A Journey Through Air Under Gravity: Unveiling the Dynamics of the Flight of a Sponge Dart

or ... Ballistics Modeling with a Sponge Dart

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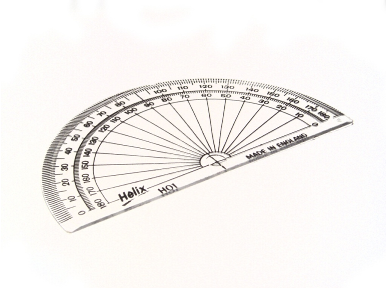
Given at SIMIODE Expo 2024

Goal; Equipment and Measurement

Project Goal: model the flight of a sponge dart and predict its landing position; find the angle that maximizes distance traveled.



The angle of the gun and muzzle height were set and the gun was fastened with duct tape in the correct position on an upright stand for firing.

Firing Data

Initial height (m)	Dropped/Fired	Time to reach ground (s)
4.06	dropped	0.95
0.39	fired	2.13

Initial height for the angle of inclination measurements is 0.18 m.

Angle of inclination (deg)	5	10	...	80	85
Distance traveled (m)	4.37	5.23	...	3.34	2.13

Four measurements were averaged to make each data point.

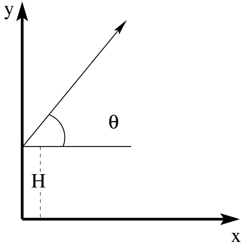
All code (MATLAB and Python) and calculations are included with the Teacher Version:

Howard, P., Linhart, J. M. (2022). 3-043-BallisticModeling-SpongeDart-ModelingScenario. SIMIODE, QUBES Educational Resources. doi:10.25334/SPEP-6T76

No air resistance: Newton's Laws of Motion

$$\frac{d^2}{dt^2}y(t) = -g, \qquad \frac{d^2}{dt^2}x(t) = 0$$

Let v_0 be the initial velocity of the dart, firing angle θ, and initial height H. Our initial conditions are $y'(0) = v_0 \sin\theta$, $x'(0) = v_0 \cos\theta$, $y(0) = H$, $x(0) = 0$.

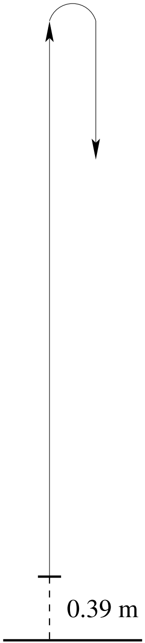


We find

$$y(t) = \frac{1}{2}gt^2 + v_0 t \sin\theta + H$$
$$x(t) = v_0 t \cos\theta$$

Student background at TAMU: engineering, lots of physics, MATLAB .
Student background at CWU: little physics; little coding.

No air resistance: Parameter estimation



The dart was fired from a height of 0.39 meters and took 2.13 seconds to hit the ground. We use $y(2.13) = 0$ to find $v_0 \approx 10.26$m/s.

No air resistance: Distance the dart travels

We set $y(t_f) = 0$ and solve

$$t_f(\theta) = \frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gH}}{g}$$

Then $t_f(\theta)$ can be substituted into $x(t)$ to find $d(\theta)$, the distance the dart travels as a function of the firing angle θ.

$$d(\theta) = v_0 \cos\theta \left[\frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gH}}{g} \right]$$

No air resistance: Maximizing the distance traveled

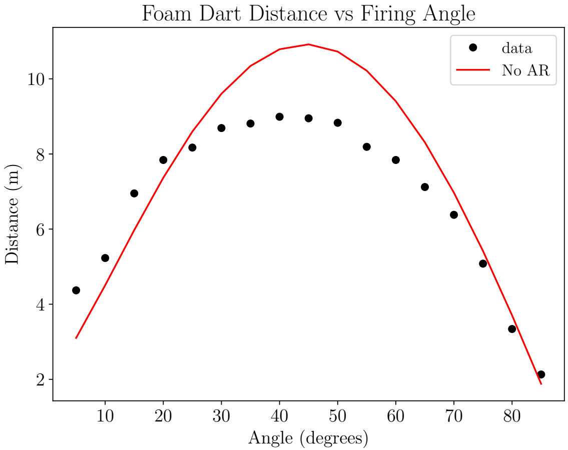
We observe our data and know this will not happen at an endpoint value for θ, so we differentiate $d(\theta)$ (not easy) knowing our maximum will be where $d'(\theta) = 0$.

$$d'(\theta) = -v_0 \sin\theta \left[\frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gH}}{g} \right] \tag{1}$$

$$+ v_0 \cos\theta \left[\frac{v \cos\theta}{g} + \frac{v_0^2 \sin\theta \cos\theta}{g\sqrt{(v_0 \sin\theta)^2 + 2gH}} \right] \tag{2}$$

This is not something we are going to solve by hand, and so students have a first experience using a numerical root-finding algorithm (`SciPy.optimize.fsolve()`). We find $\theta = 44.53^\circ$ maximizes the distance travelled.

No air resistance: Data and model



Getting my students the basics of Python and getting this far in the project often takes me 3 weeks.

Linear Air Resistance: Differential equations

$$\frac{d^2}{dt^2}y(t) = -g - b\frac{d}{dt}y(t) \tag{3}$$

$$\frac{d^2}{dt^2}x(t) = -b\frac{d}{dt}x(t), \tag{4}$$

To solve, we have to recognize that the equation for $y(t)$ is a first order linear differential equation for $\frac{dy}{dt}$ that we solve with an integrating factor.

$$w'(t) + bw(t) = -g$$

We find the integrating factor $\mu = e^{bt}$, and

$$y'(t) = w(t) = -\frac{g}{b} + Ce^{-bt}$$

Linear air resistance: Solving for $y(t)$

Since $w(0) = y'(0) = v_0 \sin\theta$, we find $C = v_0 \sin\theta + \frac{g}{b}$ and

$$y'(t) = -\frac{g}{b}\left(v_0 \sin\theta + \frac{g}{b}\right) e^{-bt}$$

integrate again, using $y(0) = H$,

$$y(t) = H - \frac{g}{b} t \left(\frac{v_0 \sin\theta}{b} + \frac{g}{b^2}\right)\left(1 - e^{-bt}\right) \tag{5}$$

Linear air resistance: Solving for $x(t)$

We recognize that the equation for $x(t)$ is a separable first order differential equation. We separate variables

$$\frac{dx'}{x'} = -b\,dt$$

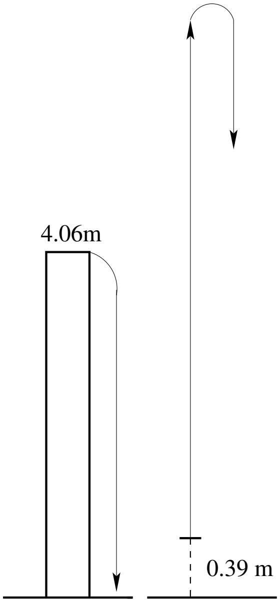
Integrating and using $x'(0) = v_0 \cos\theta$ we find

$$x'(t) = (v_0 \cos\theta)e^{-bt}$$

Now we integrate again using $x(0) = 0$ to solve for the unknown constant of integration.

$$x(t) = \frac{v_0 \cos\theta}{b} - \frac{v_0 \cos\theta}{b} e^{-bt}. \qquad (6)$$

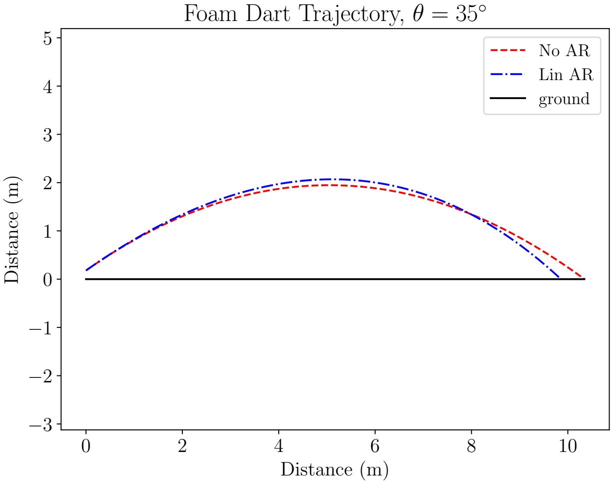
Linear air resistance: Parameter estimation



- To find b, we use our equation for $y(t)$ and the data from the dropped sponge dart ($v_0 = 0$, $H = 4.06$, $t_f = 0.95$, $\theta = \frac{\pi}{2}$) with a numerical root finder to find $b = 0.2792$.
- Now we use our equation for $y(t)$ with the data from the sponge dart fired straight up ($H = 0.39$, $t_f = 2.13$, $\theta = \frac{\pi}{2}$) with a numerical root finder to to find the initial velocity of the gun $v_0 = 11.23$m/s (compare: 10.26m/s with no air resistance).

Linear air resistance: Finding the distance traveled

We use the numerical root finder to find t_f from $y(t_f) = 0$ for a given firing angle θ. Substituting this into $x(t)$ allows us to create the following graph of the dart's trajectory at $\theta = 35°$ or any other angle.



Linear air resistance: Finding the maximum distance traveled

We let $t(\theta)$ to be the time it takes for a dart to hit the ground as a function of the firing angle. Then $x(t(\theta)) = d(\theta)$

$$d(\theta) = \frac{v_0}{b} \cos\theta \left(1 - e^{-bt(\theta)}\right) \tag{7}$$

which implies

$$\left(1 - e^{-bt(\theta)}\right) = \frac{bd(\theta)}{v_0 \cos\theta}, \tag{8}$$

and

$$t(\theta) = -\frac{1}{b} \ln\left(1 - \frac{bd(\theta)}{v \cos\theta}\right). \tag{9}$$

Linear air resistance: Implicit equation for distance traveled

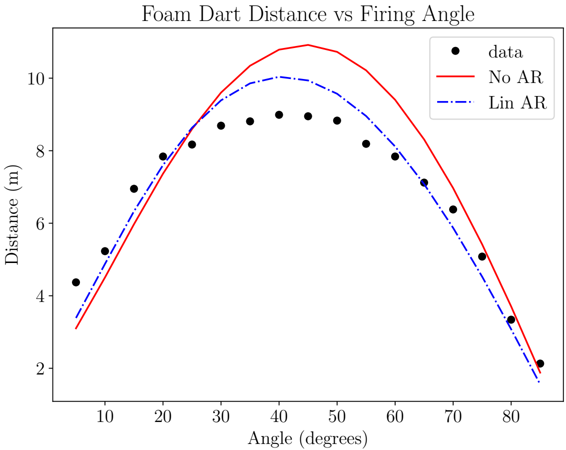
We substitute this formula for $t(\theta)$ into $y(t)$ and set $y(t) = 0$ to get an implicit equation for the distance traveled.

$$0 = H + \frac{g}{b^2} \ln\left(1 - \frac{bd(\theta)}{v_0 \cos\theta}\right) + \left(\frac{v_0 \sin\theta}{b} + \frac{g}{b^2}\right) \frac{bd(\theta)}{v \cos\theta} \quad (10)$$

Given θ, we can use a numerical root finding algorithm (`SciPy.optimize.fsolve()`) to find $d(\theta)$.

To find the maximum distance traveled, we must couple this to a numerical minimization algorithm (`SciPy.optimize.fmin()`) to find θ that minimizes $-d(\theta)$ and hence maximizes $d(\theta)$. We find $\theta = 40.68^\circ$ (compare: 44.53° with no air resistance).

Linear air resistance: Data and Models



It's usually taken me 5-6 weeks to get through this much of the project in a 10-week mathematical modeling course teaching students with little physics background.

Physical Air Resistance

Use Buckingham's π Theorem of dimensional analysis.

Let F be the force due to air resistance and start with the assumption that it depends on the air density ρ, surface area of the object, S and velocity of the object, v,

$$F = \kappa \rho^a S^b v^c,$$

here κ is a dimensionless constant. We equate dimensions where M represents mass, L length, and T time:

$$MLT^{-2} = M^a L^{-3a} L^{2b} L^c T^{-c},$$

Which results in the system

$$\begin{aligned} L \text{ length} &: \quad 1 = -3a + 2b + c \\ M \text{ mass} &: \quad 1 = a \\ T \text{ time} &: \quad -2 = -c. \end{aligned}$$

We find $a = 1$, $b = 1$ and $c = 2$, giving

$$F = \kappa \rho S v^2, \qquad \text{thus we let } b = \kappa \rho S. \tag{11}$$

Physical air resistance: First attempt at a model

Following the work with linear air resistance, students might try to incorporate an air resistance term proportional to the square of the velocity like this, but there are several problems here:

$$\frac{d^2}{dt^2}y(t) = -g - b\left(\frac{d}{dt}y(t)\right)^2 \qquad \frac{d^2x}{dt^2} = -b\left(\frac{d}{dt}x(t)\right)^2 . \quad (12)$$

Physical air resistance: First attempt at a model

Following the work with linear air resistance, students might try to incorporate an air resistance term proportional to the square of the velocity like this, but there are several problems here:

$$\frac{d^2}{dt^2}y(t) = -g - b\left(\frac{d}{dt}y(t)\right)^2 \qquad \frac{d^2x}{dt^2} = -b\left(\frac{d}{dt}x(t)\right)^2. \quad (13)$$

- The squared terms are always positive, and so air resistance can act in the same direction as the velocity.
- This should be a vector model.

Physical air resistance: Vector differential equation model

With $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{g} = \begin{bmatrix} 0 \\ g \end{bmatrix}$, our model becomes

$$\frac{d^2}{dt^2}\vec{r}(t) = -\vec{g} - b\,\|\vec{r}\,'(t)\|\,\vec{r}\,'(t) \tag{14}$$

Initial conditions

$$\vec{r}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \quad \vec{r}\,'(0) = \begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{bmatrix}$$

where H is the height at which the dart is fired, and v_0 is the initial speed of the dart.

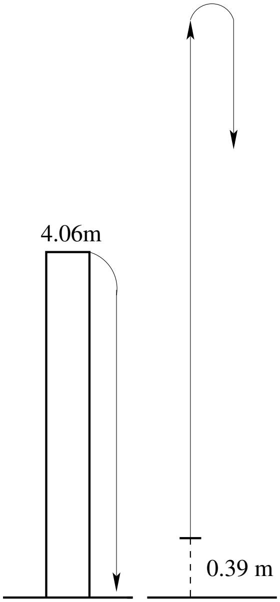
Analytic solution is either impossible or very cumbersome, and so we rely on technology to solve and complete the analysis.

Physical air resistance: First-order system

We translate our vector second-order differential equation into a linear first-order system in components, where $R_1(t) = x(t)$ and $R_2(t) = \frac{dx}{dt}(t)$, $R_3(t) = y(t)$ and $R_4(t) = \frac{dy}{dt}(t)$:

$$\begin{aligned}
\frac{dR_1}{dt} &= R_2 \\
\frac{dR_2}{dt} &= -b\left[\sqrt{(R_2)^2 + (R_4)^2}\right] R_2 \\
\frac{dR_3}{dt} &= R_4 \\
\frac{dR_4}{dt} &= -g - b\left[\sqrt{(R_2)^2 + (R_4)^2}\right] R_4.
\end{aligned} \tag{15}$$

Physical air resistance: Parameter estimation



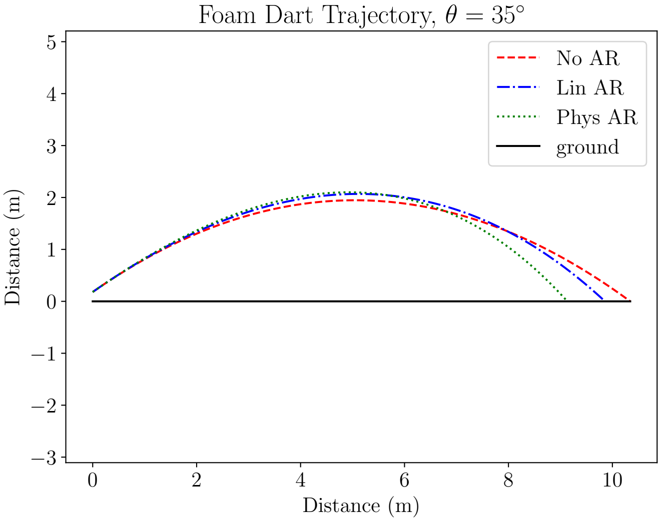
Two methods:

1. Use an IVP solver and a root finding algorithm to estimate parameters.
2. Use a BVP solver to estimate parameters.

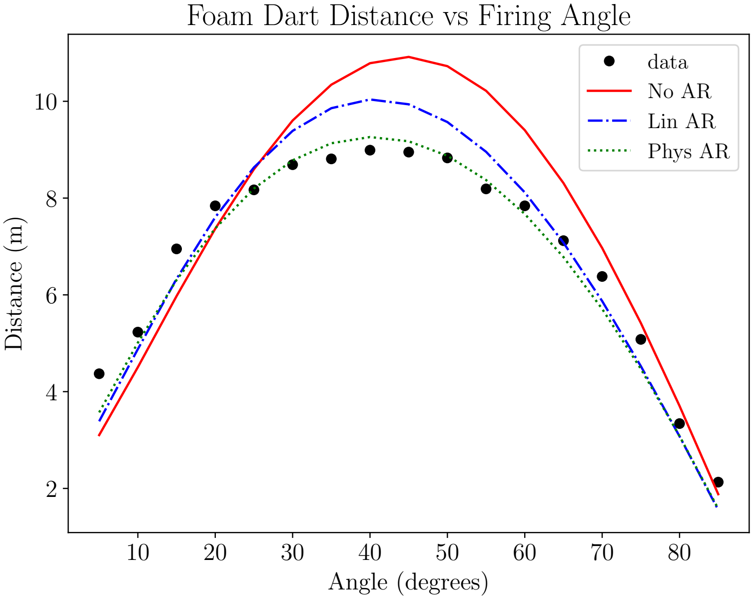
Results agree to 3 significant figures.
$b_0 \approx 0.0645$ (compare 0.2792 with linear air resistance)
$v_0 \approx 12.3$ m/s (compare 10.26 and 11.23 m/s with no air resistance and linear air resistance respectively).

Physical air resistance: 35° trajectories

We use *event location* functionality with an IVP solver to plot trajectories.



Physical air resistance: Data and Models



Physical air resistance: Maximizing distance traveled

Again we use that $d(\theta)$ is maximized when $-d(\theta)$ is minimized. Python's `SciPy.optimize.fmin()` can be used to find the value of θ for which $-d(\theta)$ is a minimum. There's some focused thinking work to be done here to set up the equations so that the numerical tools will work.

Using an initial guess of 45°, we found $\theta = 40.45^\circ$ maximizes the distance traveled. With linear air resistance we found $\theta = 40.68^\circ$, and with no air resistance we found $\theta = 44.53^\circ$.

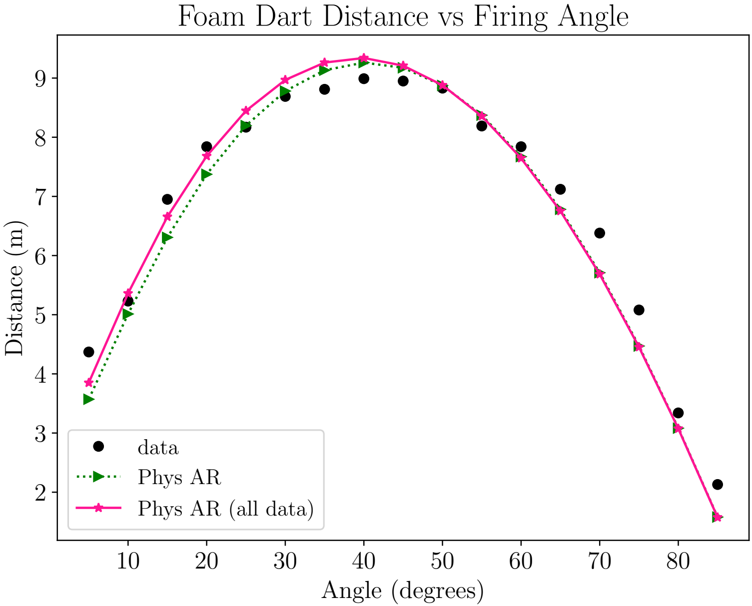
Physical air resistance: Regression approach

We use the linear first-order differential equation model with physical air-resistance and event location to return an array of distances the dart travels for each firing angle. Then Python's `SciPy.optimize.curve_fit()` is used to find the values of parameters b and v_0 to minimize the least squared errors between the model distances and actual distances the dart travels.

We find $b = 0.08018$ and $v_0 = 13.27$.

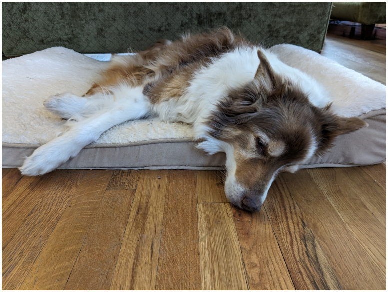
Model	b	v_0 m/s	θ_{max}
Physical AR with regression	0.0802	13.27	39.33°
Physical AR w/o regression	0.0645	12.3	40.45°
Linear air resistance	0.2792	11.23	40.68°
No air resistance	–	10.26	44.53°

Physical air resistance: using all the data



Thanks

Thank you for listening!



The full modeling scenario is:

Howard, P., Linhart, J. M. (2022). 3-043-BallisticModeling-SpongeDart-ModelingScenario. SIMIODE, QUBES Educational Resources. doi:10.25334/SPEP-6T76

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