The birth rate of the zebras.

b = The death rate per encounter of the zebras, when interacting with lions.

c = The efficiency of turning predated zebras into new lions per encounter with the zebra population.

d = The multiplier of which lions die due to starvation.

UNITED STATES MILITARY ACADEMY WEST POINT.

Solve

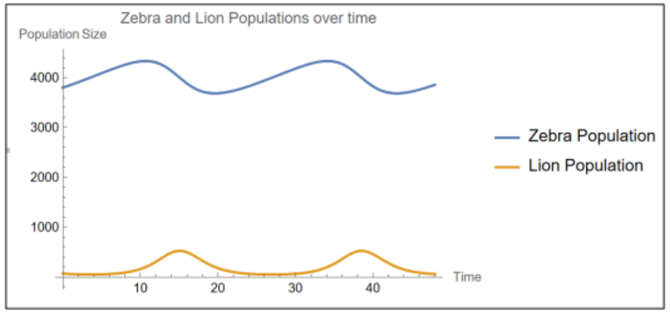


Figure 2: Modeled Zebra-Lion Population

Z(0)= 3,800

L(0)= 70

UNITED STATES MILITARY ACADEMY WEST POINT.

Interpret

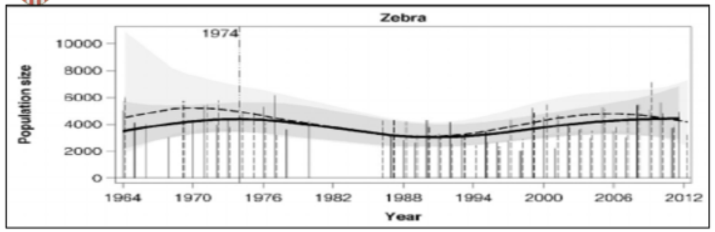


Figure 3: Recorded Zebra Data set

Fits well, but limited by the following:

- **No disease**
- **No human interaction**
- **Lion population over years is unknown**
- **Food availability**

UNITED STATES MILITARY ACADEMY WEST POINT.

Conclusion/Summary

Based upon our analysis, there are many other factors that influence the populations of the lions and zebras such as:

- **Human Interaction**
- **Disease**
- **Drought**
- **Food availability**

Overall, it has a great rough model for population over time. It allows us to create a general model of the zebra and lion populations.

Project Submission 2

Introduction

UNITED STATES MILITARY ACADEMY WEST POINT.

Purpose

- By modeling BAC, we can see when spikes in BAC occur after drinking, and how to space out additional drinks to keep one's BAC in a safe range.

Approach

- To model BAC, we treated the human digestive tract as a mixing tank. Separating it into three "tanks": the stomach, gut (colon, small intestine, and large intestine), and bloodstream.

"Untreated alcohol problems are also at the root of problems that devastate unit morale—up to 50% of suicides, sexual assaults and intimate personal violence incidents are alcohol-related."
- Charles Milliken

Transform – Variables and Equations

UNITED STATES MILITARY ACADEMY WEST POINT.

$s(t)$ ≡ Quantity of alcohol in mg in the stomach.

$g(t)$ ≡ Quantity of alcohol in mg in the gut.

$b(t)$ ≡ Quantity of alcohol in mg in the bloodstream.

$f(t)$ ≡ function representing additional drinks.

- X ≡ Amount of alcohol, in mg, per drink.
- n ≡ Number of drinks per hour.

$A(t)$ ≡ function representing the absorption of alcohol into the bloodstream

- k_a ≡ absorption constant for the bloodstream

$$s'(t) = f(t) - k_1 s, \quad s(0) = s_0$$

$$g'(t) = k_1 s - k_2 g, \quad g(0) = 0$$

$$b'(t) = A(t) * k_2 g - k_3 b, \quad b(0) = 0$$

$$f(t) = X * n * t * U(t)$$

$$A(t) = e^{-k_a t}$$

k_1, k_2, k_3 ≡ alcohol transfer constants for the stomach, gut, and bloodstream, respectively.

Assumptions:

- Negligible amounts of alcohol are metabolized in the stomach and gut.
- Alcohol is metabolized at a constant rate and distributed uniformly throughout the body.
- Additional alcohol is consumed at a constant rate with a constant amount of alcohol per drink.

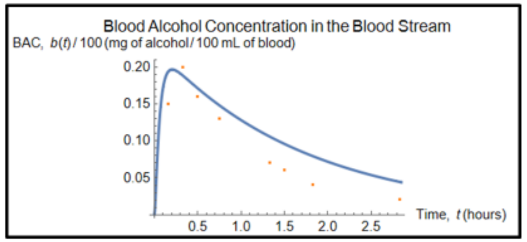
Solve

UNITED STATES MILITARY ACADEMY WEST POINT.

$s_0 = 11265\ mg$
$k_a = 0.38$
$k_1 = 0.2$
$k_2 = 0.18$
$k_3 = 100$

Time (hours since initial drink)	Experimental BAC (mg per 100ml of Blood)
0	0
1/6	0.15
1/3	0.2
1/2	0.16
3/4	0.13
4/3	0.07
3/2	0.06
11/6	0.04
17/6	0.02

Table 1: Blood alcohol level for a 75kg subject after drinking 15mls of 95% alcohol.



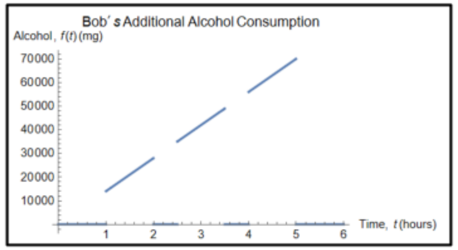
$$b(t) = 22.9161e^{-118.96t}(0.218084e^{18.96t} - 1.21808e^{100.58t} + e^{118.38t})$$

- The solved equation and graph are reasonable. The graph of b(t)/100 fits the data set decently well. When f(t) is integrated, we expect multiple spikes in the curve representing the introduction of more alcohol.

Interpret

UNITED STATES MILITARY ACADEMY WEST POINT.

Scenario: Specialist Bob is a 75kg man having a standard drink at 19:00. He plans to have several more drinks over the next 6 hours and stop drinking at 01:00. How will Bob have to space apart additional standard drinks to keep his BAC under 0.31%? This is important as anything over 0.31% has been dangerous for Specialist Bob in the past and caused him to be hospitalized.



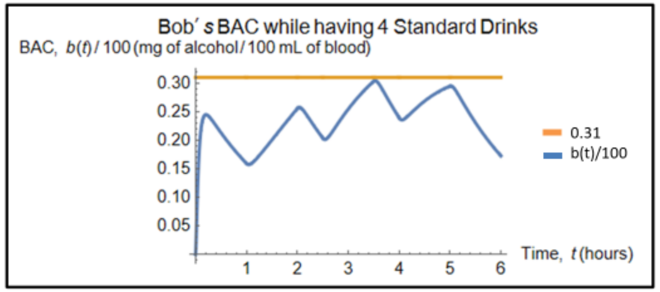
$X = 14000\ mg$
$n = 1$

- Additional drinks at 20:00, 21:30, and 23:00,

$$f(t) = 14000tU(t-1) - 14000tU(t-2) + 14000tU(t-2.5) - 14000tU(t-3.5) + 14000tU(t-4) - 14000tU(t-5)$$

Interpret Continued

UNITED STATES MILITARY ACADEMY WEST POINT.



$s_0 = 14000\ mg$
$k_a = 0.38$
$k_1 = 0.2$
$k_2 = 0.18$
$k_3 = 100$

- The extrema in the graph show how additional drinks spike Bob's BAC. The scenario demonstrates how the model can be utilized to predict BAC when drinking and keep one's BAC in a healthy range to minimize risk.

Conclusion/Summary

UNITED STATES MILITARY ACADEMY WEST POINT.

- Ultimately, by altering a mixing tank model we were able to effectively model BAC in the human body. Additionally, through our model it is possible to predict BAC spikes in the human body for safer drinking.

Possible improvements:

- Once alcohol enters the bloodstream, more than 90 percent of it is metabolized in the liver at a rate varies from person to person.
- This missing 10 percent is metabolized in systems such as the stomach and the gut.
- By accounting for alcohol metabolized in organs other than the liver the model's accuracy can be improved.

Project Grading/Scoping

DATA

- No data – Zombie fight, Jurassic Park ecosystem, etc (0)

COMPLEXITY

- First Order (0)

CREATIVITY

- Model from the Textbook or directly from SIMIODE (0)

Project Grading/Scoping

DATA

- No data – Zombie fight, Jurassic Park ecosystem, etc (0)
- End point or parameters researched (1)

COMPLEXITY

- First Order (0)
- Second order or 2x2 System (1)

CREATIVITY

- Model from the Textbook or directly from SIMIODE (0)
- 1-2 justified terms added (1)

Project Grading/Scoping

DATA

- No data – Zombie fight, Jurassic Park ecosystem, etc (0)
- End point or parameters researched (1)
- Time Series Data (2)

COMPLEXITY

- First Order (0)
- Second order or 2x2 System (1)
- Third or higher order (or 3x3 system) (2)

CREATIVITY

- Model from the Textbook or directly from SIMIODE (0)
- 1-2 justified terms added (1)
- 3 or more justified terms added (2)

Project Grading

	Data	Complexity	Creativity
Start Point (0)	No data (Zombie fight, Jurassic Park ecosystem)	First Order	Model from Zill or SIMIODE
Solid (1)	End point or parameters researched	Second Order (or 2 x 2 system)	1-2 justified terms added
Exceptional (2)	Time Series Data	Third or higher order (or 3 x 3 or bigger system)	3 or more justified terms added

MAD1

MAD2

"Hard enough" metric

- 1 point: 80=perfectly executed
- 2 points: 90=perfectly executed
- 3 points: 100=perfectly executed
- 4 points: a little more gentle grading

SCUDEM

- SIMIODE Challenge Using Differential Equations Modeling
- Annual Competition in October/November timeframe
- Offer extra credit for teams that perform in the Meritorious or Outstanding categories
- 2023 teams:
  - Outstanding: 6
  - Meritorious: 8
  - Successful: 1

SCUDEM VIII 2023

Problem A: Kangaroo Care

A review of infant mortality for kangaroos indicates that there is a wide range of mortality rates for infant kangaroos, and to better understand the variance in mortality a number of different studies have been conducted. A meta-analysis [1] of some of the different studies indicates that direct care of an infant by a kangaroo mother plays an important role in the infant's first 28 days of life. The quality of care may have limited impact on a number of factors such as neurodevelopmental outcomes, but there may be an impact on other factors such as infections and mortality. The authors of the report recommend that all kangaroo infants, as well as human infants, receive care from their mothers within twenty-four hours of their birth and receive at least eight hours of contact each day.

SCUDEM VIII 2023

Problem B: Punishing Infants

Researchers recently sought to examine the origins of the tendency to punish anti-social behaviors [1]. They found that infants as young as 19 months old have the capacity to want to punish others for how they interact with a third-party. The researchers claim that some infants have a propensity to try to discipline those who they believe are hurting others, and evidence is provided that this can occur even before language skills develop.

SCUDEM VIII 2023

Problem C: Dog Cannot Catch

A popular video [1] features a dog, Fritz, who has difficulties catching food that is thrown to him. The video features multiple scenes in which someone throws a food item to the dog, and the dog struggles to catch the item in his mouth. Some dogs are quite adept at catching items, but this particular pet struggles to learn the trick. How difficult is this task?

Fritz must use the changes in the path of the object to predict where the item will be and how to respond. How precisely does the dog's brain have to estimate the orientation of his head and eyes in order to best predict the location of the item, and how much time will the dog have to orient its mouth to the correct position? Are certain foods more difficult to catch due to different effects of air friction? Does the height at which the object is thrown matter? What are the most important aspects of the dog's perceptions and behaviors for the dog's success?

Challenges

Getting the "right fit" for each student

Timing everything well

- Homework assignments
- Technical Labs
- Exams
- Project Midpoints

Balancing Instructor Load for projects

How do we fit it all in?

55 MINUTE CLASS PERIODS

MEET 4-5 TIMES PER WEEK (TOTAL 64 SESSIONS)

18 WEEK SEMESTERS

SMALLER CLASS SIZES (19 CADETS MAX)

TEACH 2-3 SECTIONS PER SEMESTER

Follow On Courses

- ODEs/Math Modeling
- Vector Calculus
- Statistics
- Major Dependent

Motivation

BUT WHY?

- Additional ODE/PDE class in career
- Teach cadets not WHAT to think but HOW to think

Questions?

Contact:

Brittany Oletti

Brittany.Oletti@westpoint.edu