Using a Sand Tank Groundwater Model to Investigate a Groundwater Flow Model

Christopher Evrard, Callie Johnson, Michael A. Karls, and Nicole Regnier

Department of Mathematical Sciences
Ball State University

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- This work appears as part of the **Third Special Issue** of the CODEE Journal: *Engaging the World: Differential Equations Can Influence Public Policies*.

One-Dimensional Heat Flow

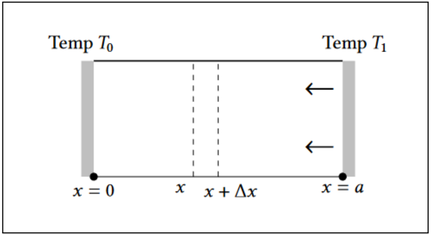

Figure 1: Rod of heat conducting material with insulated lateral surface.

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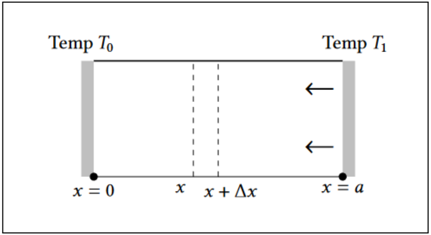

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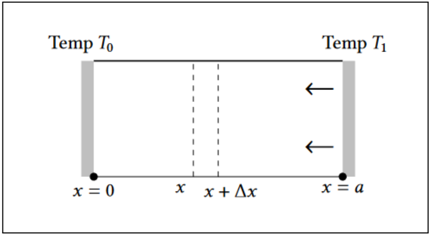

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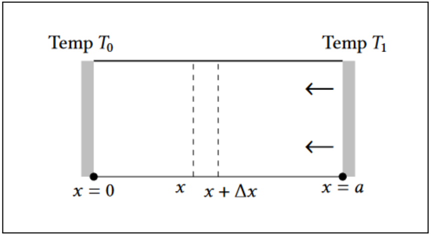
- Consider a rod such as the one depicted in Figure 1.



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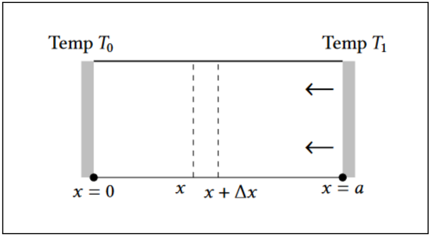

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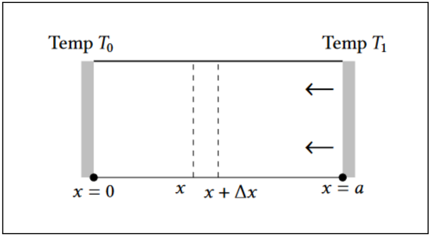

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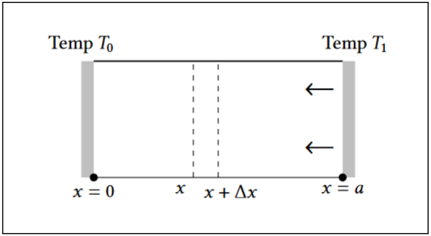

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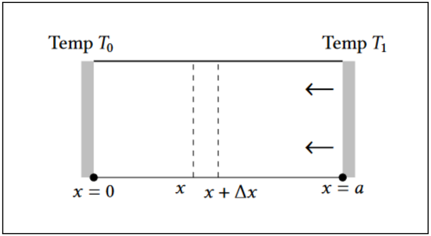
- The diagram in Figure 1 shows heat flowing from right to left through the rod (we are assuming that temperature T_1 is greater than temperature T_0), along with a small representative cross–sectional slice of width Δx and cross sectional area A.



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- Both density ρ and heat capacity c can depend on x and t, but since we are considering a rod made up of homogeneous material, we can assume that these parameters are constant.

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- The minus sign in (4) indicates that heat flows from higher temperatures to lower temperatures.

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- Finally, with *thermal diffusivity* k defined by

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we arrive at the *one–dimensional heat equation*,

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- In some texts, one may also find applications that involve a fixed temperature at one end and the other end insulated (a *no flow condition*).

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- Just as with the heat conducting rod, for applications, we need to specify appropriate boundary conditions and initial condition.
- Instead of fixed temperatures (or insulated boundary) at each end and a known initial temperature distribution, there would be fixed head levels (or a no-flow boundary) and known head levels at each point of the aquifer at a given time.

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Heat flux, $q(x,t)$	Volumetric flux, $q(x,t)$
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Fourier's Law, $q = -\kappa \frac{\partial u}{\partial x}$	Darcy's Law, $q = -K \frac{\partial h}{\partial x}$
Thermal conductivity, κ	Hydraulic conductivity, K
Heat generation, $g(x,t)$	Recharge, $R(x,t)$
Heat storage, $\rho c A \Delta x \frac{\partial u}{\partial t}$	Groundwater storage, $S_s A \Delta x \frac{\Delta h}{\Delta t}$
Density (ρ), Heat capacity (c)	Specific storage (S_s)
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shows that the *volumetric flux* or volumetric flow rate per unit cross-sectional area of the aquifer, q, is proportional to the gradient of the hydraulic head, with proportionality quantity, K, the *hydraulic conductivity*.

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- Using Darcy's Law, (8), and the idea of continuity (conservation of mass), one can use essentially the same argument as above for deriving the one-dimensional heat equation to obtain a partial differential equation that describes hydraulic head level h at any point in an aquifer, at any time, known as the *one-dimensional groundwater flow equation*,

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- As we see, the one-dimensional groundwater flow equation, (9), is the "same" as the one-dimensional heat equation, (7).

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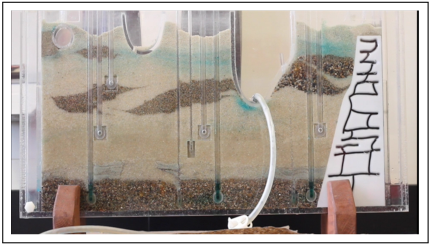

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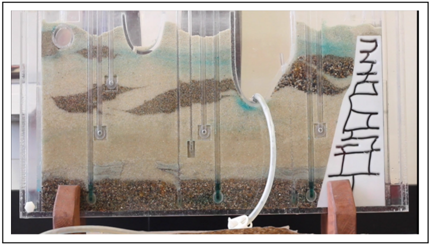

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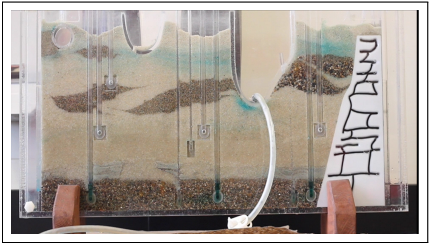

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- ... Through the use of water tinted with food coloring or grape Kool-Aid, it is possible to observe a wide range of groundwater movements."

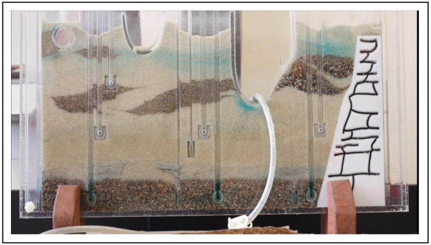

Figure 2: Sand Tank Groundwater Model.

Sand Tank Groundwater Model

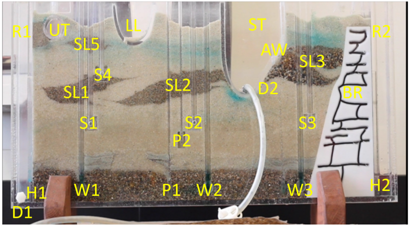

Figure 3: Sand Tank Groundwater Model Components: Underground Tank (UT), Leaky Lagoon (LL), Stream (ST), Bedrock Aquifer, Sand Lenses (SL1 - SL5), Shallow Wells (S1 - S4), Artesian Well (AW), Pumping Wells (P1 - P2), Deep Wells (W1 - W3), Drain Outlets (D1 - D2), Recharge Columns (R1 - R2), Access Holes into Aquifer (H1 - H2).

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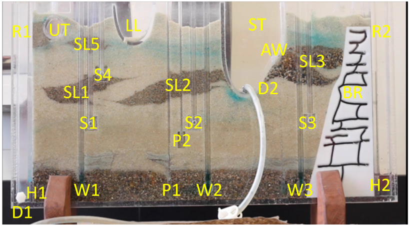

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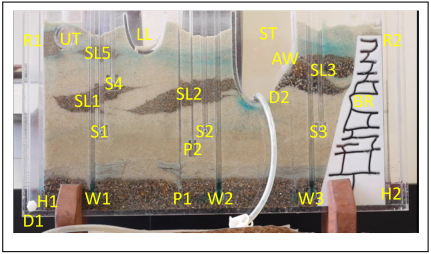
- Sand tanks can also be used in an "unconventional manner" to tie together groundwater flow modeling and more advanced differential equations concepts and solution techniques.
- By leaving a sand tank in unmodified state (i.e. leaving in the drain plug D1 in Figure 3) we were able to physically simulate one-dimensional groundwater flow through an aquifer with a fixed head level at one boundary (right) and no flow at the other boundary (left).



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Sand Tank Groundwater Model

Figure 4: Collecting Well Head Data via Tracker.

Sand Tank Groundwater Model

- We added a drop of green food coloring to each well and used an upside down bottle with stopper and tube to establish a fixed head level and introduce clean water at a fixed rate into the aquifer via the access hole at the base of the right recharge column.

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- Using a video camera, we were able to film the sand tank and use TRACKER software to collect approximately 70 seconds worth of well head data from the resulting video recording.

Figure 4: Collecting Well Head Data via Tracker.

A Model for Our Sand Tank Aquifer

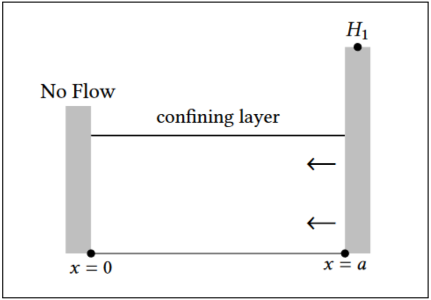

Figure 5: One dimensional confined groundwater flow in our Sand Tank aquifer.

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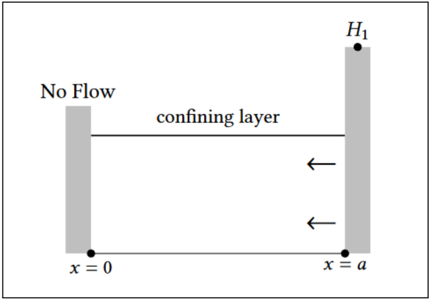

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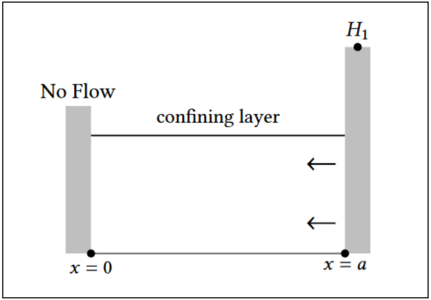

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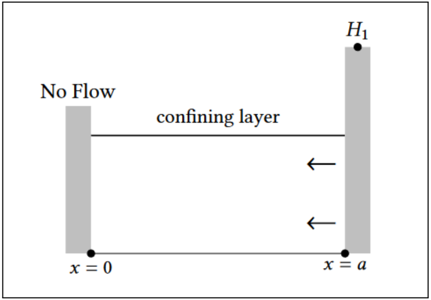

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$$h(a, t) = H_1, \quad \text{for} \quad t > 0. \tag{11}$$

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- To guarantee a solution to the IVBVP (10)–(13), we assume that $f(x)$ is *sectionally smooth* on $[0, a]$, i.e. f has at most a finite number of removable jumps, discontinuities, and corners, with the function and its derivatives being continuous between such points.

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$$f(x) \equiv H_0 = \frac{1.02919 + 1.16587 + 1.39256}{3} = 1.19587 \text{ in}, \tag{17}$$

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- Using graphical, RMSE, and square error comparisons, we find that a partial sum with 50 terms in our model should be sufficient.

Model Coefficients and Parameters

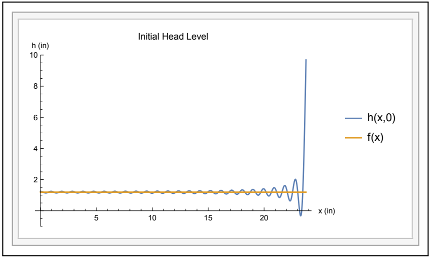
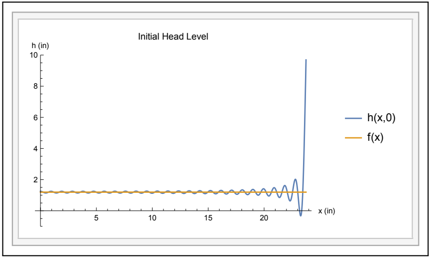


Figure 6: Initial Head Level.

Model Coefficients and Parameters

- Figure 6 compares our model at time $t = 0$ sec, $h(x, 0)$, to the initial head level $f(x)$ on the interval $0 < x < a$.



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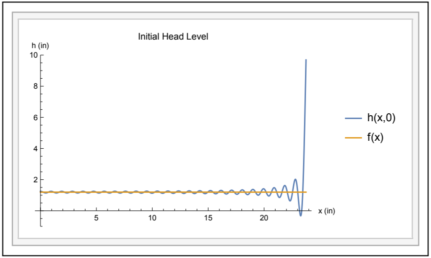

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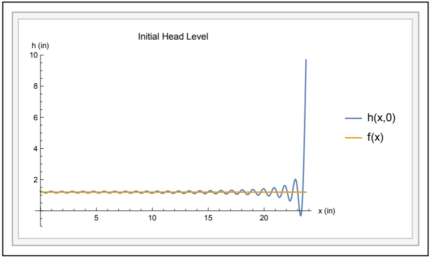
- We also compute RMSE between model and measured data for $m = 1$ to 100 terms in the model to see how different numbers of terms impact the choice of k and resulting error.



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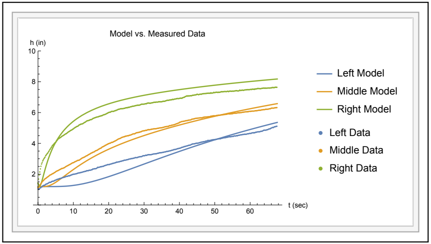
Comparison of Model to Measured Data

- What we find is that the resulting RMSE and choice of k are essentially the same for most choices of m, with the smallest error occuring with $m = 11$.



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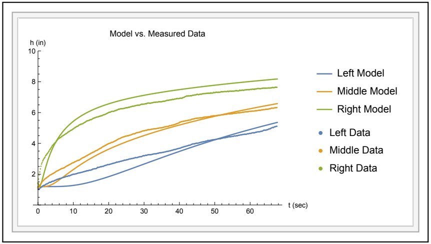

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- Figure 7 shows that with $k = 3.01356\ \text{in}^2/\text{sec}$ and an RMSE of 0.451039 inches, our model overestimates head level at the right well and underestimates the head level at the left well.

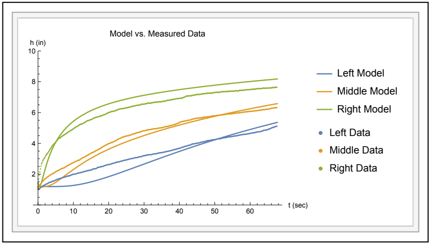


Figure 7: Comparison of Model to Measured Data.

First Revision: Variable Head Level at Right Boundary

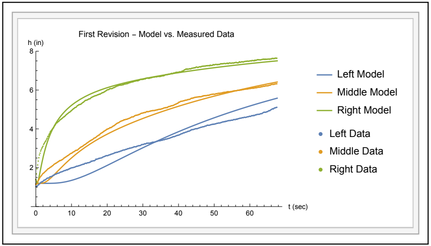
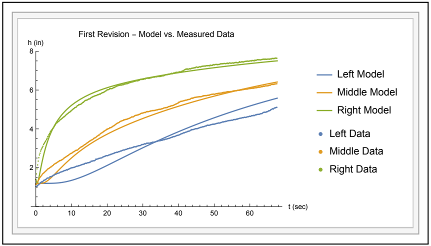


Figure 8: First Revision: Variable Head Level at Right Boundary.

First Revision: Variable Head Level at Right Boundary

- One way to take into consideration the discrepancy between the model and measured data at the right well is to treat fixed head level H_1 at $x = a$, as an unknown parameter to be determined.



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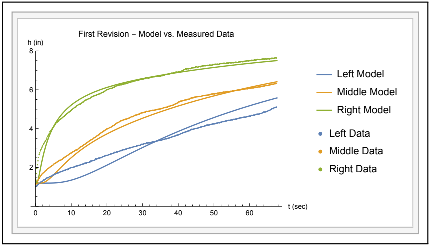
- Again, starting with choices of k and H_1 to get a good graphical fit, followed by an application of Mathematica's FindMinimum command, we obtain an RMSE of 0.297491 in with $k = 3.81486$ in^2/sec and $H_1 = 8.53077$ in.



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- It is clear from Figure 8 that we get a much better fit for the right well, about the same for the middle well, but still not a great match at the left well.

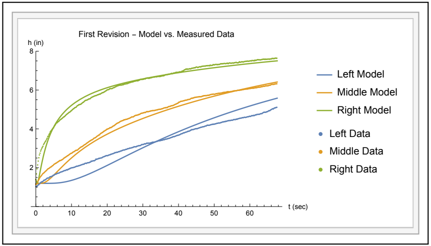


Figure 8: First Revision: Variable Head Level at Right Boundary.

Second Revision: Variable Head Level at Right Boundary and Variable Initial Head Level

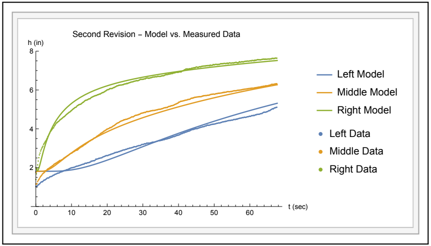
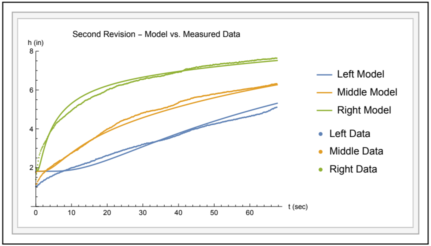


Figure 9: Second Revision: Variable Head Level at Right Boundary and Variable Initial Head Level.

Second Revision: Variable Head Level at Right Boundary and Variable Initial Head Level

- To see if we can get a better fit at the left well, we make another model revision by also treating the initial head level as another unknown parameter H_0 to be determined along with parameters k and H_1 to minimize RMSE.



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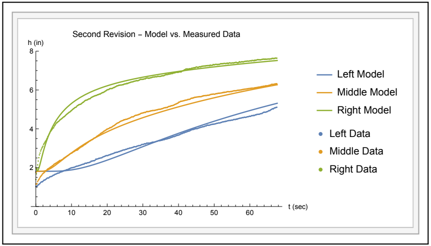

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- For this revision, RMSE is reduced to 0.196507 in.

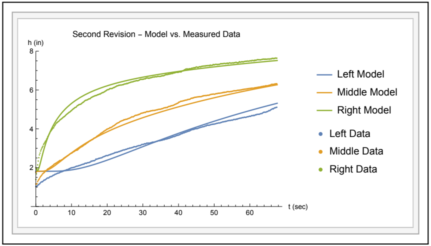

Figure 9: Second Revision: Variable Head Level at Right Boundary and Variable Initial Head Level.

Third Revision: Variable Head Level at Right Boundary, Variable Initial Head Level, and Recharge Term

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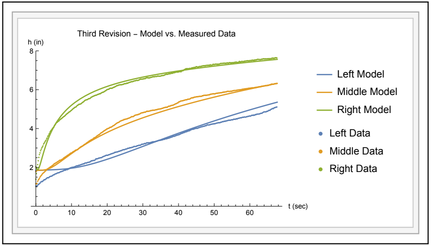
with

$$\begin{aligned} b_n &= \frac{2}{a}\int_0^a \left(H_0 - \frac{a^2 R + 2H_1 K - Rx^2}{2K}\right)\cos\left(\frac{(2n-1)\pi}{2a}x\right)\, dx \\ &= \frac{2(-1)^n\left(8a^2 R + 2K(\pi - 2\pi n)^2(H_1 - H_0)\right)}{\pi^3 K(2n-1)^3} \end{aligned} \quad (20)$$

Third Revision: Variable Head Level at Right Boundary, Variable Initial Head Level, and Recharge Term



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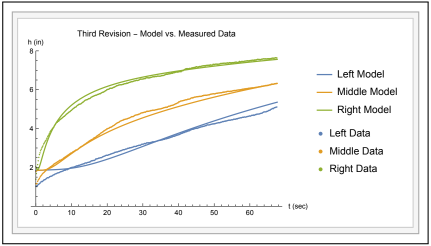

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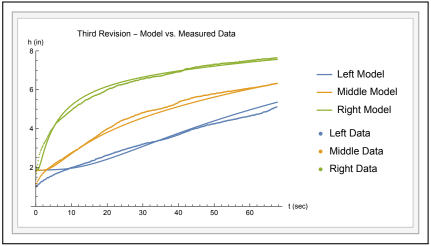
- Figure 10 reinforces this, as the graphs are nearly indistinguishable from those in Figure 9.



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- Using our model, we can answer the following questions:
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  3. How do these model time estimates, if any, compare to the actual time needed for the contaminated groundwater to reach the drinking wells?

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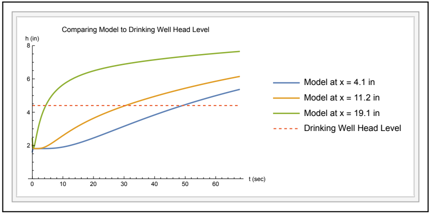

Figure 11: Comparing Model to Drinking Well Head Level.

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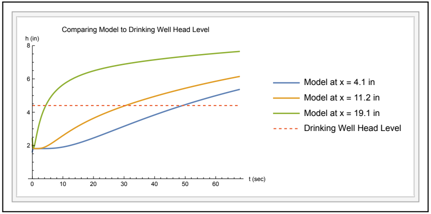
- Plotting our model head levels at each drinking well location, we see from Figure 11 that the model predicts that the contaminated groundwater will reach the left, middle, and right drinking wells at approximately $t = 48$ sec, $t = 30$ sec, and $t = 4$ sec, respectively.



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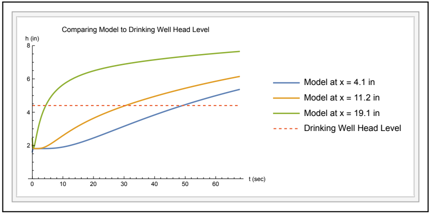
- Using Mathematica's FINDROOT command with our model, we can get more accurate numerical estimates for these times, namely $t = 49.1323$ sec, $t = 30.4799$ sec, and $t = 4.39004$ sec, at the left, middle, and right drinking water wells, respectively.



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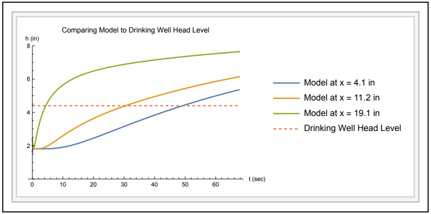

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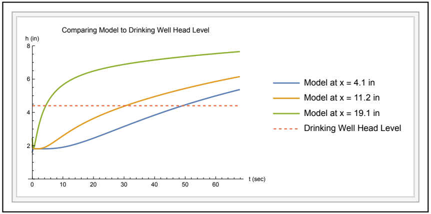

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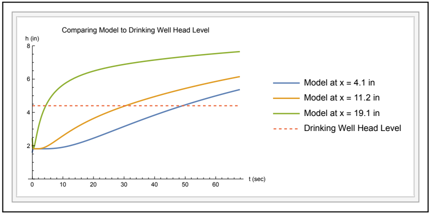

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- The results obtained are surprisingly good (to us), especially considering the fact that the aquifer we worked with does not have uniform material throughout.

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- This would lead to a much more complicated model that is beyond the scope of this investigation.

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- Projects such as the one outlined in this paper can be used for this purpose.

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