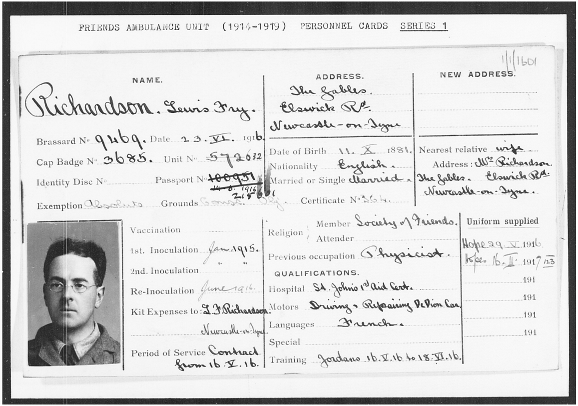
Using An Arms Race Model To Illustrate Key Concepts and Techniques in a First Differential Equations Course

Michael Olinick
Middlebury College
molinick@middlebury.edu

SIMIODE EXPO 2024
February 9, 2024

What Causes Nations To Wage War?



FRIENDS AMBULANCE UNIT (1914–1919) PERSONNEL CARDS SERIES 1

NAME. Richardson. Lewis Fry.

ADDRESS. The Gables. Elswick Rd. Newcastle-on-Tyne

NEW ADDRESS.

Brassard No 9469. Date 23.VI. 1916.

Cap Badge No 3685. Unit No 572632

Identity Disc No ..... Passport No 100951 14-8-1916 215661

Exemption Absolute Grounds Consc. Obj.

Date of Birth 11. X. 1881. Nationality English. Married or Single Married. Certificate No 364.

Nearest relative wife. Address: Mrs Richardson. The Gables. Elswick Rd. Newcastle-on-Tyne.

Vaccination

1st. Inoculation Jan. 1915.

2nd. Inoculation " "

Re-Inoculation June 1916.

Kit Expenses to: L. F. Richardson. Newcastle-on-Tyne.

Period of Service Contract from 16.V.16.

Religion: Member Society of Friends. Attender

Previous occupation Physicist.

QUALIFICATIONS.

Hospital St. John's 1st Aid Cert.

Motors Driving & Repairing De Dion Car.

Languages French.

Special

Training Jordans 16.V.16 to 18.VI.16.

Uniform supplied

Hope 29. V. 1916

Hopes 16. II. 1917

191

191

191

191

What Causes Nations To Wage War?

History shows that the existence of large military arsenals increases the likelihood of violent conflict.

Without destructive weapons, nations might sometimes settle disputes by other means.

If nations were increasing their expenditures on defense budgets then a small spark could start a major conflagration.

If two nations were decreasing their defense budgets, then a small incident might not trigger a war.

Ultimately, Richardson wanted to build a model to examine certain conditions in order to predict whether an arms race was "stable" or "unstable"

The Richardson Arms Race Model describes the dynamics of arms expenditures of two nations, each of which is driven to increase its spending by the spending level of its rival, but faces internal resistance as its own expenditures grow and experiences external drivers of good or ill will.

Lewis Fry Richardson
1881 – 1953
Arms and Insecurity, 1949

Mutual Fear

$$x' = ay$$

$$y' = bx$$

$$a > 0, \quad b > 0$$

Mutual Fear and Internal Constraints

$$x' = ay - mx$$

$$y' = bx - ny$$

$$a,\ m > 0, \quad b,\ n > 0$$

Mutual Fear, Internal Constraints, Underlying Good/ill Will

$$x' = ay \;\; mx + r$$

$$y' = bx - ny + s$$

$$a, m > 0, \;\; b, n > 0$$

r , s Constant

The original formulation of the model is

x' = ay - mx + r

y' = bx - ny + s

where

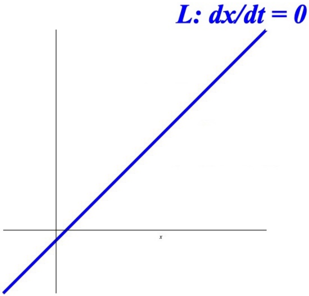
- x and y are arms expenditures as functions of time,
- a, b, m, n are positive constants and
- r, s are constants of arbitrary sign;
- differentiation is with respect to time.

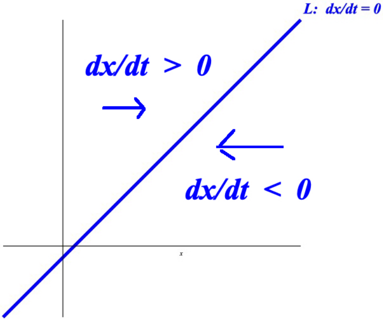
Phase plane analysis of the model early in a differential equations course reveals the possibilities of sinks, sources, and saddle points.

Prerequisite: Derivative is a Rate of Change

Positive Derivative Implies Quantity Increasing
Negative Derivative Implies Quantity Decreasing

$$dx/dt = ay - mx + r$$

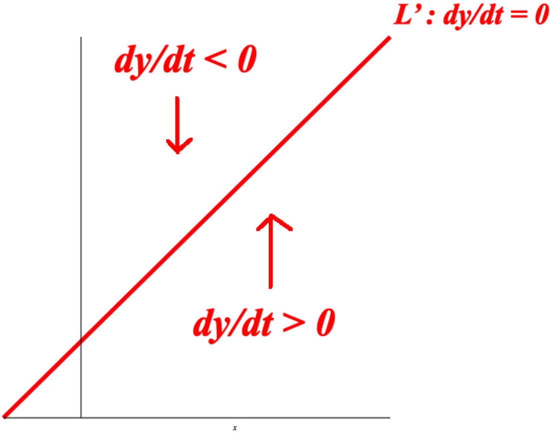


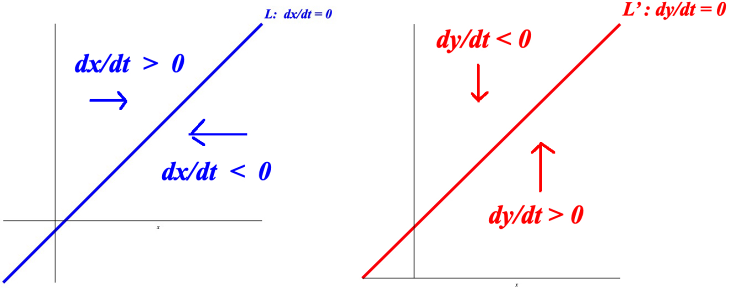


Examine Other Nullcline:

$$dy/dt = bx - ny + s$$

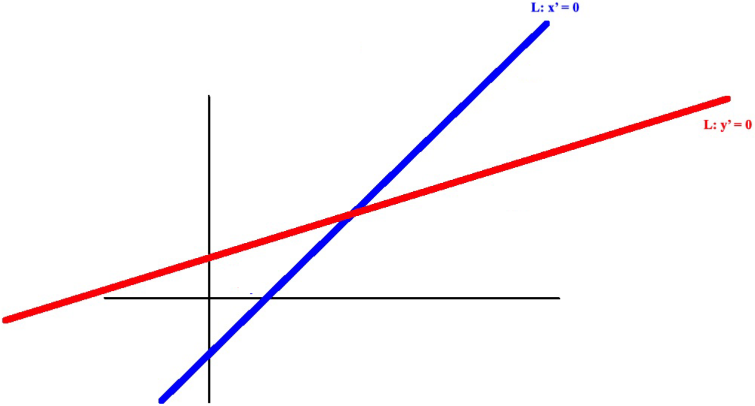
$$L' : dy/dt = 0 \implies bx - ny + s = 0$$



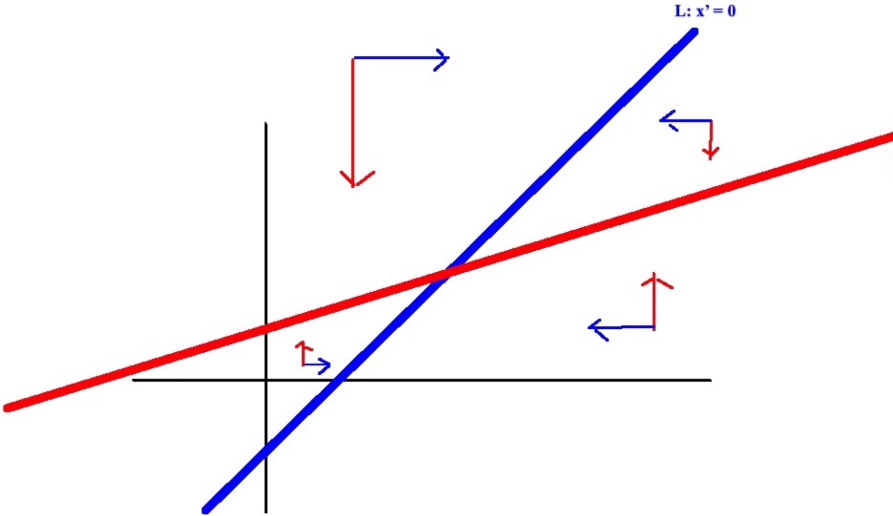


Each Nation Tries To Move Back Toward Its Stable Line

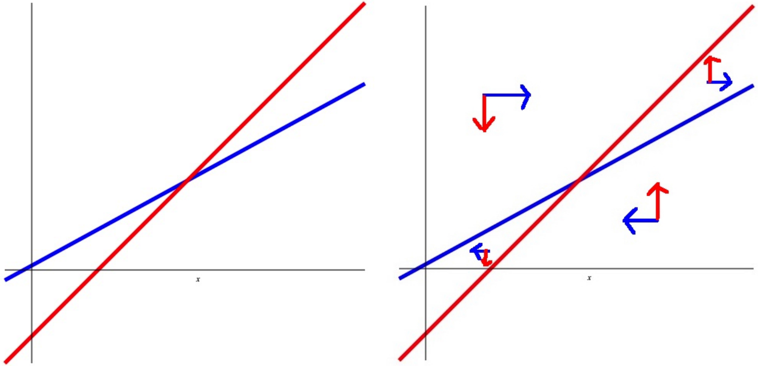
Combine Nullclines

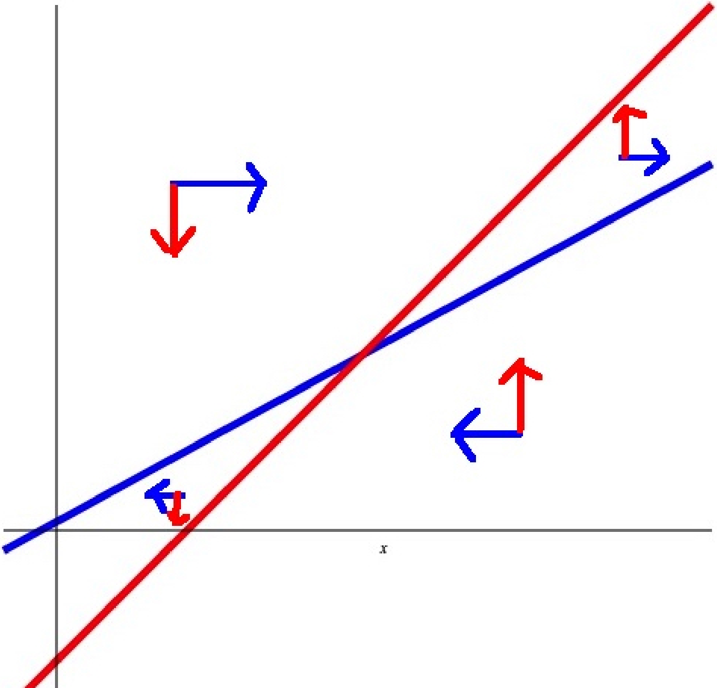


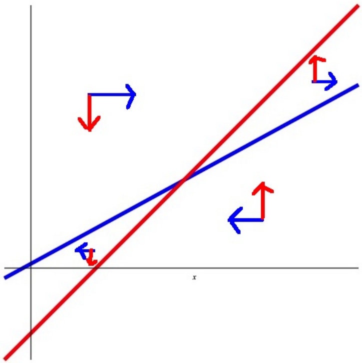
A Stable Arms Race



An Unstable Arms Race

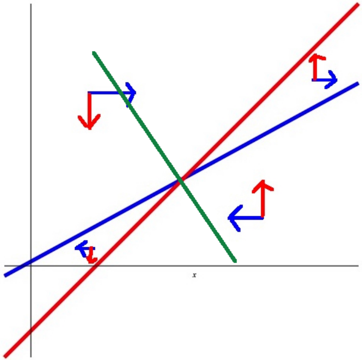



x



Outcome Depends on Initial Conditions

Is There a Dividing Line?

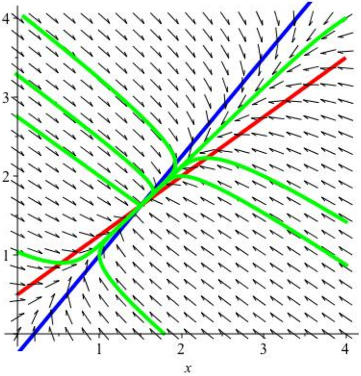


Is There a Dividing Line?

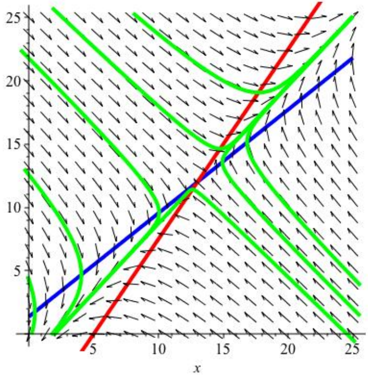
$$\frac{dy}{dx} = \frac{y - y_s}{x - x_s}$$

$$\frac{bx - ny + s}{ay - mx + r} = \frac{y - y_s}{x - x_s}$$

$$ay^2 + (-x_s n - y_s a - mx + nx + r)y + x_s bx + y_s mx - bx^2 + x_s s - y_s r - xs = 0$$

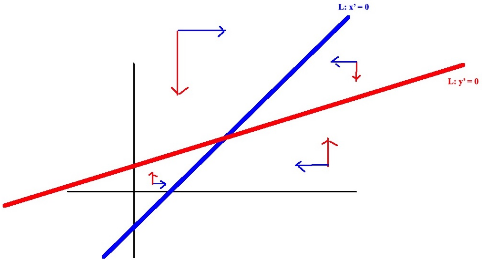


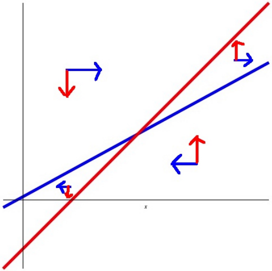
Stable Arms Race



Unstable Arms Race

What's The Difference Between Stable and Unstable Arms Race?





Stable: Slope of $L >$ Slope of L'

$$\frac{m}{a} > \frac{b}{n} \text{ so } mn > ab \text{ or } ab - mn < 0$$

Identify Trajectory for Pure Mutual Fear Model

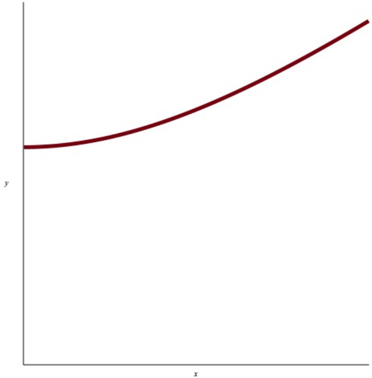
dx/dt = ay, dy/dt = bx

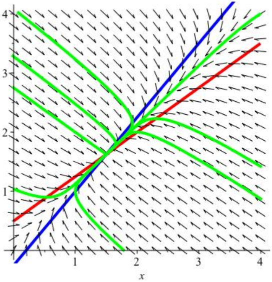
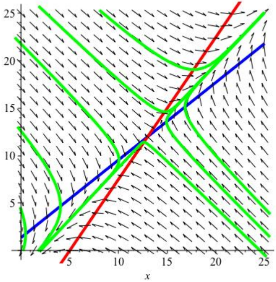
Use Chaiin Rule to obtain:

$$\frac{dy}{dx} = \frac{bx}{ay}$$

Variables Separate

$$ay^2 = bx^2 + C$$



$x' = -5x + 4y + 1$ $y' = 3x - 4y + 2$	$x' = 11y - 9x - 15$ $y' = 12x - 8y - 60$
	

Projects

- Investigate Other Configurations of Nullclines
- Upper Limits on Arms Expenditures

$$dx/dt = \left(1 - \frac{x}{x_M}\right)(ay - mx + r)$$

$$dy/dt = \left(1 - \frac{y}{y_M}\right)(bx - ny + s)$$

- Extensions to Three Nations
- Lanchester Combat Models

Middle of the Term

Students can find exact solutions using eigenvalues and eigenvectors.

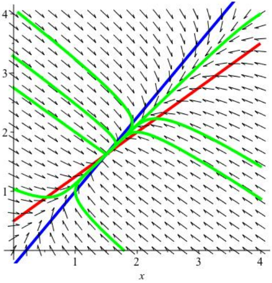
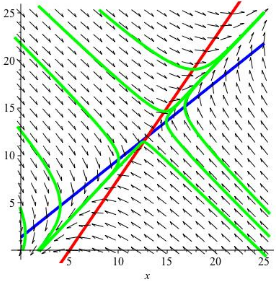
$$dx/dt = -mx + ay + r, dy/dt = bx - ny + s$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -m & a \\ b & -n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix}$$

Characteristic Polynomial: $\lambda^2 + (m+n)\lambda + (mn - ab)$

Eigenvalues: $\lambda = \dfrac{-(m+n) \pm \sqrt{4ab + (m-n)^2}}{2}$

Both Eigenvalues are Real
At Least One is Negative.

$x' = -5x + 4y + 1$ $y' = 3x - 4y + 2$	$x' = 11y - 9x - 15$ $y' = 12x - 8y - 60$
	
$x = Ae^{-8t} + Be^{-t} + \frac{3}{2}$ $y = -A\frac{3}{4}e^{-8t} + Be^{-t} + \frac{13}{8}$	$x = Ae^{-20t} + Be^{3t} + 13$ $y = -Ae^{-20t} + \frac{12}{11}Be^{3t} + 12$

Replace Constants By Continuous Functions of Time

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -m & a \\ b & -n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$$

Variation of Parameters

$$\mathbf{x} = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t) \int_{t_0}^{t} \mathbf{X}^{-1}(s)\, \mathbf{f}(s)\, ds$$

where $\mathbf{X} = \begin{pmatrix} e^{\lambda_1 t}\mathbf{v_1} & e^{\lambda_2 t}\mathbf{v_2} \end{pmatrix}$

or

$$\mathbf{x} = e^{At}\mathbf{c} + e^{At} \int_{t_0}^{t} e^{-As}\, \mathbf{f}(s)\, ds$$

More Generalized Linear Systems

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -m(t) & a(t) \\ b(t) & -n(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -m(x,y) & a(x,y) \\ b(x,y) & -n(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

Late in the Semester

Students can explore more realistic models by

- replacing the constant parameters with their own choices of continuous functions,
- exploring nonlinear representations,and
- applying numerical solution methods to simulate the behavior.

Delay differential equations can replace the ODEs as neither nation may be able to change its arms budget instantaneously in response to its opponent.

$$\mathbf{x'(t) = ay(t\text{-}D) - mx(t\text{-}D) + r}$$

$$\mathbf{y'(t) = bx(t\text{-}D) - ny(t\text{-}D) + s}$$

Conclusion

A relatively simple mathematical model meant to help understand an important real world problem can also motivate the consideration of many central topics in an introductory differential equations course.