Thoughts on the content of ODE courses

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I have done research linked with differential equations most of my life, but I have never taught the introductory ODE course until now.

Now I am trying to design an introductory ODE course, and I am mostly thinking about *content*, less about *methods of teaching.*

I'll tell you some of my thoughts, but I am here mostly to learn from people who have done this many times.

My primary guiding principle:
Almost every equation should have an applications background.

Certainly nothing like

$$\frac{dy}{dt} = \sec(t)(y^2 + 4),$$

that's just silly!

We practice separation of variables on equations like

$$\frac{dP}{dt} = 0.01\left(1 - \frac{t}{60}\right)P$$

(a decent model of the world's human population) or

$$\frac{dP}{dt} = r\,\left(\ln\frac{K}{P}\right)\,P$$

(the Gompertz law of tumor growth).

I make an exception for

$$\frac{dy}{dt} = y^2, \qquad y(0) = 1.$$

It means nothing, but illustrates blow-up in finite time. I then mention the open problem of whether or not the Navier-Stokes equations allow blow-up in finite time.

I might then also discuss

$$\frac{dy}{dt} = \frac{y^2}{1+t^2} \qquad \text{or} \qquad \frac{dy}{dt} = \frac{y^2}{e^t}.$$

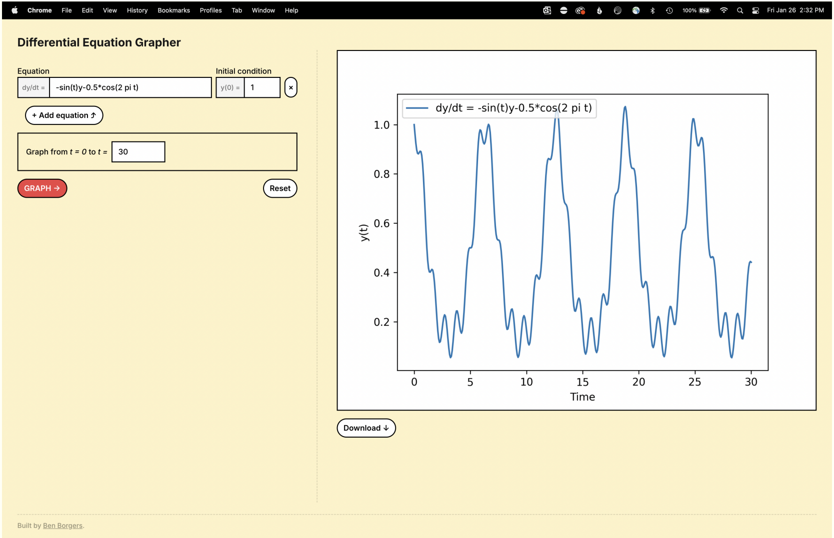
Do they result in blow-up in finite time? (The first does, the second does not.)

But those equations are perilously close to silly again already.

In real life, ODEs are solved numerically, not analytically.
This has to be crystal-clear throughout the course.

I do discuss Euler's method (nothing more sophisticated) in my course, but mostly I let the students use websites that plot solutions of ODEs for them, from the first day on.

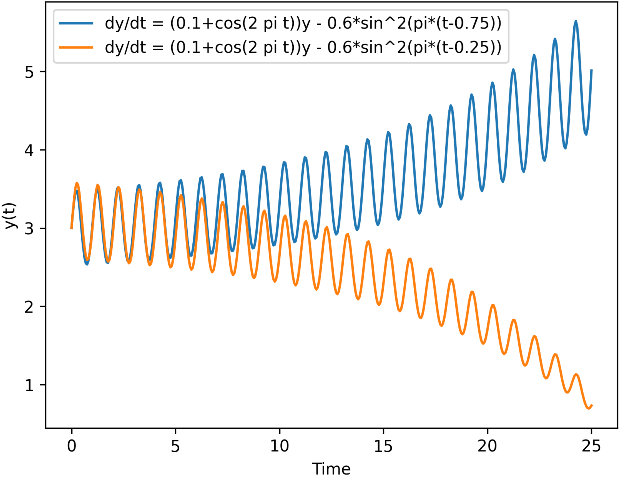
https://diffeqgrapher.com
(created by Ben Borgers with the help of ChatGPT4)



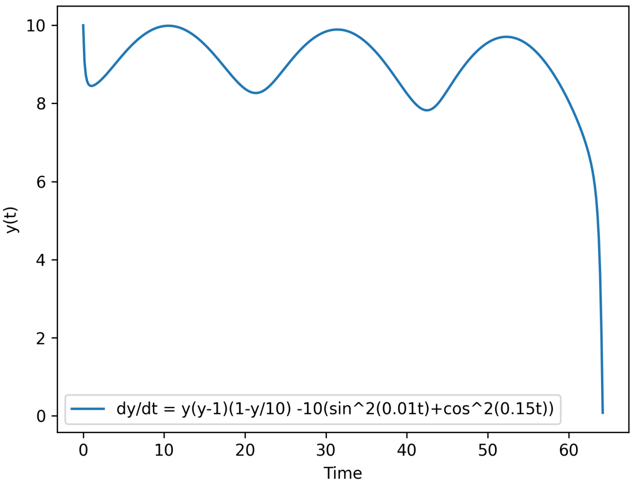
Making my guiding principle stricter:

Each model should show or suggest something unexpected.

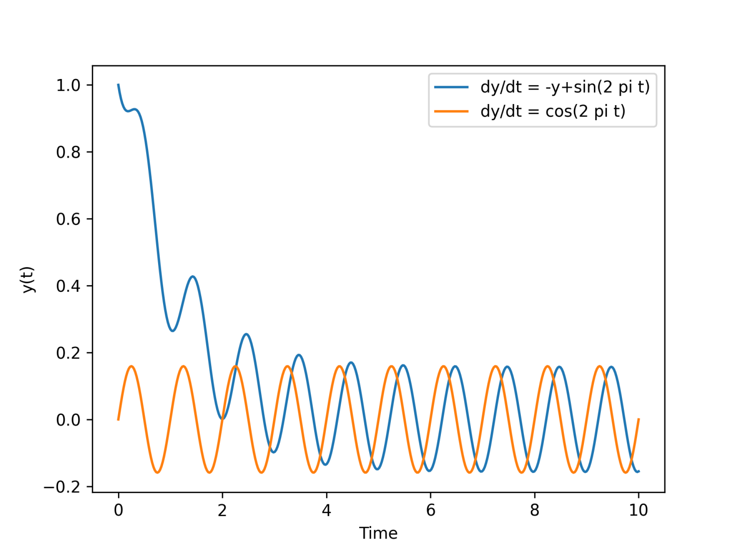
Example 1: The timing of hunting season matters.



Example 2: What might have happened to the passenger pigeon.



Example 3: Phase shift generated by an RC circuit.



This is one of the very few situations when integrating factors are useful.

Applications should be honest.

From Gian-Carlo Rota's essay "Ten lessons I wish I had learned before I started teaching differential equations":

> *The word problems that we find in differential equations textbooks are [...] artificial, dishonest, [...] irrelevant. [...] I cannot see how a student can learn anything by being forced to solve [...] Rube Goldberg flows of salt water in communicating tanks. [...]*

I share Rota's distaste for fake applications. No "application" should start like this:

> *"A fly travels along the curve given by $x = \cos t$, $y = \sin t$, $z = t$, from time $t = 0$ to time $t = 4\pi$. At time 4π, she has a heart attack and dies."*

(Quote, more or less verbatim, from a Calculus III exam.)

Applications should come from a broad range of fields, including natural sciences, engineering, and even social sciences, in a single course.

Gian-Carlo Rota again:

> *Avoid word problems [...]*
>
> *The "word problems" a student of economics will meet are drastically different from the "word problems" of a student of chemical engineering.*

This time, I completely disagree with Rota! Economists should not be protected from chemical engineering, nor vice versa.

That's precisely why the math department should teach this course. It should be an inter-disciplinary course, showing what the different disciplines have in common.

An economic policy example: Income tax vs. wealth tax

> Assumption: Without government intervention, the rich will become richer until they own everything. As the rich become richer, the pace at which they acquire further wealth accelerates.

Some theoretical justification in the "yard-sale model".
The effect is due to randomness, not to superior talent.

A. Chakraborti, Int. J. Mod. Phys. C, 2002 (numerical simulations)
B. Boghosian, Int. J. Mod. Phys. C, 2014 (continuum version)
C. Chorro, Stat. Prob. Letters 2016 (probabilistic analysis)
C. Börgers and C. Greengard, ArXiv Preprint 2023
(simpler, more general probabilistic analysis)

A simple model is the logistic equation:

$$\frac{dw}{dt} = gw(1 - w)$$

with w = fraction of society's wealth owned by the super-wealthy, $g > 0$.

Boghosian *et al.*, *Physica A*, 2017
(derivation of the logistic equation from microscopic principles)

Income taxation

$$\frac{dw}{dt} = (1 - \tau) g w (1 - w)$$

with w = fraction of society's wealth owned by the super-wealthy, $g > 0$, $0 < \tau < 1$.

This obviously just changes the time scale.
w still converges to 1.

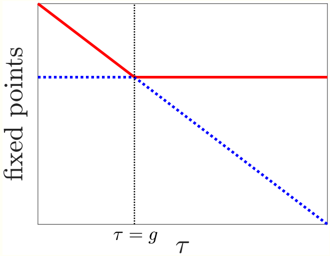
Wealth taxation

$$\frac{dw}{dt} = gw(1 - w) - \tau w$$

with $g > 0$, $\tau > 0$.

This is the most popular example of a transcritical bifurcation, more commonly interpreted as a logistically growing animal population with a constant rate (per unit time and per capita) of hunting.

$$\frac{dw}{dt} = gw(1-w) - \tau w$$



The fixed point $w = 0$ (no super-wealthy class) becomes stable when the wealth taxation rate overpowers the wealth-acquired advantage.

B. Boghosian and C. Börgers, ODE models of wealth concentration and taxation, *CODEE Journal*, January 2024

B. Boghosian and C. Börgers, Mathematics of poverty, inequality, and oligarchy, *SIAM News*, October 2023

A political science example: Discontinuities in politics

C. Börgers, B. Boghosian, N. Dragovic, and A. Haensch,
Am. Math. Monthly, December 2023

C. Börgers, N. Dragovic, and A. Haensch, A. Kirshtein, and L. Orr,
CODEE Journal, January 2024

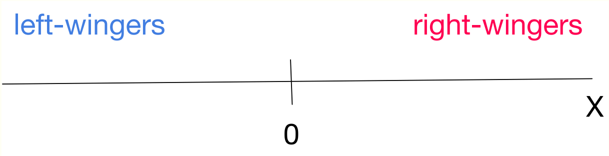
C. Börgers, N. Dragovic, and A. Haensch, *SIAM News*, January 2024

I don't have time to present the whole story, but I'll present a simplification.

Simplifying assumption:

Everybody's political views can be characterized by a single number, their position on the left-right axis.

For instance, a person situated at $x = -1$ is a left-winger, a person at $x = 0$ is a centrist, a person at $x = 1$ is a right-winger.



Three groups of voters:

- hard right-wingers, firmly situated at $x = -1$
- hard left-wingers, firmly situated at $x = 1$
- people in between, distributed evenly in the interval $(-1, 1)$

Two candidates:

- a right-winger, named R, firmly situated at 1, capturing the votes of the hard right-wing voters, and some in-between voters, and
- a left-winger, named L, opportunistically maneuvering. Denote L's position by $X = X(t)$.

Assume that L, by assuming the position $X \in (-1, 1)$, gets a the fraction

$$e^{-(X+1)^2/\gamma^2}$$

of the hard left-wing vote (the other hard left-wing voters abstain). Here $\gamma > 0$ is a model parameter measuring hard left-wing voters' patience with centrist candidates.

Further sssume that L gets a fraction of the in-between vote that rises linearly with X.

It is then not unreasonable to assume that the fraction of the vote captured by the left-winger is proportional to

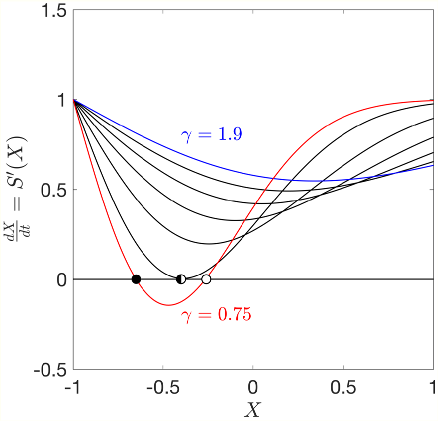
$$S(X) = e^{-(X+1)^2/\gamma^2} + X + C$$

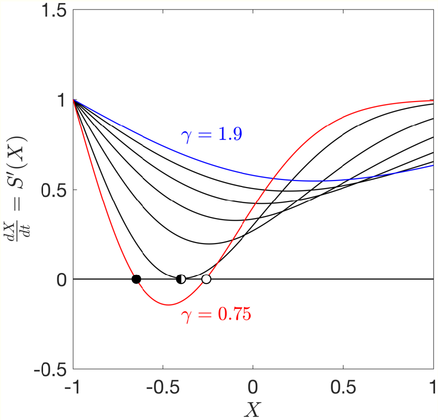
for some constant C.

(We might introduce constants reflecting the sizes of the hard left-wing and in-between camps of voters. I'm skipping that for simplicity.)

Assume: L optimizes using continuous steepest ascent:

$$\frac{dX}{dt} = S'(X) = -\frac{2}{\gamma^2}(X+1)e^{-(X+1)^2/\gamma^2} + 1$$





It's an example of a saddle-node bifurcation. If the left-winger starts at $X = -1$, the position at which they end up depends discontinuously on γ — it is 1 for $\gamma > 0.85$, and it is near -0.5 for γ only barely larger than 0.85.

Conclusion from this model:

Optimal political strategies can depend discontinuously on the political environment.

The underlying mechanism is (or in any case can be) a saddle-node bifurcation.

In an extension of he model in which voter views are dynamic as well, there are numerous saddle-node bifurcations with respect to various different parameters.

(C. Börgers, N. Dragovic, A. Haensch, and A. Kirshtein, in preparation)

Summary

- Almost every ODE in an introductory course should have an applications background.
- Computers should be used at every step to plot solutions.
- Omit models that don't have anything unexpected to say about the world.
- Omit childish nonsense like flies that have heart attacks or dragon infestations.
- Many applications areas in a single course.
- Applications areas can very well include economics and political science.

Thank you!