Contrasting Discrete and Continuous Density Dependent Population Models

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Discrete and Continuous Density Dependent Models

$$
\begin{aligned}
p_{n+1} - p_n &= r(p_n)p_n \\
p'(t) &= r(p(t))p(t) \\
p(t+h) - p(t) &= hr(p(t))p(t)
\end{aligned}
$$

Per capita growth rate r: Compensation model

1. intrinsic per capita growth rate, $r(0) = b$
2. r is decreasing
3. carrying capacity, $r(L) = 0$, $r(1) = 0$

Equilibrium

$$\begin{aligned} p_{n+1} &= p_n + r(p_n)p_n \\ p'(t) &= r(p(t))p(t) \end{aligned}$$

- $E = 0$
- $r(E) = 0$, so $E = 1$.

Stability of Equilibrium, continuous case

$$
\begin{aligned}
dp(t)/dt &= f(p(t)) \\
\frac{df(p)}{dp}\Big|_{p=E} &= a \\
p(t) \sim E + c \cdot e^{at}
\end{aligned}
$$

E is stable if $a < 0$, unstable if $a > 0$.
Summary for $dp/dt = r(p)p$

- $f(p) = r(p)p$
- $f'(E) = r(E) + r'(E)E$
- $f'(0) = r(0) = b > 0$, unstable
- $f'(1) = r'(1) < 0$, stable

Stability of Equilibrium, discrete case

$$\begin{aligned} p_{n+1} &= f(p_n) \\ \frac{df(p)}{dp}\Big|_{p=E} &= a \\ p_n \sim E + c \cdot a^n \end{aligned}$$

E is stable if $|a| < 1$, unstable if $|a| > 1$.
Summary for $p_{n+1} = p_n + r(p_n)p_n$

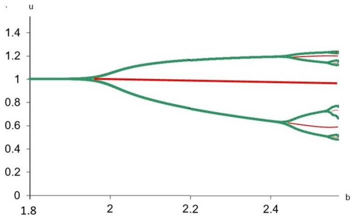
- $f(p) = p + r(p)p$
- $f'(E) = 1 + r(E) + r'(E)E$
- $f'(0) = 1 + r(0) = 1 + b > 1$, unstable
- $f'(1) = 1 + r'(1) < 1$,
- stable if $-2 < r'(1)$

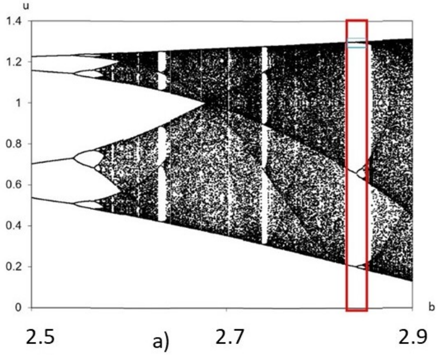
Logistic Equation: Linear r

$$
\begin{aligned}
r(p) &= b - bp \\
p'(t) &= (b - bp)p \\
p_{n+1} &= p_n + (b - bp_n)p_n
\end{aligned}
$$

- $E = 0$ unstable for both
- Continuous:
  - solution using partial fractions.
  - $E = 1$ is stable for all b.
- Discrete:
  - $E = 1$ is stable for $-2 < r'(1) = -b$ or $b < 2$.
  - period doubling to chaos.
  - if $b > 3$, $p_n \to -\infty$
  - if $p_0 > 1 + 1/b$, $p_n \to -\infty$

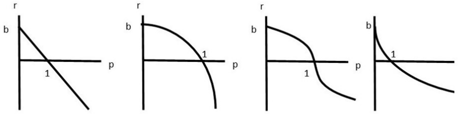
Period Doubling to Chaos





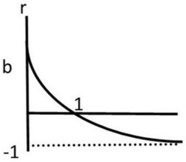
Per capita growth rate

1. $r(0) = b$
2. r is decreasing
3. $r(1) = 0$

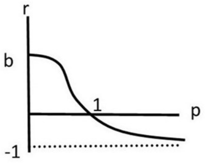


Scramble competition   Scramble competition   Contest competition

4. $\lim_{p \to \infty} r(p) = -1.$



Contest competition



scramble competition

Rational r

$$r(p) = \frac{a_1 + a_2 p^{j_2}}{a_3 + a_4 p^{j_4}}$$

$r(0) = b$ implies

$$r(p) = \frac{b + a_2 p^{j_2}}{1 + a_4 p^{j_4}}$$

$r(1) = 0$ implies

$$r(p) = \frac{b - b p^{j_2}}{1 + a_4 p^{j_4}}$$

r decreasing implies

$$a_4 \geq 0 \text{ and } j_2 \geq j_4.$$

$r \to -1$ implies $j_2 = j_4$ and $a_4 = b$, so

$$r(p) = \frac{b - b p^{j}}{1 + b p^{j}}$$

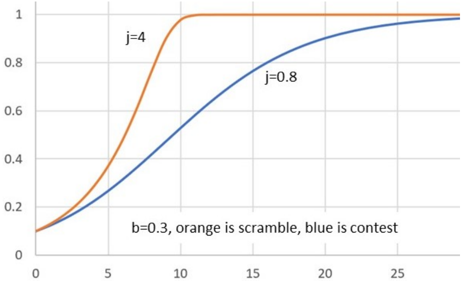
Beverton/Holt, 1957, $j = 1$: Maynard-Smith/Slatkin, 1973, $j \neq 1$.

Competition

$$r''(p) = \frac{(1 - j + b(1+j)p^j)b(1+b)jp^{j-2}}{(1+bp^j)^3}.$$

$$1 - j + b(1+j)p^j$$

- $j \leq 1$, concave up, contest competition (Beverton/Holt) exploited fish. (blue)
- $j > 1$, point of inflection so scramble competition Maynard-Smith/Slatkin, prey species. (orange)



Finding a solution, continuous

$$p' = \frac{bp - bp^{j_2}}{1 + a_4 p^{j_4}}$$

Separation of Variables

$$\frac{1 + a_4 p^{j_4}}{p - p^{j_2}} dp = bdt.$$

partial fractions, integrate
Opinion: Solutions are overrated.

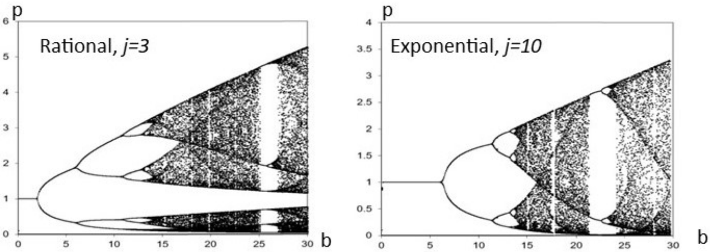
Exponential per capita growth rate

$$
\begin{aligned}
r(p) &= c_1(c_2)^{p^j} + c_3 \\
r(0) = b &\Rightarrow c_1 = b - c_3 \\
r(1) = 0 &\Rightarrow c_2 = \frac{-c_3}{b - c_3} \\
r'(p) < 0 &\Rightarrow c_3 < 0 \\
r(p) \to -1 &\Rightarrow r(p) = (1 + b)^{1-p^j} - 1
\end{aligned}
$$

Discrete model is the Ricker(1954, $j = 1$)/Bellows (1981, $j \neq 1$) model.

When is $E = 1$ stable?

- Always for continuous model.
- Rational model: For $0 < j \leq 2$ or if $b < \frac{2}{j-2}$.
- Exponential model: For $b < e^{2/j} - 1$.
- Period doubling to chaos,
  - $j > 2$ rational,
  - all $j > 0$ exponential.



Thank you!

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