Providing contextual learning experiences of differential equations through mathematical modelling

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Background

New Zealand study involving 3 case studies

Each case study:

- Mathematical modelling course
- Lecturer participant
- Student participants

Each course:

- Different approach to teaching modelling
- Different type of modelling activity
- All modelling activities occurred within student groups

Table 1. Summary of modelling activities for each course

Course	Modelling Activity used	Focus on modelling cycle	Students' responsibility for decisions
Course One	Open ended	Explicit	All decisions, including foundation model to use
Course Two	Case Studies	Implicit	Modification of model only
Course Three	Popular Culture	Explicit	All decisions except for foundation model

Outline

Information on each course

Main modelling activity used in the course

Examples of students' use of differential equations in the activities

Course One: Open ended modelling questions

Set up around activity to make modelling possible

During lectures and tutorials

Developing techniques for modelling:

Mathematical techniques, methods and tools including

- Dimensional analysis, optimisation, rates of change, curve fitting
- Standard models (including physical laws such as Newton's laws)
- Differentiation techniques (including analytical, numerical and integration techniques)

Acquiring knowledge of the process of modelling:

- Explicit exposure to the modelling cycle
- Frequently alluding to the process of modelling
- Heuristic strategies

- Pre modelling day tutorial

  Skills for effective group work

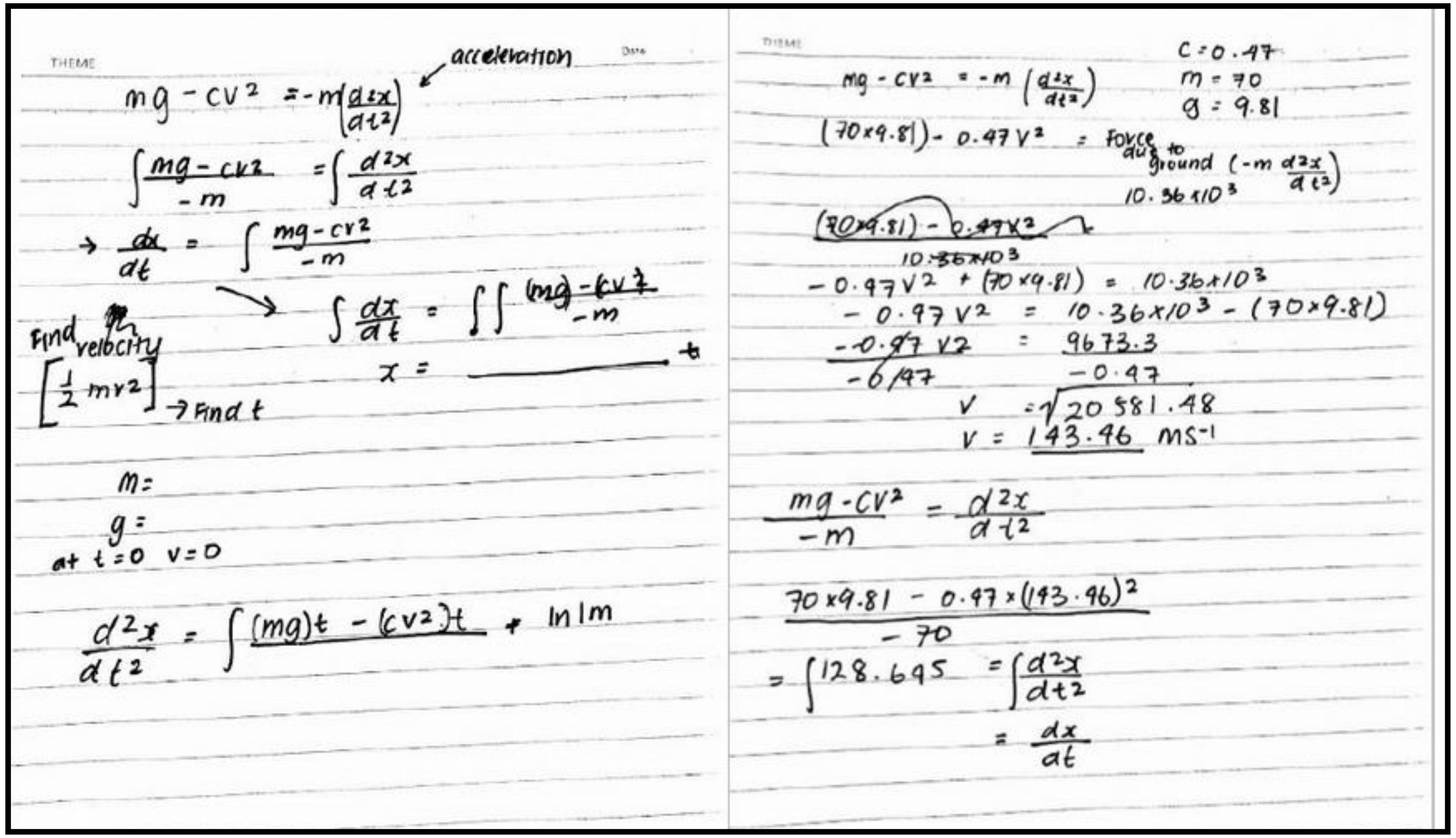
  Understanding the criteria for the assessing the open-ended modelling project

Additional Student work:

- Assignments (past and present assignment solutions)
- Past worked exam problems

How high can you jump from a building without causing injury?

How high can you jump from a building without causing injury?



THEME

Date

$$mg - cv^2 = -m\left(\frac{d^2x}{dt^2}\right)$$ acceleration

$$\int \frac{mg - cv^2}{-m} = \int \frac{d^2x}{dt^2}$$

$$\rightarrow \frac{dx}{dt} = \int \frac{mg - cv^2}{-m}$$

$$\int \frac{dx}{dt} = \iint \frac{(mg) - (cv^2)}{-m}$$

$$x = \underline{\hspace{3cm}}\, t$$

Find velocity $\left[\frac{1}{2}mv^2\right]$ → Find t

$m=$

$g=$

at $t=0$ $v=0$

$$\frac{d^2x}{dt^2} = \int \frac{(mg)t - (cv^2)t}{} + \ln|m$$

THEME

$c = 0.47$

$m = 70$

$g = 9.81$

$$mg - cv^2 = -m\left(\frac{d^2x}{dt^2}\right)$$

$$(70 \times 9.81) - 0.47v^2 = \text{force due to ground } \left(-m\frac{d^2x}{dt^2}\right)$$

$$10.36 \times 10^3$$

$$\frac{(70 \times 9.81) - 0.47v^2}{10.36 \times 10^3}$$

$$-0.47v^2 + (70 \times 9.81) = 10.36 \times 10^3$$

$$-0.97v^2 = 10.36 \times 10^3 - (70 \times 9.81)$$

$$\frac{-0.97v^2}{-0.97} = \frac{9673.3}{-0.47}$$

$$v = \sqrt{20581.48}$$

$$v = \underline{143.46}\ ms^{-1}$$

$$\frac{mg - cv^2}{-m} = \frac{d^2x}{dt^2}$$

$$\frac{70 \times 9.81 - 0.47 \times (143.46)^2}{-70}$$

$$= \int 128.695 = \int \frac{d^2x}{dt^2}$$

$$= \frac{dx}{dt}$$

Course Two: Case Studies

Set up around activity to make modelling possible

During lectures and tutorials:

Modelling tools and mathematical tools including

- Discrete equations
- Ordinary differential equations in general
- Models of population dynamics using ordinary differential equations
- Recap on solving differential equations
- Difference between difference equations and differential equations.

Acquiring knowledge of the process of modelling:
Studying case studies and modifying a case study

During presentations of case studies:
Engagement with case study, feedforward from lecturer, questions from audience

Case Studies

Students selected a case study from case studies using similar base models and ranked case studies in order of preference.

Groups formed through common interest in a particular case study

Three presentations:

- First present model for case, explaining terms of model
- Second present model an mathematics of model
- Third present limitations and a modification to model. Present mathematical workings of new model and its implications.

Case study provides internal guidance
Lecturer provides steps forward with areas to work on for following presentation

First Presentation

Present background for problem

Present model for the case study

Explaining terms of model

Success Story – Prickly pear is controlled by Cactoblastis cactorum moth

- In 1864 Captain Phillip introduced prickly pear to Australia.
- The prickly pear thrived in the arid environment and spread like wild fire.
- It became a leech on the land destroying farmland viability.

- In 1925 the Argentinian moth was introduced to Australia.
- The moth's larvae (killing machines) bore through the plant.
- Their breeding is restricted solely to the prickly pear plant.
- They were distributed to farmers on egg-sticks (approx. 80 eggs per stick).

By 1927 the moths had won the fight and the prickly pear was virtually wiped out.

Moth vs Prickly pear

$$f_1 = \frac{dV}{dt} = r_1 V\left(1 - \frac{V}{K}\right) - c_1 H\left(\frac{V}{V + D}\right)$$

$$f_2 = \frac{dH}{dt} = r_2 H\left(1 - \frac{JH}{V}\right)$$

V = unit population of pear per unit area - unit mass per acre
t = unit time
$\mathbf{r_1}$ = rate of growth multiplier for pears - (unit time)$^{-1}$
K = unit mass per acre (this is a maximum population per area)
$\mathbf{c_1}$= unit population of pear per unit population of moth per unit area - unit mass per egg stick per acre
H = unit population of moths per unit area - egg sticks per acre
D = unit mass per acre (grazing efficiency factor)
$\mathbf{r_2}$ = rate of growth multiplier for moths - (unit time)$^{-1}$
J = unit population of pears per unit population of moths - unit mass per egg stick

Second Presentation

Mathematics of model

Model

$$f_1(Cacti) = \frac{dV}{dt} = r_1 V\left(1 - \frac{V}{K}\right) - c_1 H\left(\frac{V}{V+D}\right)$$

$$f_2(Larvae) = \frac{dH}{dt} = r_2 H\left(1 - \frac{JH}{V}\right)$$

Fixed Points

$$f_2 = r_2 H\left(1 - \frac{JH}{V}\right) = 0$$

$$So\ H = 0\ or\ r_2\left(1 - \frac{JH}{V}\right) = 0\ which\ gives\ V = JH$$

$$When\ H = 0\ gives\ V = 0\ or\ V = K$$

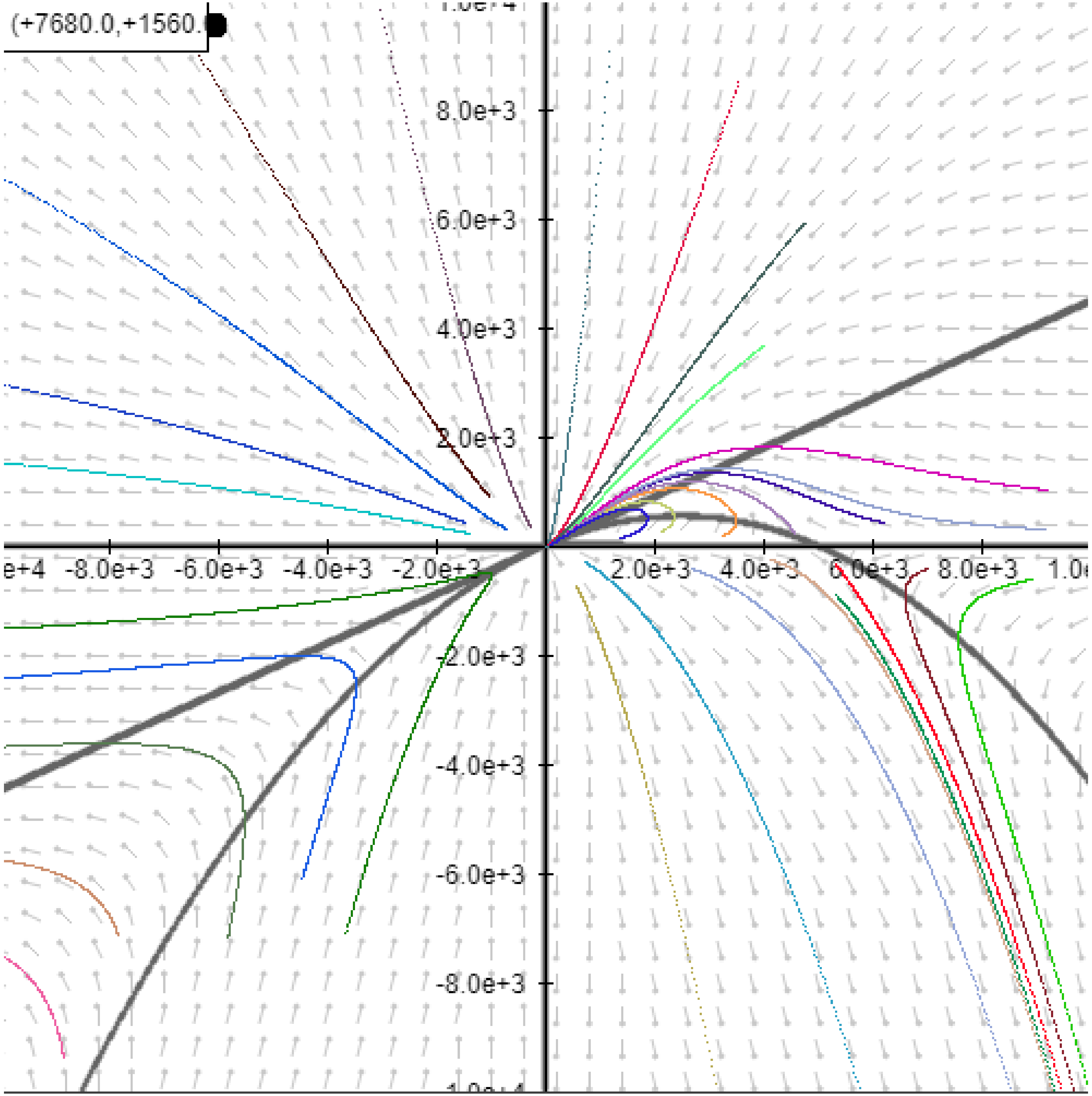
$$When\ V = JH\ and\ assume\ H \neq 0$$

$$H = -\frac{r_1 JD - r_1 KJ + c_1 K \pm \sqrt{2KDr_1^2 J^2 + K^2 J^2 r_1^2 + 2KDc_1 r_1 J - 2K^2 c_1 r_1 J + D^2 J^2 r_1^2 + K^2 c_1^2}}{2r_1 J^2}$$

$$V = -\frac{r_1 JD - r_1 KJ + c_1 K \pm \sqrt{c_1^2 K^2 + K^2 J^2 r_1^2 - 2K^2 c_1 r_1 J + 4KDr_1{}^2 J + 2KDc_1 r_1 J - 2KDr_1^2 J^2 + D^2 r_1^2 J^2}}{2r_1 J^2}$$

Phase Portrait

Nullclines in grey



Stability

$$J = \begin{bmatrix} r_1 - \frac{2r_1V}{K} - \frac{c_1H}{(V+D)} - \frac{c_1HV}{(V+D)^2} & -c_1\left(\frac{V}{V+D}\right) \\ \frac{r_2JH^2}{V^2} & r_2 - \frac{2r_2JH}{V} \end{bmatrix}$$

$If\ H = V = 0, then\ stability\ of\ fixed\ point\ is\ undeterminable.$

$$If\ H = 0, V = K, then\ J = \begin{bmatrix} r_1 - 2r_1 & -c_1\left(\frac{K}{K+D}\right) \\ 0 & r_2 \end{bmatrix}$$

$\lambda_{21} = r_1 - 2r_1 = -r_1 \Rightarrow stable\ for\ r_1 > 0, which\ is\ always\ true.$

$\lambda_{22} = r_2 \Rightarrow stable\ for\ r_2 < 0, which\ is\ never\ true\ as\ r_2 > 0.$

$So\ this\ is\ a\ unstable\ point\ (saddle).$

$If\ H = quaratic = V, we\ saw\ from\ the\ phase\ portrait$

$that\ the\ positive\ fixed\ point\ is\ asymtoticaly\ stable\ so\ we\ assume\ \lambda_{31} and\ \lambda_{32}\ are\ negative.$

Fixed Points and Stability

1 $If\ H = 0, V = \varepsilon,\ J = \begin{bmatrix} r_1 - \frac{2r_1\varepsilon}{K} & -c_1\left(\frac{\varepsilon}{\varepsilon+D}\right) \\ 0 & r_2 \end{bmatrix}$

$then \lim_{\varepsilon\to 0} J = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$

$\lambda_{11} = r_1 \Rightarrow unstable,\ since\ r_1 > 0,\ which\ is\ always\ true\ so\ \lambda_{11} > 0.$

$\lambda_{12} = r_2 \Rightarrow unstable,\ since\ r_2 > 0,\ which\ is\ always\ true\ so\ \lambda_{12} > 0.$

$So\ this\ is\ a\ unstable\ point.$

2 $If\ H = 0, V = K, then\ J = \begin{bmatrix} r_1 - 2r_1 & -c_1\left(\frac{K}{K+D}\right) \\ 0 & r_2 \end{bmatrix}$

$\lambda_{21} = r_1 - 2r_1 = -r_1 \Rightarrow stable,\ since\ r_1 > 0, which\ is\ always\ true,\ \lambda_{21} is\ always < 0.$

$\lambda_{22} = r_2 \Rightarrow unstable\ for\ r_2 < 0, which\ is\ never\ true\ as\ r_2 > 0.$

$So\ this\ is\ a\ unstable\ point\ (saddle).$

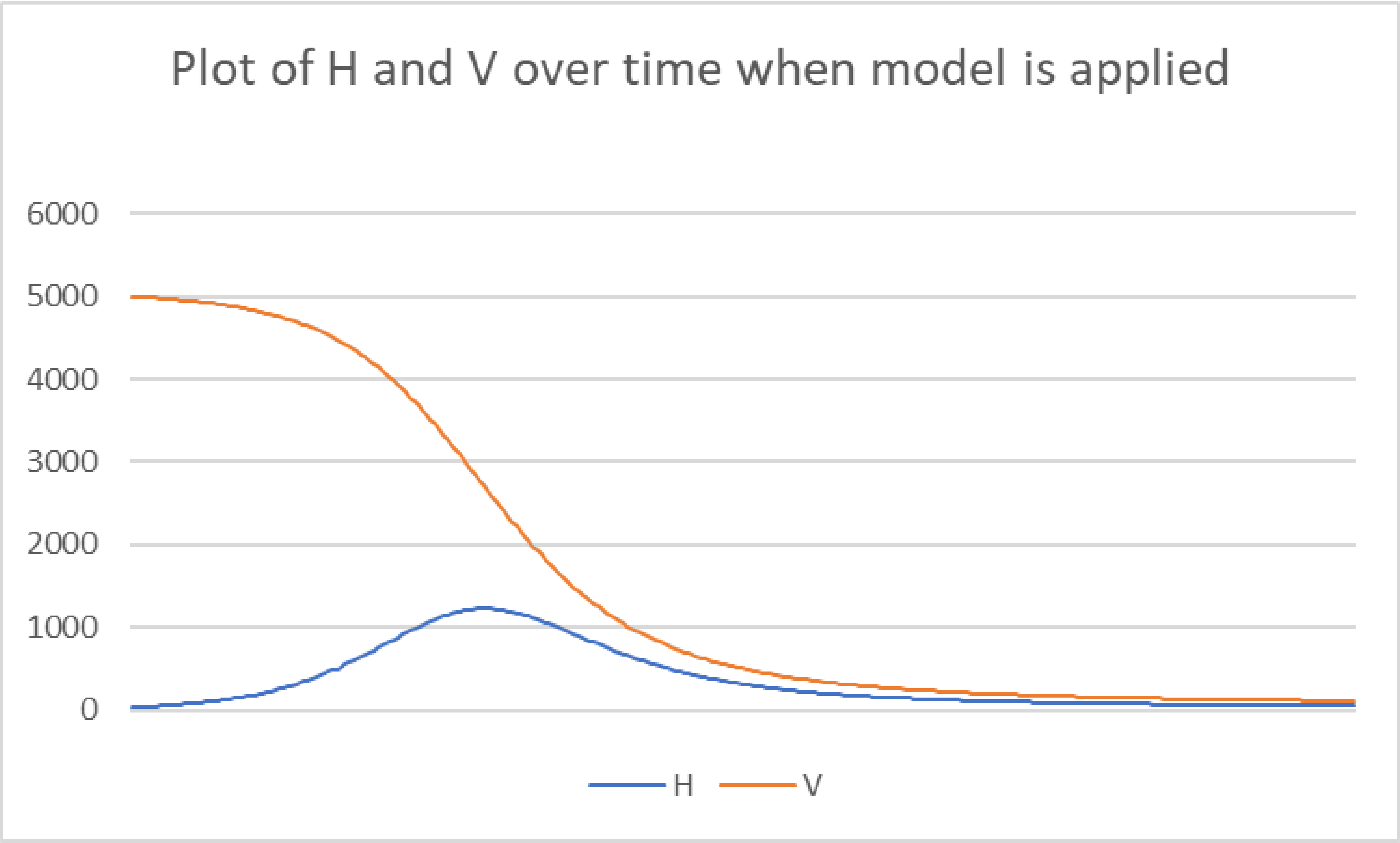
3 $If\ H = 37.838, V = 84.379,\ we\ saw\ from\ the\ phase\ portrait$

$that\ the\ positive\ fixed\ point\ is\ stable\ so\ using\ the\ exstimated\ values$

$\lambda_{31} = -3.2898 + 3.1801i\ and\ \lambda_{32} = -3.2898 - 3.1801i$

$So\ Re(\lambda_{31}) = -3.2898 < 0\ \Rightarrow stable\ and\ Re(\lambda_{32}) = -3.2898 < 0 \Rightarrow stable$

$So\ this\ is\ a\ stable\ point.$


Plot of H and V over time when model is applied
6000
5000
4000
3000
2000
1000
0
H
V

Third Presentation

Present limitations of model

Suggest and justify a modification to the model

Present new model, including mathematical workings, that includes modification

Limitations

The model does not account for the effects that naturally occurring fluctuations in temperature would have on the Moth population, particularly on the birth rate (r_2).*

*Study by Jesusa C. and Benjamin C. Legaspi; Life Table Analysis for Catoblastis cactorum Immatures and Female Adults under Five Constant Tempratures: Implications for Pest Management, *Annals of the Entomological Society of America,* Volume 100, Issue 4, 1 Jul 07, Pages 497-505

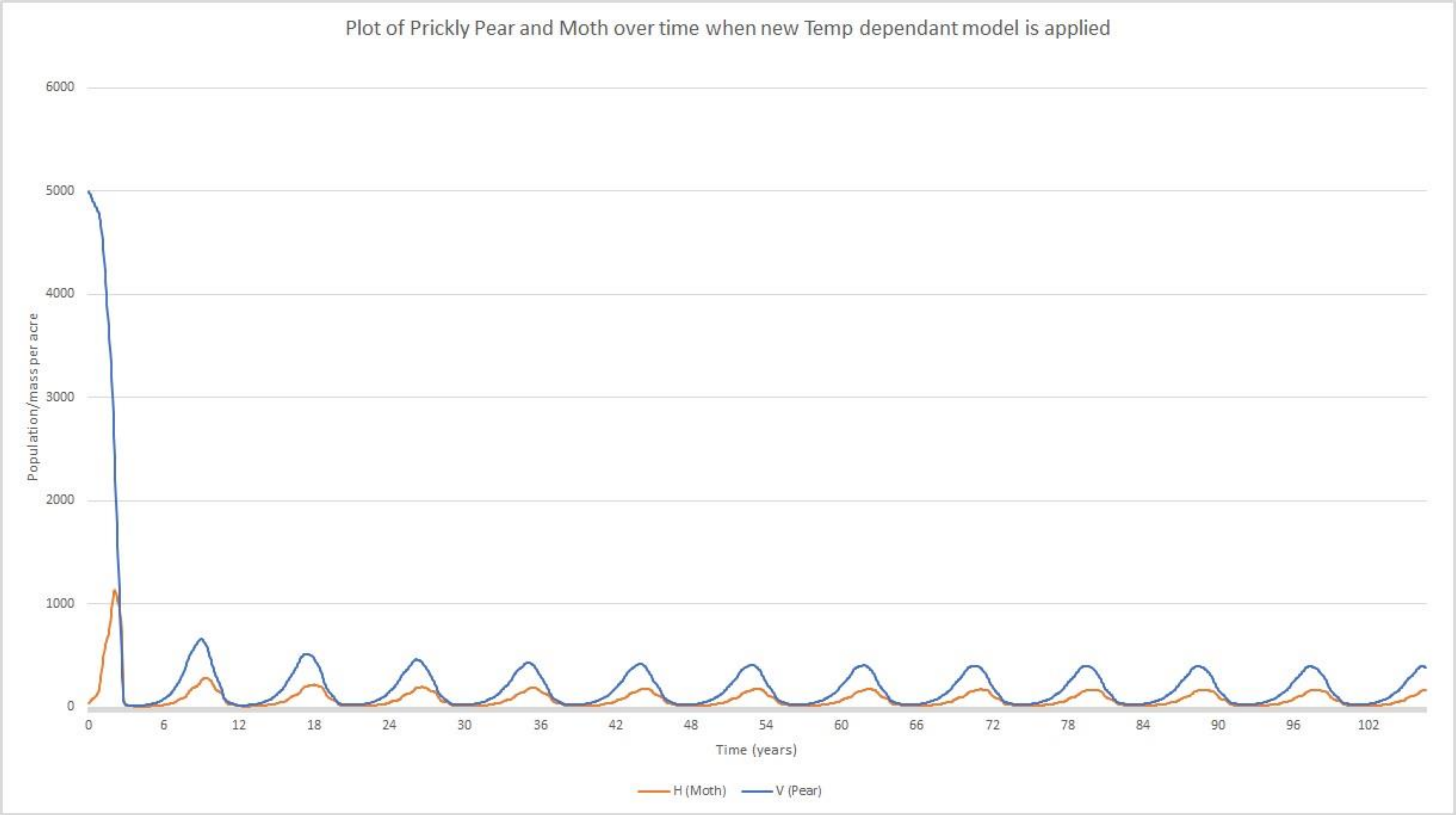
Modification

We will amend the system of equations by including a function of temperature over time (f(T)) to the equation for the change of moth population over time.

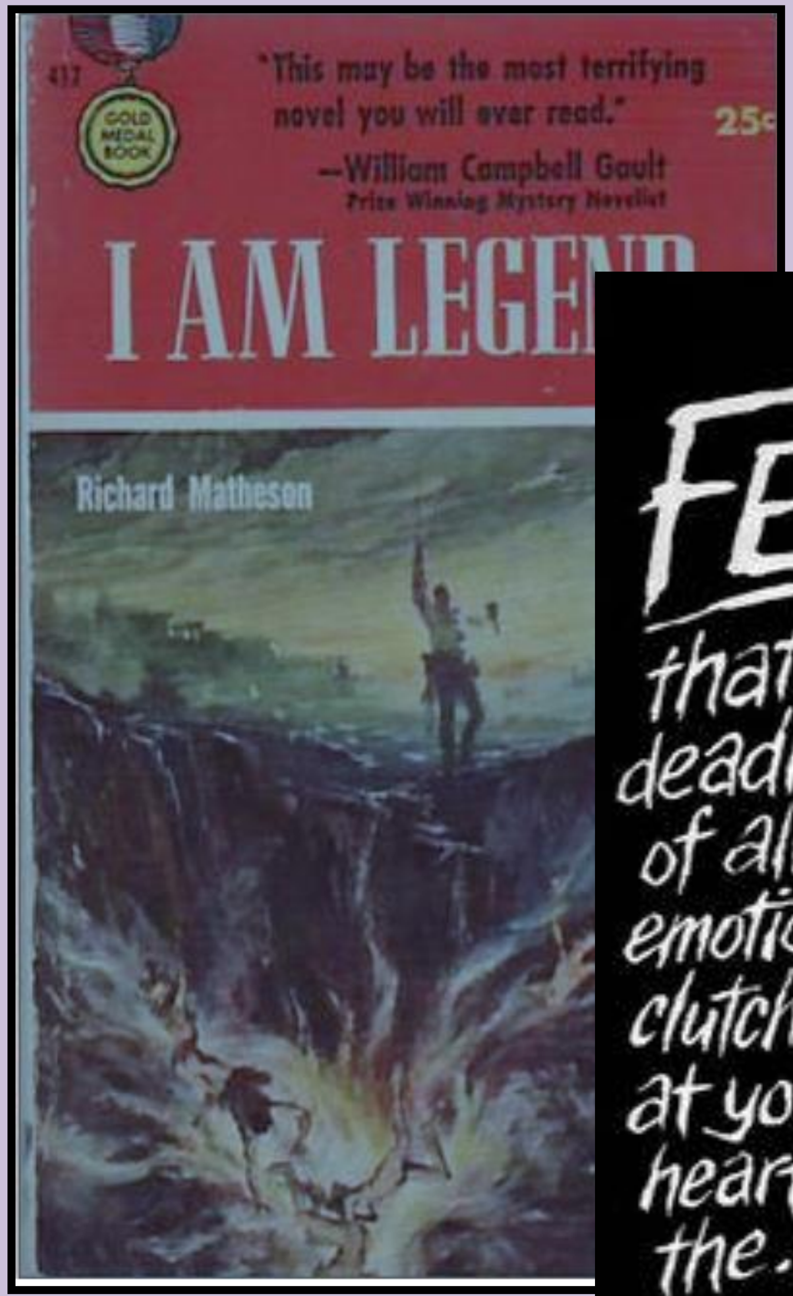
$$f_1 = \frac{dV}{dt} = r_1 V\left(1 - \frac{V}{K}\right) - c_1 H\left(\frac{V}{V + D}\right)$$

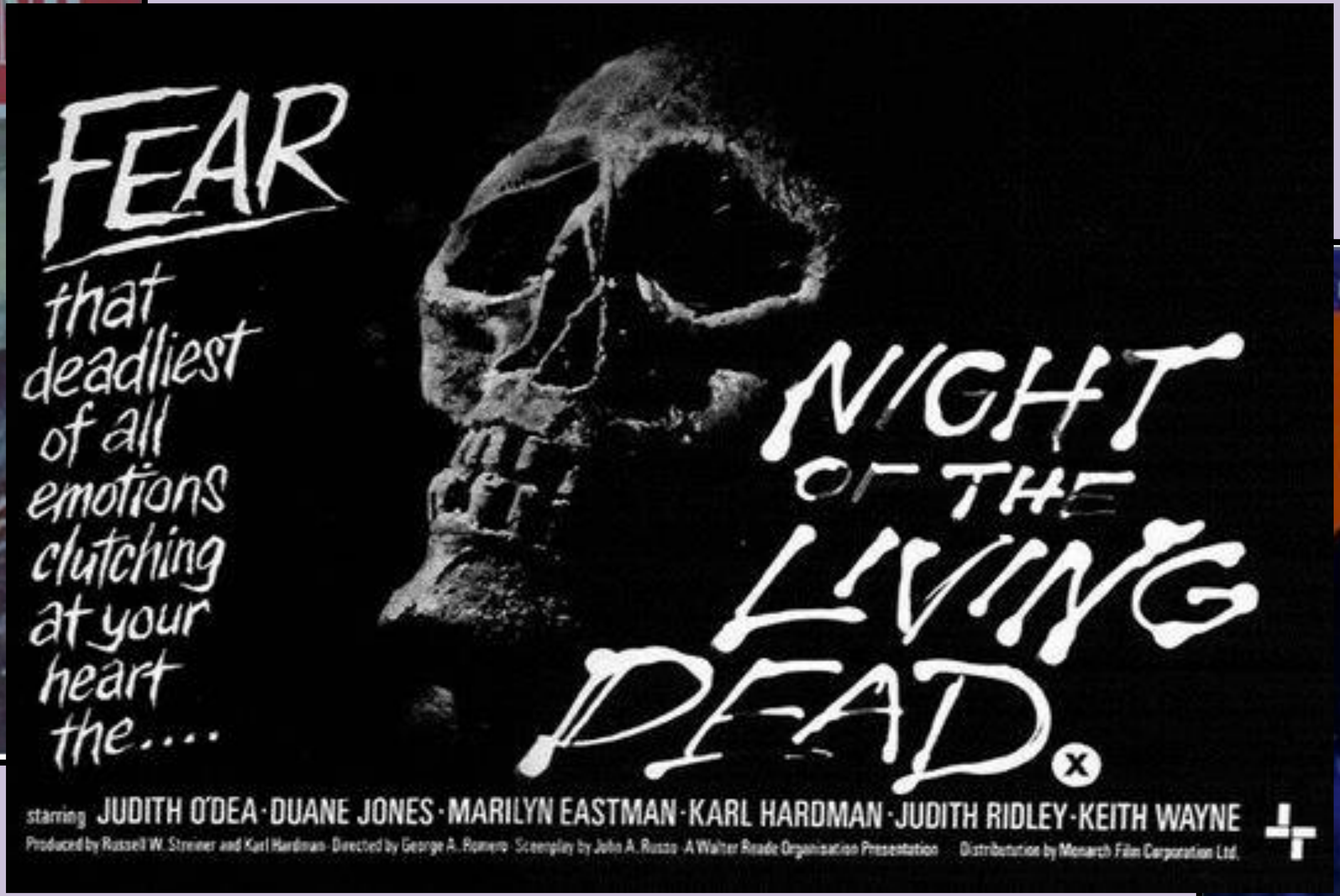
$$f_2 = \frac{dH}{dt} = f(T) r_2 H\left(1 - \frac{JH}{V}\right)$$

$$Model\ of\ temprature\ f(T) = \left\{\begin{matrix} 1 & 299.15\text{K} \leq T \leq 307.15\text{K} \\ \frac{10}{127}T - 22.5551 & T < 299.15\text{K} \\ \frac{-1}{16}T + 20.1969 & T > 307.15\text{K} \end{matrix}\right\}$$


Plot of Prickly Pear and Moth over time when new Temp dependant model is applied
Population/mass per acre
6000
5000
4000
3000
2000
1000
0
0
6
12
18
24
30
36
42
48
54
60
66
72
78
84
90
96
102
Time (years)
H (Moth)
V (Pear)

Course Three: Popular Fiction





Set up around activity to make modelling possible

During lectures and tutorials:

Before the modelling activity:

- Two lessons on S-I-R model and how to manipulate the model.

Acquiring knowledge of the process of modelling:

- First introduced to modelling cycle through lecturer developing simple models using known mathematics.

Modelling experienced through creating models for fictional scenarios.

Set up around activity to make modelling possible

Modelling cycle

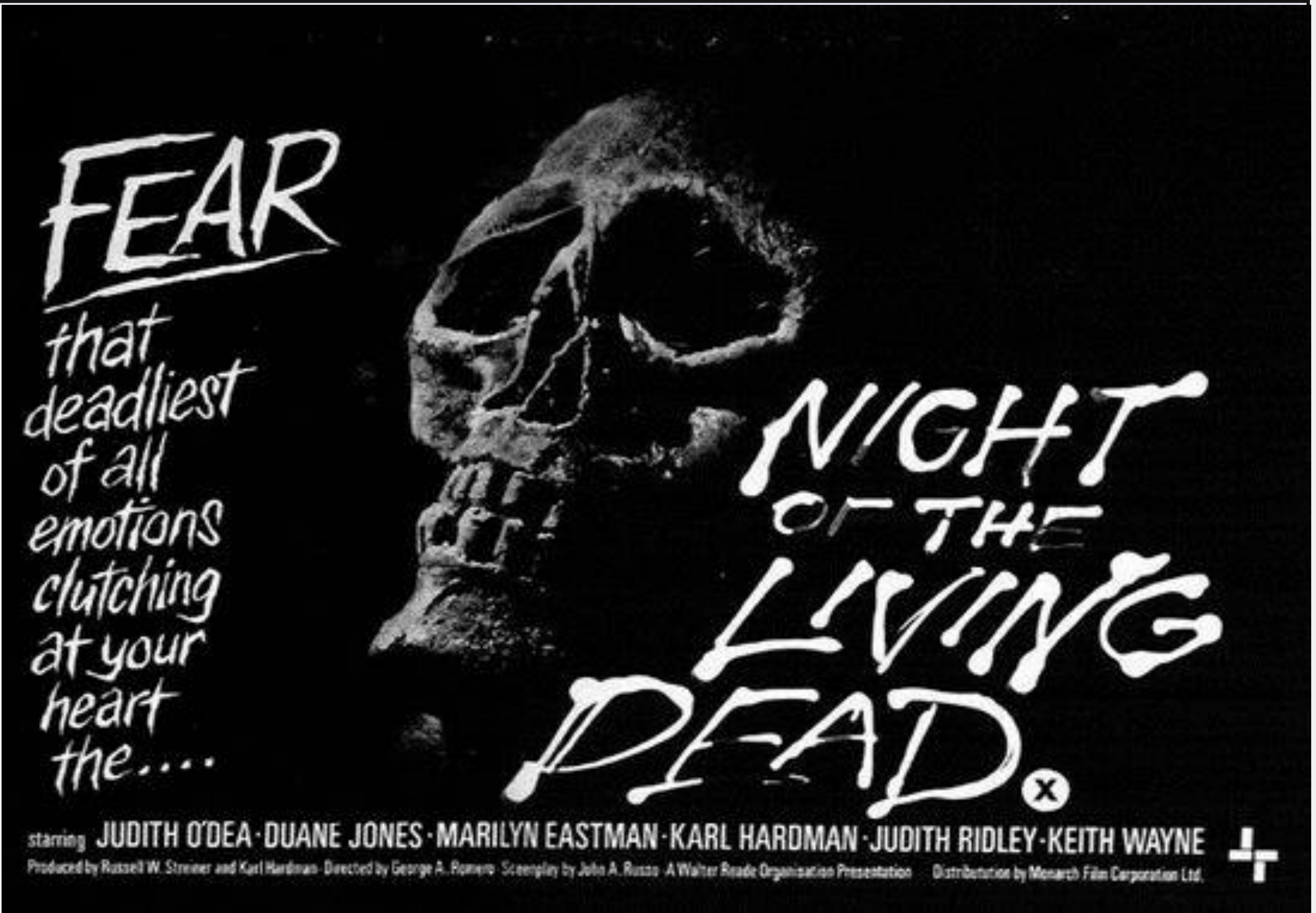
- Learnt standard infectious disease model beforehand in lecture.
- Role modelling of manipulating how to fit model to a different situation.
- Computing based tutorials to explore behaviour of model.

Topics taught included:

Estimation and mathematical modelling, Difference equations and dynamical models, Models of population dynamics, randomness and stochastic models, data fitting and numerical methods, uncertainty quantification.

Groups formed by:

Lecturer handing out different brands of sweets as students enters lecture theatre
Brand A sweets = group A
Brand B sweets = group B etc



A radioactive leak at a nearby military base leads to dead bodies becoming reanimated as zombies. The zombies attack and kill humans, and can only be stopped by decapitation or by burning.

Common for all student groups was to **“take the S-I-R model and change it”** (Leo).

Hazel said **“we had to determine what were the important factors for our situation and incorporate them into the model”.**

Similarly, Leo and Luke both talked about **“identifying the variables and then assigning them to susceptible, infected or recovered”** (Luke).

Both Leo and Luke discussed how they then used their modified models to determine **“how the numbers of susceptible people and infected people and dead, and removed people changed over time”** (Leo).

Luke went on to say **“once we assigned the variables to the S-I-R model, we had to come up with another variable for the people that had been bitten but weren’t infected or didn’t die but they came back as a zombie”**

These student comments illustrate that modifying the S-I-R model for the situation was not straightforward and involved students actively having to think and apply themselves.

Students enjoyed the modelling experience and got a lot out of the experience as Hannah illustrates:

“We got to apply modelling to, I wouldn’t say a real situation, a situation and actually have a go at trying to model something. It made me feel smart. Ours was insanely complicated because we tried to incorporate all the variables, but we did alright. We ended up writing all the differential equations out and everything” (Hannah)

Hannah summarized the student experience of the activity by saying **“we learned about the SIR model and how to manipulate it to fit together”** (Hannah)

Students commented on how participating by watching the presentations allowed them to **“listen and see how others would approach it’** (Luke) commenting on how they **“could learn that way as well”** (Hope).

This is similar to course 2 students’ experience who found watching other students present their work a valuable learning experience.

Insights into common lecturer practices across all three courses

All lecturers designed and delivered their instruction such that it included opportunities for lecturers to *role model mathematical modelling behaviours, provide resources, and promote independence* for students.

All lecturers created opportunities for students to work independently, while also promoting independence as a key modelling behaviour for students to develop.

Thank you

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