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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

TEACHER VERSION

How High?! Modeling Free Fall with Air Drag

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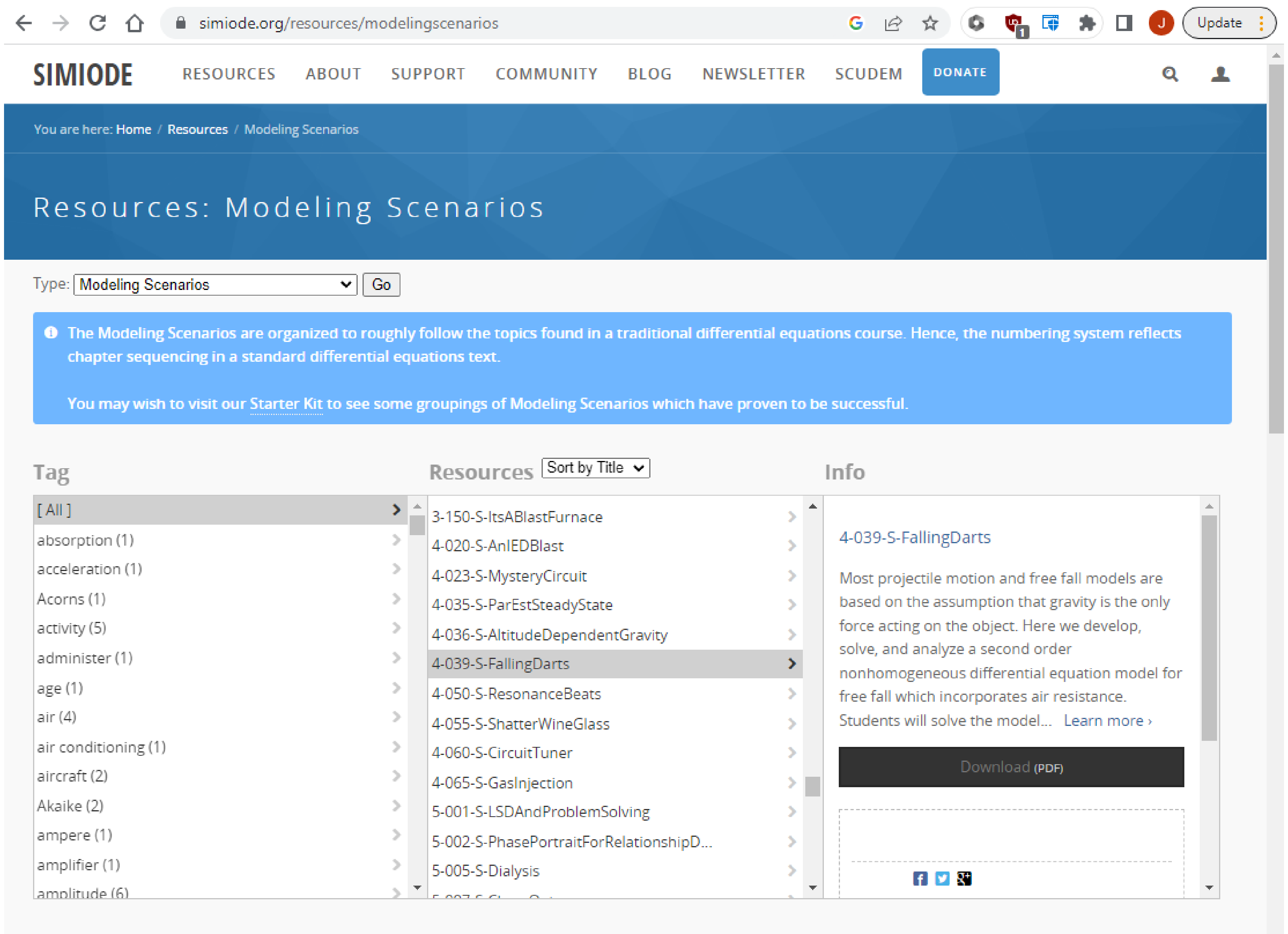
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The Modeling Scenarios are organized to roughly follow the topics found in a traditional differential equations course. Hence, the numbering system reflects chapter sequencing in a standard differential equations text.
You may wish to visit our Starter Kit to see some groupings of Modeling Scenarios which have proven to be successful.
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4-065-S-GasInjection
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4-039-S-FallingDarts
Most projectile motion and free fall models are based on the assumption that gravity is the only force acting on the object. Here we develop, solve, and analyze a second order nonhomogeneous differential equation model for free fall which incorporates air resistance. Students will solve the model... Learn more ›
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Overview

Students will …

1. **Derive** a *second-order nonhomogeneous differential equation model* of free fall incorporating air resistance

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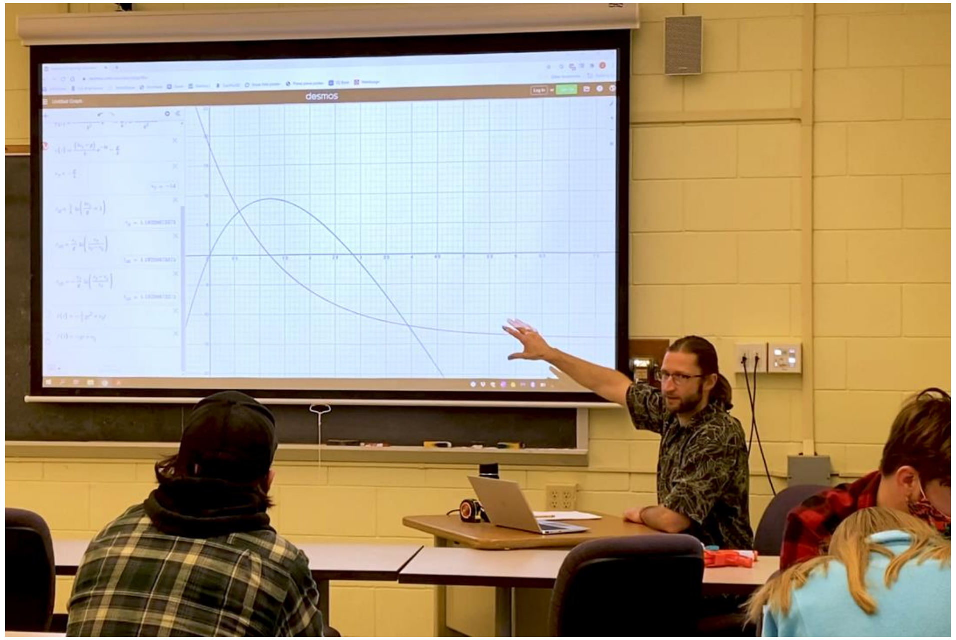
1. **Derive** a *second order nonhomogeneous differential equation model* of free fall incorporating air resistance

2. **Solve** the model using the following two methods:
    a. Reduction of Order & Separation of Variables
    b. Method of Undetermined Coefficients



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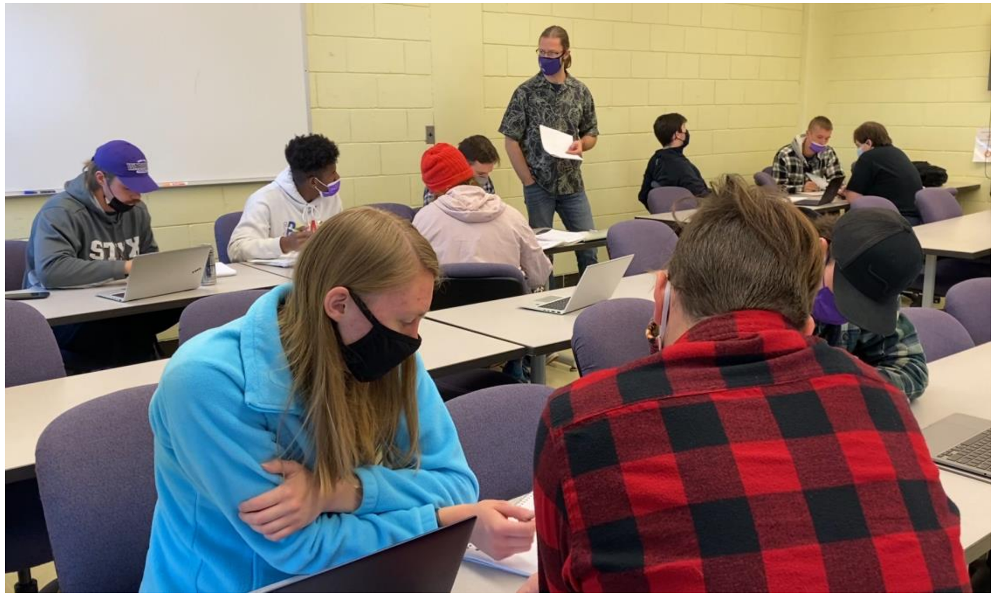
3. **Predict** terminal velocity and maximum height



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   b. Method of Undetermined Coefficients

3. **Predict** terminal velocity and maximum height

4. **Parameterize** the model through experiments with a Nerf dart launcher

5. **Validate** the model by comparing predicted maximum height and observed maximum height.

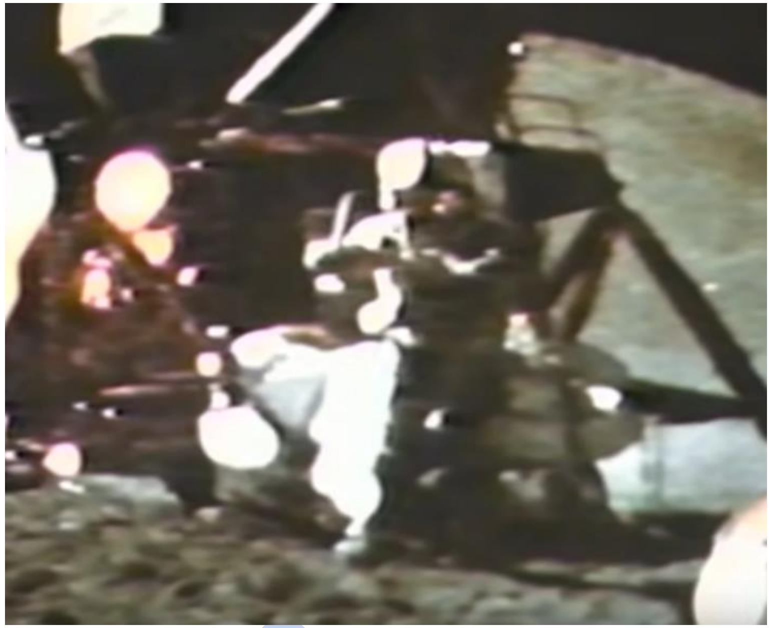
Project Introduction

1 Introduction

Free fall refers to any motion of a body where gravity is the only force acting upon it. Galileo's law of free fall states that, in the absence of air resistance, all bodies fall with the same acceleration, independent of their mass. See the following videos illustrating this concept:

– Hammer vs Feather: Physics on the Moon
  `https://www.youtube.com/watch?v=KDp1tiUsZw8`

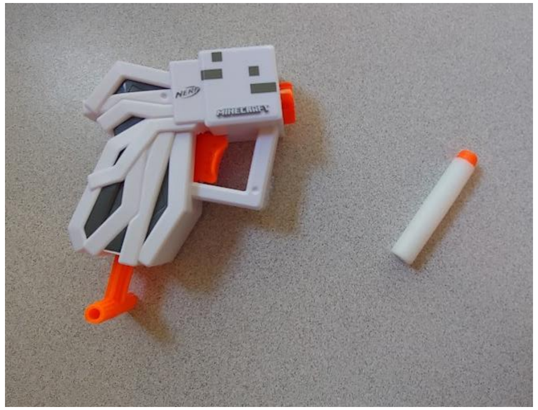
– Brian Cox visits the world's biggest vacuum
  `https://www.youtube.com/watch?v=E43-CfukEgs`



Materials and Setup

2 Materials and Setup

- Several dart guns (one for each student group) with at least 3 darts each
- Tape measure
- Video camera

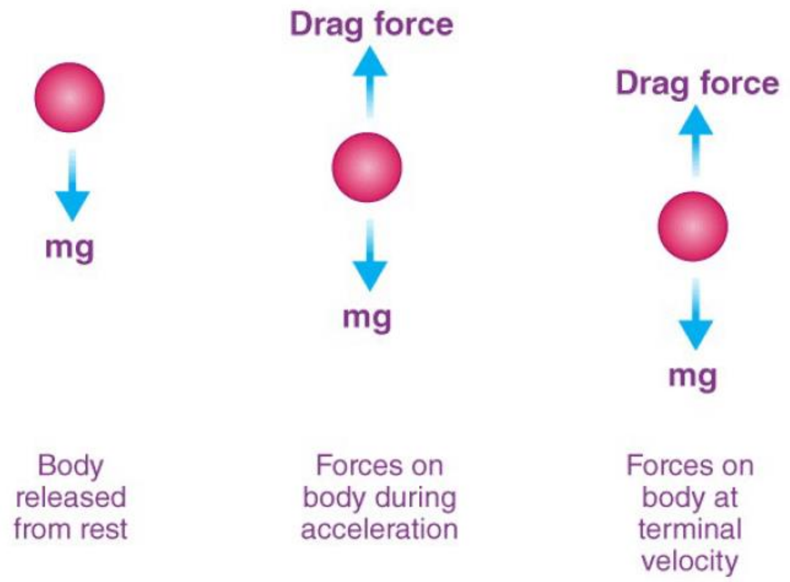


Model Development

3 Differential Equation Model

Using Figure 1 we see that the net force on an object in free fall in the atmosphere is the sum of the force due to gravity and the force due to air resistance,

$$F_{\text{net}} = F_{\text{gravity}} + F_{\text{drag}}.$$



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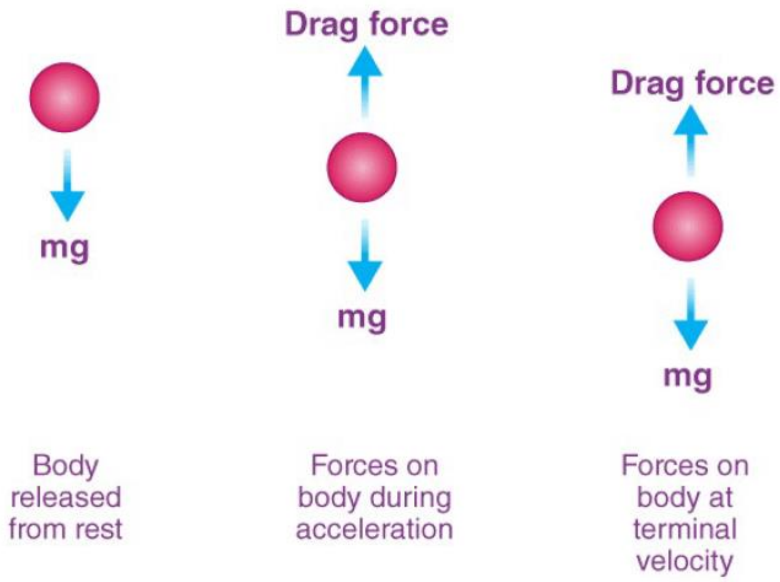

Practice Problem

$$y'' = -5 - 2y'$$

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- **Method 1**:

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  1. Let $v = y'$ and rewrite the equation in terms of v and v'.
  2. Find the general solution of the resulting differential equation for $v(t)$. Note that $v' = \frac{dv}{dt}$.

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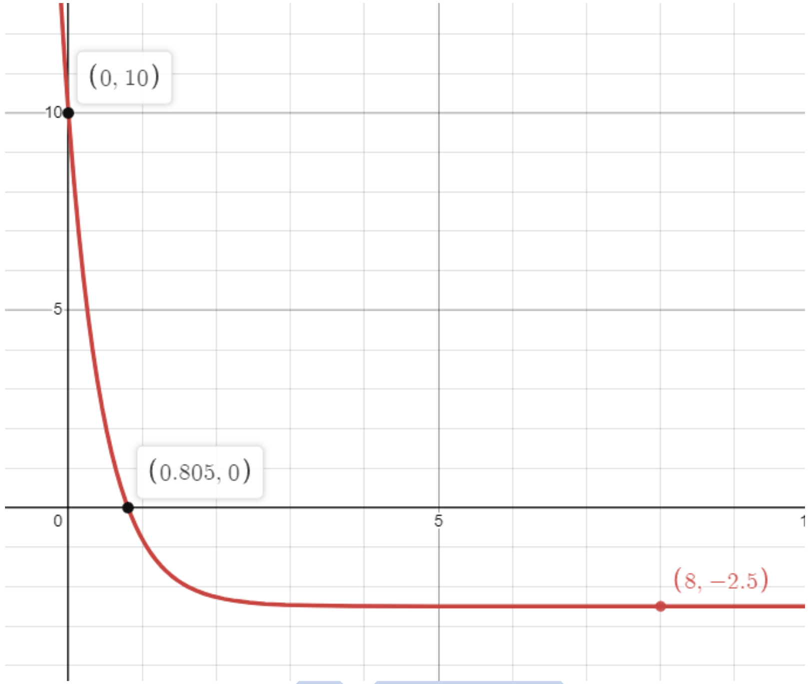
5. Graph the solution and verify your answer from part 4.



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6. Using $y' = v$ and your solution for $v(t)$ (from part 3), find the general solution $y(t)$.

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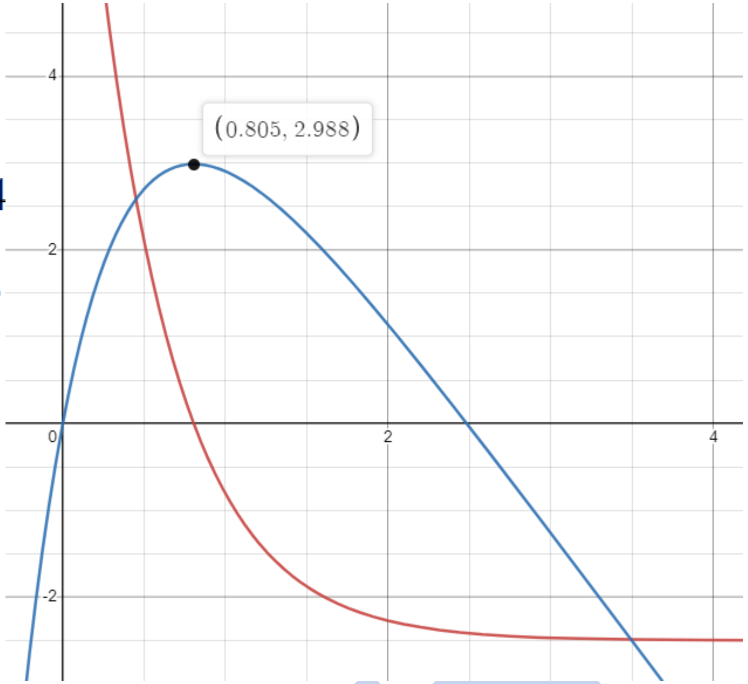

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8. Graph the particular solution $y(t)$.
9. Determine the maximum value of $y(t)$ on $[0, \infty)$ and verify your answer graphically.

Practice Problem

$$y'' = -5 - 2y'$$

- **Method 2**:

Practice Problem

$$y'' = -5 - 2y'$$

- **Method 2**:

1. Find the general solution of $y'' = -5 - 2y'$ using the method of undetermined coefficients (UC). (Hint: Use a linear function for the particular solution of the UC method.)

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4. Let $v = y'$ and find $\lim_{t\to\infty} v(t)$.
5. Graph $v(t)$ and verify your answer from part 4.

Solve the DE Model

$$y'' = -g - ky'$$

- **Method 1**:

1. Let $v = y'$ and rewrite (1) in terms of v and v'.
2. Find the general solution of the resulting differential equation for $v(t)$. Note that $v' = \frac{dv}{dt}$.
3. Use $v(0) = v_0$ to find the particular solution.
4. Find $\lim_{t\to\infty} v(t)$ (in terms of model parameters). Denote this value by v_T. What does this value represent in the context of free-fall?
5. Graph the solution (use $g = 9.8\frac{\text{m}}{\text{s}^2}$). Verify your answer from part 4.
6. Using $y' = v$ and your solution for $v(t)$ (from Part 3), find the general solution $y(t)$.
7. Find the particular solution for $y(t)$ if $y(0) = 0$.
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Solve the DE Model

$$y'' = -g - ky'$$

- **Method 2**:

1. Find the general solution of (1) using the method of undetermined coefficients (UC). (Hint: Use a linear function for the particular solution of the UC method.)
2. Using $y(0) = 0$ and $y'(0) = v_0$, find the particular solution of Equation (1).
3. Let $v = y'$. Graph $v(t)$.
4. Find $\lim_{t\to\infty} v(t)$ (in terms of model parameters). Denote this value by v_T. What does this value represent in the context of free-fall? Verify your prediction using the graph of $v(t)$.

Parameterization by Experiment

6 Estimating Terminal and Muzzle Velocity (Model Parameterization)

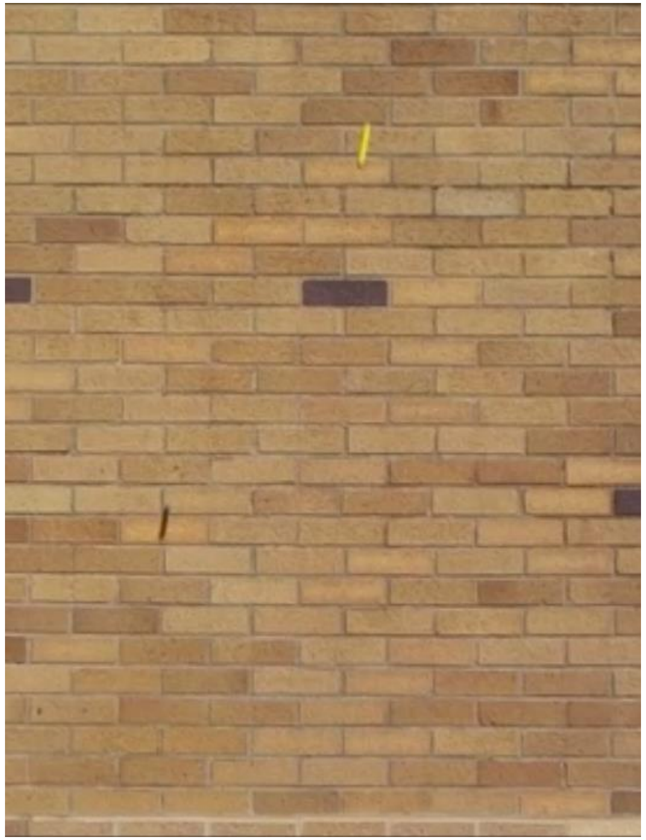
Parameterization by Experiment

6 Estimating Terminal and Muzzle Velocity (Model Parameterization)

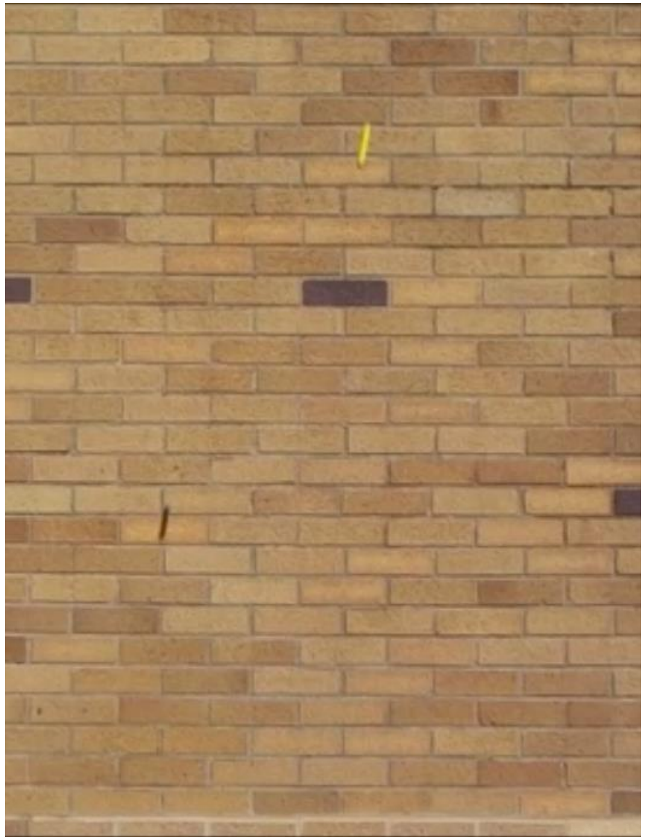
- **Terminal Velocity**: We experimentally estimate the terminal velocity of the dart using video frames of the dart being fired vertically with a distance measure in the background. We measure the dart's velocity on its descent and assume this is its terminal velocity. The example and images below illustrate this process.

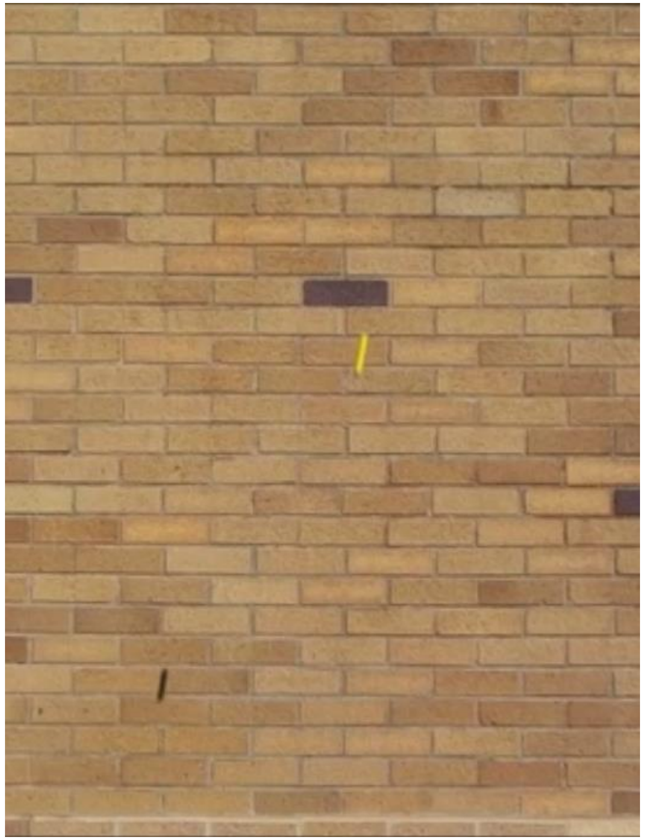
Parameterization by Experiment

Parameterization by Experiment



Parameterization by Experiment





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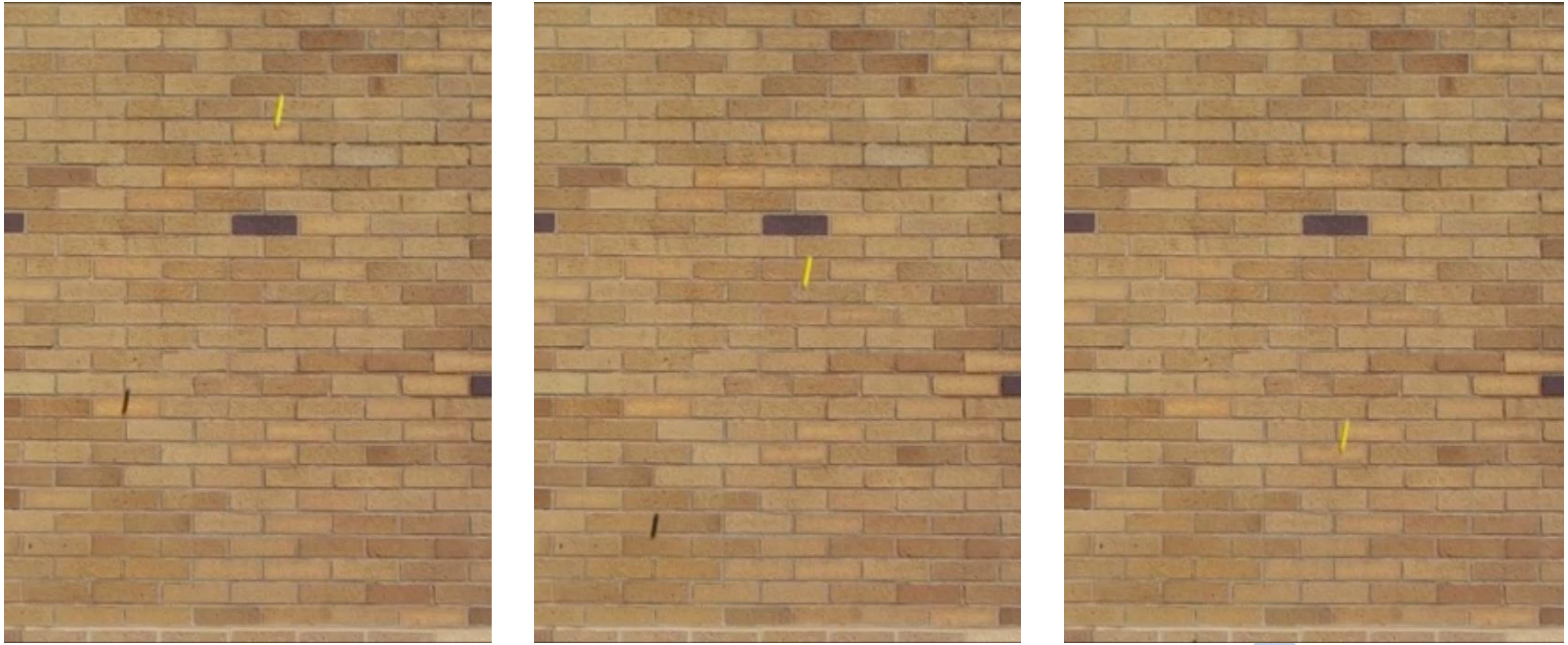

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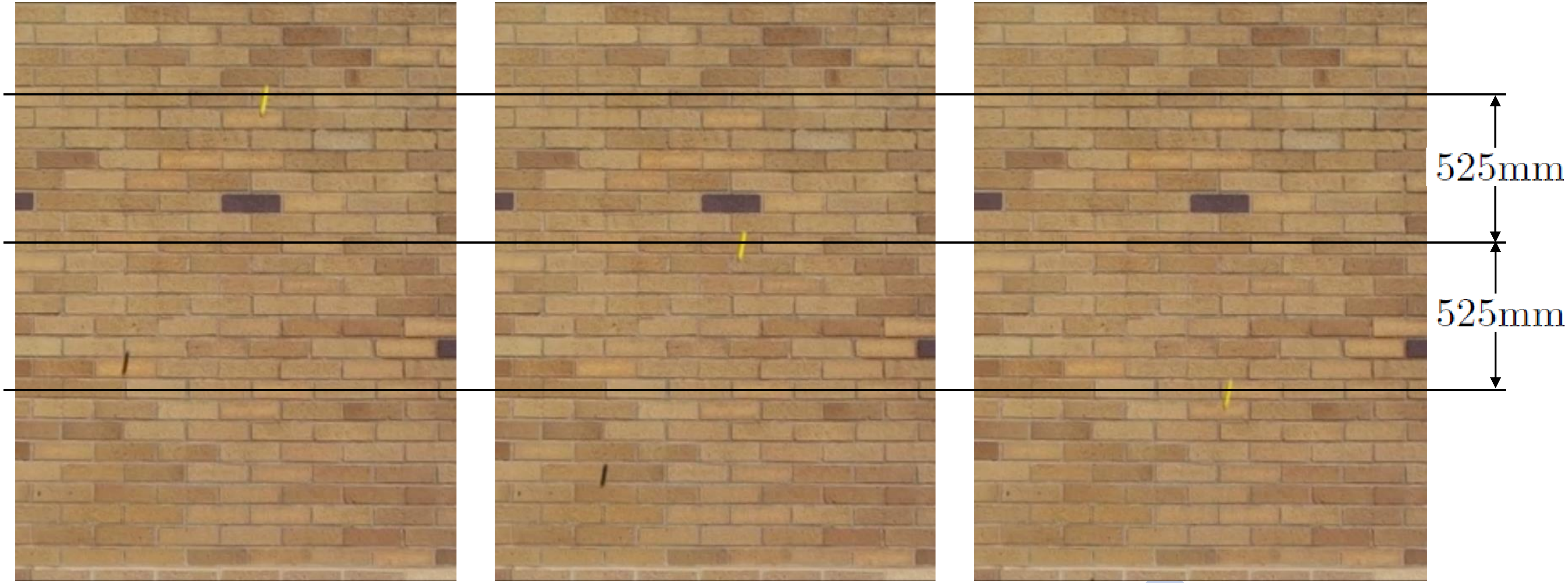


Parameterization by Experiment

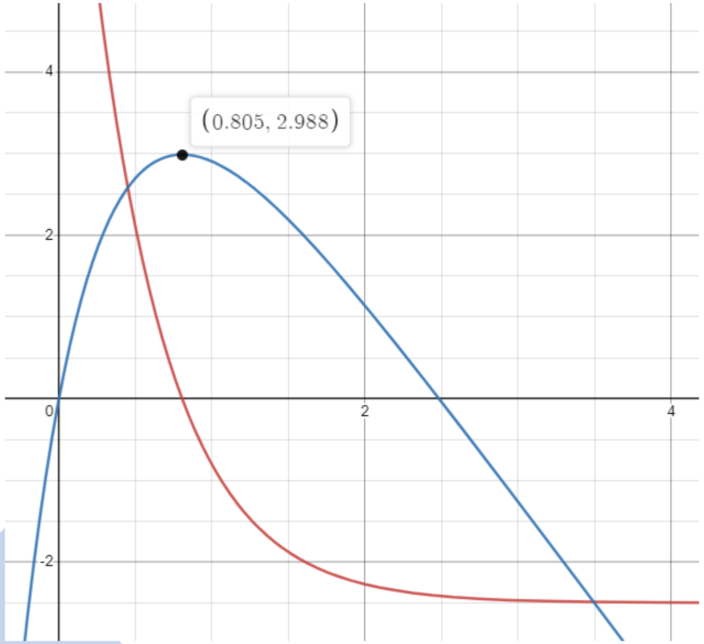
In each frame, the dart fell approximately 7 brick heights (75 mm including mortar). The frame rate of the video was 30fps. Hence, the dart fell $7 \times 75\text{mm} = 525\text{mm}$ in $\frac{1}{30}$ second. This results in a terminal velocity estimate of $v_T \approx \frac{525}{\frac{1}{30}} = 15,750\frac{\text{mm}}{\text{sec}} = 15.75\frac{\text{m}}{\text{sec}}$.

Parameterization by Experiment

- **Muzzle Velocity**: We experimentally estimate the muzzle velocity of the firing apparatus (dart gun) using video frames of the dart being fired horizontally with a distance measure in the background.

Model Validation

7 Predicting Ascent Time (Model Validation)



Model Validation

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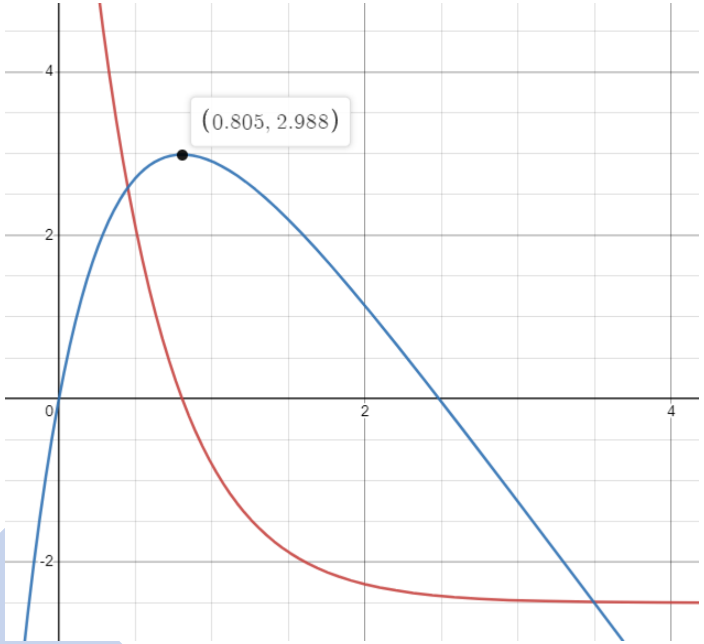
1. Use your solution from section 4 to determine the ascent time of the dart (time required for the dart to reach maximum height) in terms of the model parameters g, k, and v_0.



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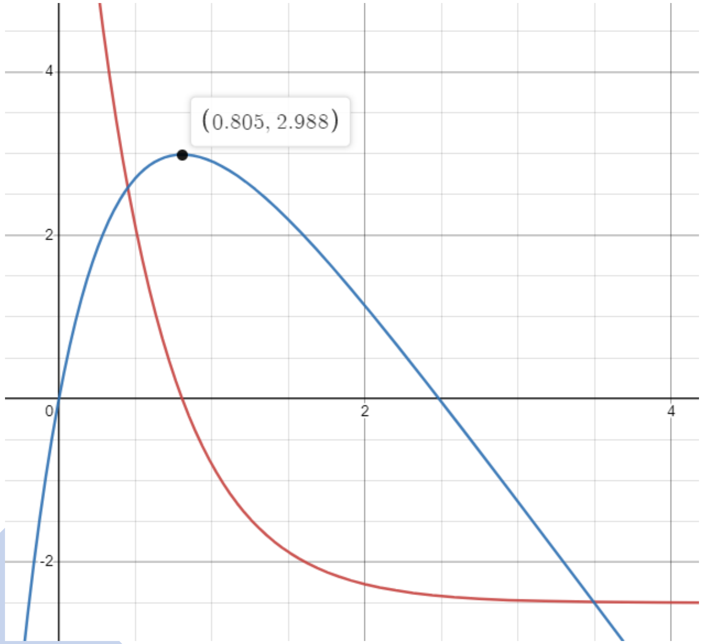
1. Use your solution from section 4 to determine the ascent time of the dart (time required for the dart to reach maximum height) in terms of the model parameters g, k, and v_0.
2. Use your formula for v_T (from section 4) to write the ascent time prediction in terms of g, v_0, and v_T.



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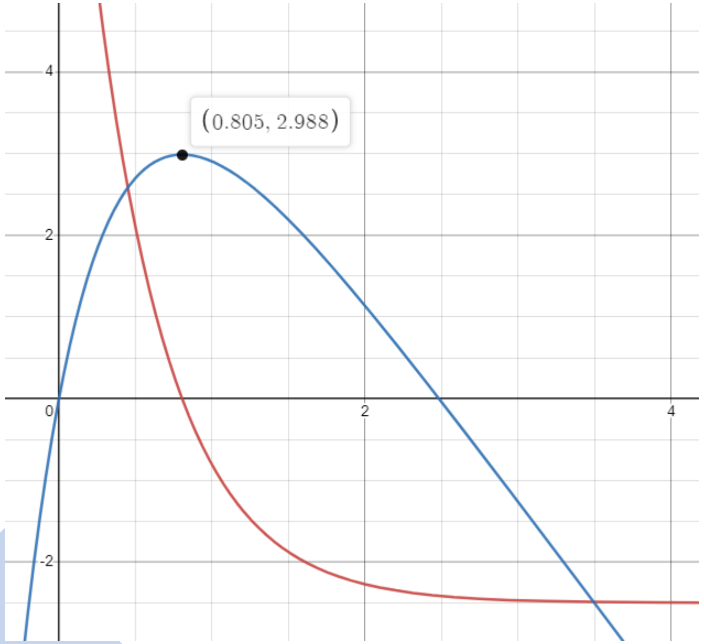
3. Verify your ascent time prediction graphically using the determined values for v_0 and v_T (use $g = 9.8\frac{\text{m}}{\text{s}^2}$).



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4. Test your prediction of ascent time by timing (with a stopwatch) the actual ascent of the dart. This should be done outside, in a stairwell, or a room with a very high ceiling.

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4. Test your prediction of ascent time by timing (with a stopwatch) the actual ascent of the dart. This should be done outside, in a stairwell, or a room with a very high ceiling.
5. Calculate the relative error of your prediction:

$$\text{relative error} = \frac{\text{observed value} - \text{predicted value}}{\text{observed value}} \times 100\%$$

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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$$y'' = -g - ky' \qquad (1)$$

If we neglect air drag in our model, Equation 1 simplifies to

$$y'' = -g \qquad (2)$$

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

$$y'' = -g - ky' \quad (1)$$

If we neglect air drag in our model, Equation 1 simplifies to

$$y'' = -g \quad (2)$$

Here we show that if the effect of air drag is less and less, i.e., $k \to 0$, the solution to Equation (1) becomes the solution to Equation (2).

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

$$y'' = -g \tag{2}$$

1. Solve Equation (2) using $v(0) = v_0$ and $y(0) = 0$.

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

$$y'' = -g \qquad (2)$$

1. Solve Equation (2) using $v(0) = v_0$ and $y(0) = 0$.
2. Graph the solutions (v and y) of both Equations (1) and (2) on the same coordinate axis. Let k approach 0 using a slider. What is the interpretation of this limit? What do you observe in the graphs?

https://www.desmos.com/calculator/pchk9ayeig

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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2. Graph the solutions (v and y) of both Equations (1) and (2) on the same coordinate axis. Let k approach 0 using a slider. What is the interpretation of this limit? What do you observe in the graphs?
3. To show that the solution of Equation (1) is equivalent to the solution of Equation (2) if we neglect air resistance, we can let k approach 0 in the solution of Equation (1). What do you observe upon attempting this limit?

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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1. Solve Equation (2) using $v(0) = v_0$ and $y(0) = 0$.
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3. To show that the solution of Equation (1) is equivalent to the solution of Equation (2) if we neglect air resistance, we can let k approach 0 in the solution of Equation (1). What do you observe upon attempting this limit?
4. Use the Taylor series expansion for e^x to expand any exonential terms in your solution of Equation (1) for velocity $v(t)$ from section 4.

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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4. Use the Taylor series expansion for e^x to expand any exonential terms in your solution of Equation (1) for velocity $v(t)$ from section 4.
5. Substitute this expansion into the velocity solution of Equation (1) and simplify.

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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6. Now let $k \to 0$.

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

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3. To show that the solution of Equation (1) is equivalent to the solution of Equation (2) if we neglect air resistance, we can let k approach 0 in the solution of Equation (1). What do you observe upon attempting this limit?
4. Use the Taylor series expansion for e^x to expand any exonential terms in your solution of Equation (1) for velocity $v(t)$ from section 4.
5. Substitute this expansion into the velocity solution of Equation (1) and simplify.
6. Now let $k \to 0$.
7. Compare the result to your velocity solution of Equation (2).

Asymptotic Analysis

8 Equivalent Model with no Air Resistance

$$y'' = -g \tag{2}$$

1. Solve Equation (2) using $v(0) = v_0$ and $y(0) = 0$.
2. Graph the solutions (v and y) of both Equations (1) and (2) on the same coordinate axis. Let k approach 0 using a slider. What is the interpretation of this limit? What do you observe in the graphs?
3. To show that the solution of Equation (1) is equivalent to the solution of Equation (2) if we neglect air resistance, we can let k approach 0 in the solution of Equation (1). What do you observe upon attempting this limit?
4. Use the Taylor series expansion for e^x to expand any exonential terms in your solution of Equation (1) for velocity $v(t)$ from section 4.
5. Substitute this expansion into the velocity solution of Equation (1) and simplify.
6. Now let $k \to 0$.
7. Compare the result to your velocity solution of Equation (2).
8. Graph

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How High?! Modeling Free Fall with Air Drag

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