Can a positive density dependent dispersal counteract an Allee effect?

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Overview

- Modeling Framework
- Model of Interest
- Main Theorem
- Computational Results
- Stability Results

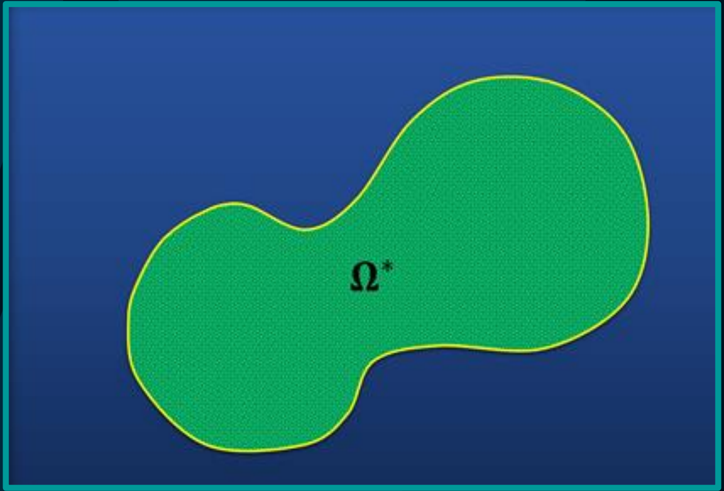
Modelling Framework

A population exhibiting a certain growth and density dependent dispersal (DDD) on the boundary.

$$\begin{cases} v_t = D\Delta v + rf(v); \Omega^*, t > 0 \\ \tilde{\alpha}(v)\dfrac{\partial v}{\partial \eta} + \sqrt{\dfrac{S_0 D_0}{\kappa}}[1 - \tilde{\alpha}(v)]v = 0; \partial\Omega^*, t > 0 \\ v(0, x) = v_0(x); \Omega^* \end{cases}$$

$\tilde{\alpha}(v) = \textit{Probability of the population staying in } \Omega^* \textit{ upon reaching boundary.}$

r = Patch intrinsic growth rate
D = Diffusion rate in the domain
D_0 = Diffusion rate in the matrix
S_0 = Death rate in the matrix
κ = Depends on the patch matrix interface

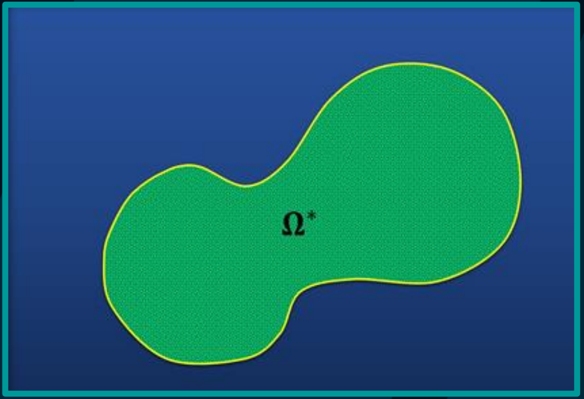


The domain

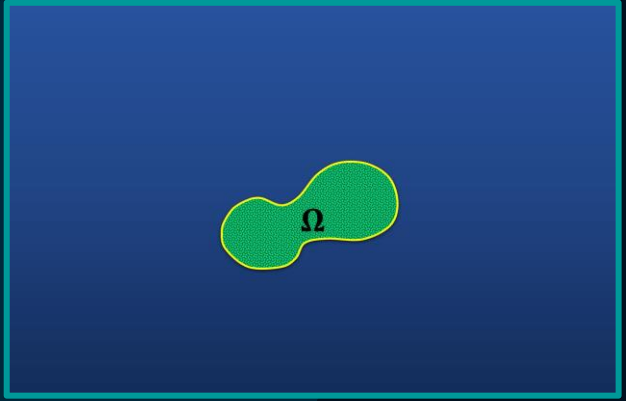
$\Omega^* = \{lx : x \in \Omega\}$ **where** $|\Omega| = 1$

⟹ **Patch size (l) + Geometry (Ω)**

$$\begin{cases} u_t = \dfrac{D}{l^2}\Delta u + rf(u); \Omega \\ \dfrac{D}{l}\alpha(u)\dfrac{\partial u}{\partial \eta} + \sqrt{\dfrac{S_0 D_0}{\kappa}}[1-\alpha(u)]u = 0; \partial\Omega \end{cases}$$



The original domain



The scaled domain

$\lambda = \dfrac{rl^2}{D}$ **and** $\gamma = \dfrac{1}{\kappa}\sqrt{\dfrac{S_0 D_0}{Dr}}$

$$g(u) = \frac{[1-\alpha(u)]}{\alpha(u)}$$

$$\begin{cases} -\Delta u = \lambda f(u); \Omega \\ \dfrac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}g(u)u = 0; \partial\Omega \end{cases}$$

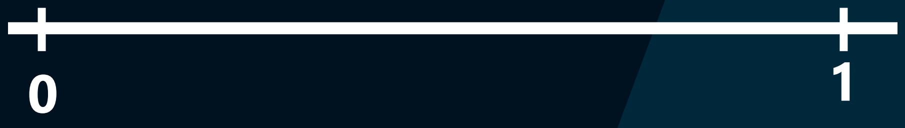
J. Cronin, J. Goddard, and R. Shivaji, Effects of patch-matrix composition and individual movement response on population persistence at the patch-level, Bull. Math. Biol., 81 (2019), no. 10, 3933–3975.

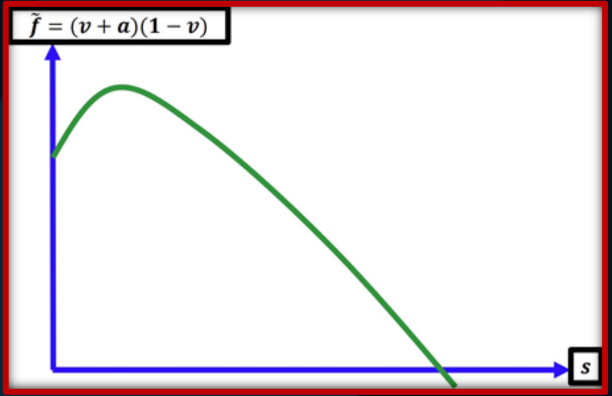
Model of Interest

(1)

$$\begin{cases} -u'' = \lambda \frac{1}{a} u(u+a)(1-u) = \lambda f(u); (0,1) \\ -u'(0) + \gamma\sqrt{\lambda} g(u(0)) u(0) = 0 \\ u'(1) + \gamma\sqrt{\lambda} g(u(1)) u(1) = 0 \end{cases}$$

$a \in (0,1)$





Per-capita growth

$$f(u) = u\tilde{f}(u)$$

Allee Effect Growth Term

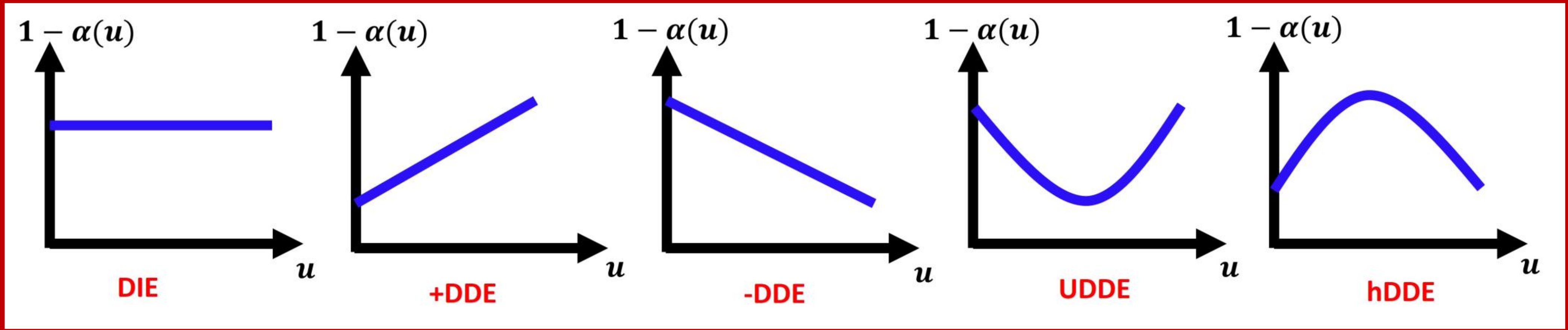
$$g(u) = \frac{[1-\alpha(u)]}{\alpha(u)}$$

$$1 - \alpha(u) = Dispersal\ (Emigration\ probability)$$

Density Dependent Emigration (DDE)

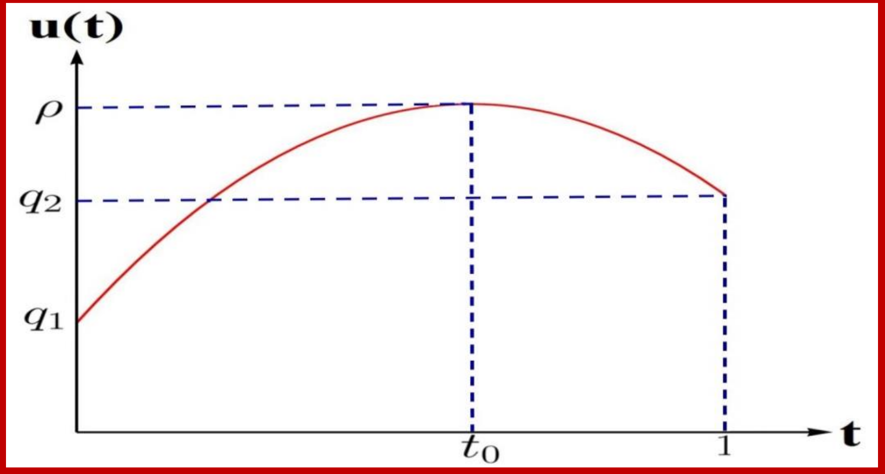
Emigration Types

DDE Form	$\alpha(u)$	$1-\alpha(u)$	Restrictions
DIE	$\alpha(u) \equiv 0.5$	$1-\alpha(u) \equiv 0.5$	none
+DDE	$\alpha(u) = \frac{M_1}{2M_1+u^2}$	$1-\alpha(u) = \frac{M_1+u^2}{2M_1+u^2}$	none
-DDE	$\alpha(u) = \frac{M_1+u^2}{2M_1+u^2}$	$1-\alpha(u) = \frac{M_1}{2M_1+u^2}$	none
UDDE	$\alpha(u) = \frac{M_1}{2M_1+u^2-2M_2u}$	$1-\alpha(u) = \frac{M_1+u^2-2M_2u}{2M_1+u^2-2M_2u}$	$M_1 > M_2^2$
hDDE	$\alpha(u) = \frac{M_1+u^2-2M_2u}{2M_1+u^2-2M_2u}$	$1-\alpha(u) = \frac{M_1}{2M_1+u^2-2M_2u}$	$M_1 > M_2^2$



Main Theorem

① $\begin{cases} -u'' = \lambda f(u); (0,1) \\ -u'(0) + \gamma\sqrt{\lambda}g(u(0))u(0) = 0 \\ u'(1) + \gamma\sqrt{\lambda}g(u(1))u(1) = 0 \end{cases}$



Theorem 1: ① **has a positive solution u such that $u(t_0) = \|u\|_\infty = \rho, u(0) = q_1, u(1) = q_2$, with $0 < q_1, q_2 < \rho$ iff λ, ρ, q_1, and q_2 satisfy:**

- $\lambda = \frac{1}{2}\left(\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right)^2$ (*)
- $2[F(\rho) - F(q_1)] = \gamma^2 q_1^2 [g(q_1)]^2$
- $2[F(\rho) - F(q_2)] = \gamma^2 q_2^2 [g(q_2)]^2$ (**)

$$t_0 = \frac{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}}{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}}$$

$$F(s) = \int_0^s \frac{1}{a} z(z+a)(1-z)dz$$

Let u be a solution of (1) such that $u(t_0) = \|u\|_\infty = \rho, u(0) = q_1, u(1) = q_2$

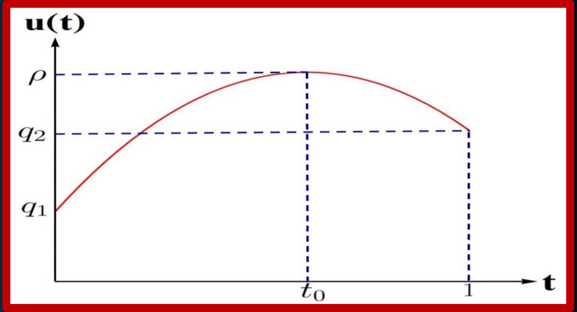
Multiplying both side of (1) by u' and integrating

$$(1)\quad \begin{cases} -u'' = \lambda f(u); (0,1) \\ -u'(0) + \gamma\sqrt{\lambda} g(u(0))u(0) = 0 \\ u'(1) + \gamma\sqrt{\lambda} g(u(1))u(1) = 0 \end{cases}$$

$$-u''u' = \lambda f(u)u' \longrightarrow -\frac{(u')^2}{2} = \lambda F(u) + C$$

$$F(s) = \int_0^s \frac{1}{a} z(z+a)(1-z)dz$$

$$u(t_0) = \rho \longrightarrow C = \lambda F(\rho)$$



$$\longrightarrow \quad u'(t) = \sqrt{2\lambda}\sqrt{[F(\rho) - \lambda F(u(t))]}\ ;\ t \in (0, t_0)$$

$$u'(t) = -\sqrt{2\lambda}\sqrt{[F(\rho) - \lambda F(u(t))]}\ ;\ t \in (t_0, 1)$$

Further integrating

$$\int_0^x \frac{u'(s)}{\sqrt{F(\rho) - F(u(s))}} ds = \sqrt{2\lambda} x; \quad x \in (0, t_0)$$

$$-\int_x^1 \frac{u'(s)}{\sqrt{F(\rho) - F(s)}} ds = \sqrt{2\lambda}[1 - x]; \quad x \in (t_0, 1)$$

Substituting $v = u(s)$

$$\int_{q_1}^{u(x)} \frac{dv}{\sqrt{F(\rho) - F(v)}} = \sqrt{2\lambda}\, x; \quad x \in (0, t_0)$$

$$\int_{q_2}^{u(x)} \frac{dv}{\sqrt{F(\rho) - F(v)}} = \sqrt{2\lambda}[1 - x]; \quad x \in (t_0, 1)$$

Taking $x \to t_0$

$$\int_{q_1}^{u(x)} \frac{dv}{\sqrt{F(\rho)-F(v)}} = \sqrt{2\lambda}\, x; \quad x \in (0, t_0)$$

$$\int_{q_2}^{u(x)} \frac{dv}{\sqrt{F(\rho)-F(v)}} = \sqrt{2\lambda}[1-x]; \quad x \in (t_0, 1)$$

$$\int_{q_1}^{\rho} \frac{dv}{\sqrt{F(\rho)-F(v)}} = \sqrt{2\lambda} t_0$$

$$\int_{q_2}^{\rho} \frac{dv}{\sqrt{F(\rho)-F(v)}} = \sqrt{2\lambda}[1-t_0]$$

$$\lambda = \frac{1}{2}\left(\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right)^2$$

$$t_0 = \frac{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}}{\left(\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right)}$$

$$\left.\begin{aligned} 2[F(\rho)-F(q_1)] &= \gamma^2 q_1^2 [g(q_1)]^2 \\ 2[F(\rho)-F(q_2)] &= \gamma^2 q_2^2 [g(q_2)]^2 \end{aligned}\right\} (**)$$

Activity 1: Prove the reverse direction with the help of implicit function Theorem.

Suppose λ, ρ, q_1, and q_2 satisfy:

$$\left.\begin{aligned} 2[F(\rho)-F(q_1)] &= \gamma^2 q_1^2 [g(q_1)]^2 \\ 2[F(\rho)-F(q_2)] &= \gamma^2 q_2^2 [g(q_2)]^2 \end{aligned}\right\} (**)$$

$$\lambda = \frac{1}{2}\left(\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right)^2 \Bigg\} (*)$$

→ (1) **has a positive solution u with $u(t_0) = \|u\|_\infty = \rho, u(0) = q_1, u(1) = q_2$**

Define u

$$\int_{q_1}^{u(x)} \frac{dv}{\sqrt{F(\rho)-F(v)}} = \sqrt{2\lambda}\, x; \quad x \in (0, t_0)$$

How to Compute Numerical Bifurcation Diagrams

(1) Fix $\gamma > 0$ and define $\rho_i = \dfrac{i}{n+1}$; $i = 1, \ldots, n$ where $n \geq 1$ is the desired number of interpolation points.

(2) Using `fzero` in MATLAB (The Mathworks, Inc. version: 9.12.0.1927505 (R2022a) Update 1), numerically find the roots of **(**)** i.e. $q_{i_1} = q_1(\rho_i)$ and $q_{i_2} = q_2(\rho_i)$, for a given ρ_i.

(3) The values of ρ_i, q_{i_1}, q_{i_2} are then substituted into **(*)** and the corresponding λ_i-value is numerically computed using `integral`.

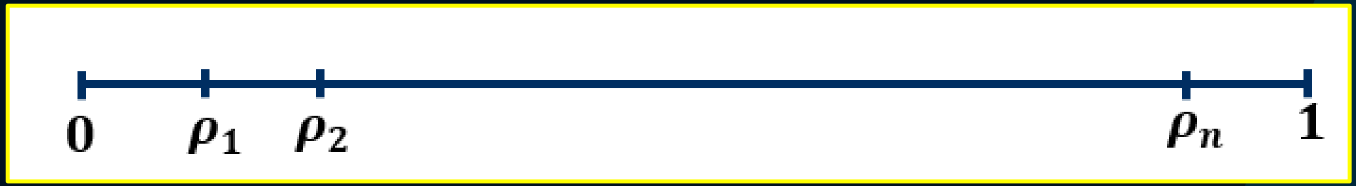
(4) Repeating (2) - (3) for $i = 1, 2, \ldots\ldots, n$, we obtain (λ_i, ρ_i) points, generating a bifurcation curve of λ vs. $\rho = \|u\|_\infty$ for positive solutions of (1.4).

$$\left.\begin{array}{l} 2[F(\rho) - F(q_1)] = \gamma^2 q_1^2 [g(q_1)]^2 \\ 2[F(\rho) - F(q_2)] = \gamma^2 q_2^2 [g(q_2)]^2 \end{array}\right\} (**)$$

$$\left.\lambda = \frac{1}{2}\left(\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right)^2 \right\} (*)$$

$$f(u) = \frac{1}{a} u(u+a)(1-u)$$

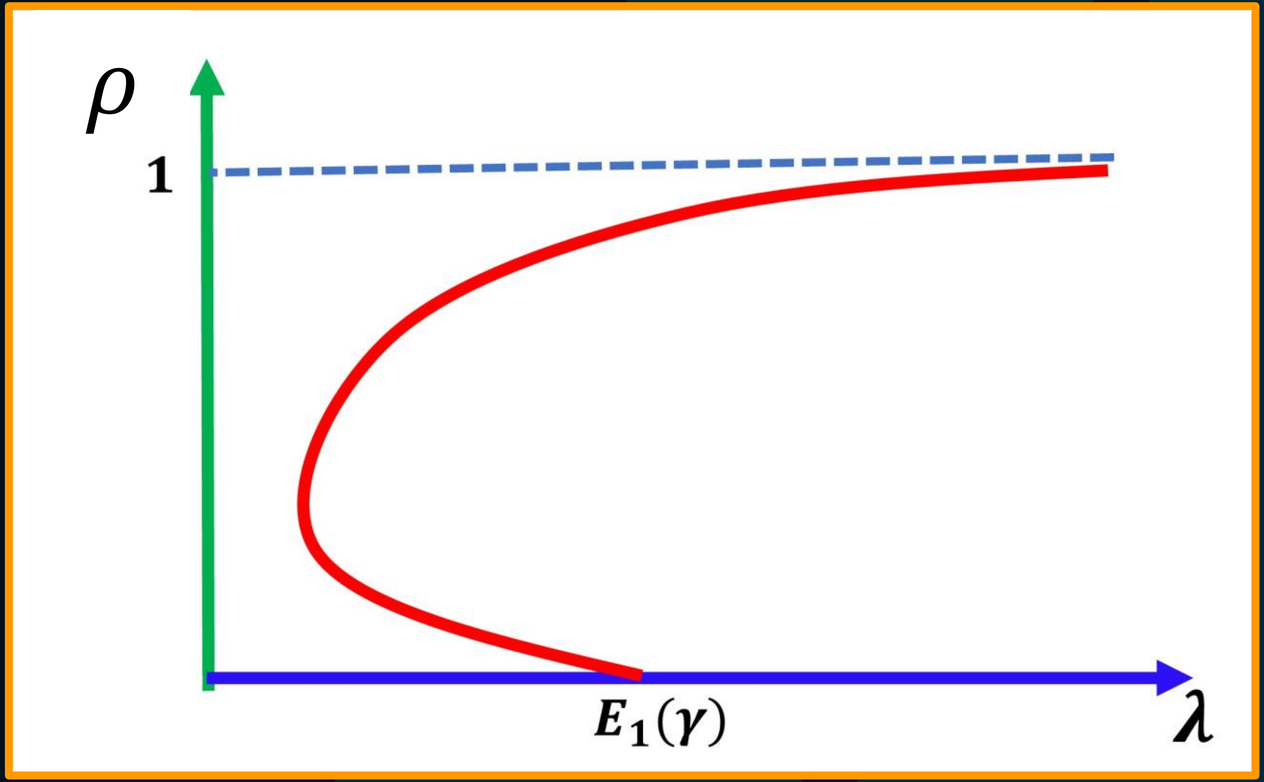
Activity 2:



$$f(u)=\frac{1}{a}u(u+a)(1-u);\ a=0.5,\ \ g(u)=1$$

- $\lambda=\frac{1}{2}\left(\int_{q_1}^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}}+\int_{q_2}^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}}\right)^2$ **(*)**
- $2[F(\rho)-F(q_1)]=\gamma^2 q_1^2[g(q_1)]^2$
- $2[F(\rho)-F(q_2)]=\gamma^2 q_2^2[g(q_2)]^2$ **(**)**

$$F(s)=\int_0^s z(1-z)dz$$



Computational Results

1
$$\begin{cases} -u'' = \lambda f(u); (0,1) \\ -u'(0) + \gamma\sqrt{\lambda}g(u(0))u(0) = 0 \\ u'(1) + \gamma\sqrt{\lambda}g(u(1))u(1) = 0 \end{cases}$$

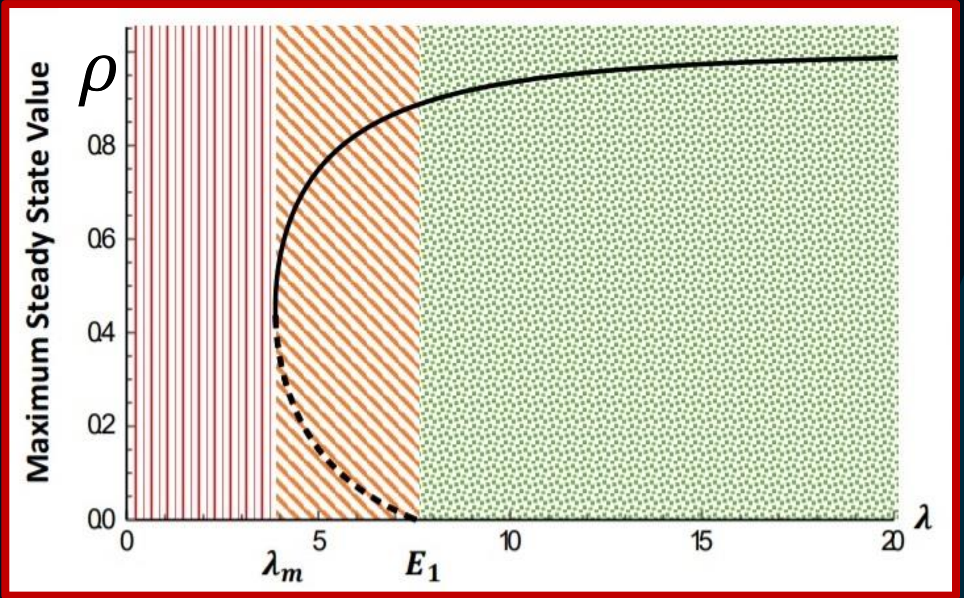
$$f(u) = \frac{1}{a}u(u+a)(1-u)$$

$$g(u) = \frac{[1-\alpha(u)]}{\alpha(u)}$$

$E_1(\gamma)$ is the P.E.V. of:

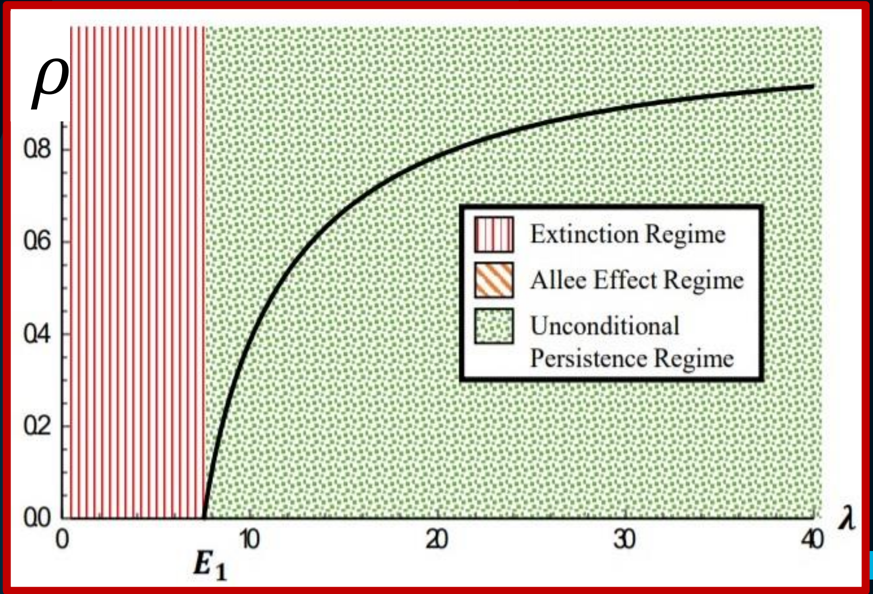
$$\begin{cases} -v'' = Ev; (0,1) \\ -v'(0) + \gamma\sqrt{E}v(0) = 0 \\ v'(1) + \gamma\sqrt{E}v(1) = 0 \end{cases}$$

Activity 3: Prove that $E_1(\gamma) = 4\left(\frac{\pi}{2} - tan^{-1}(\gamma)\right)^2$



$E_1 - \lambda_m$ = Allee effect region length

Can a DDE eliminate an Allee effect

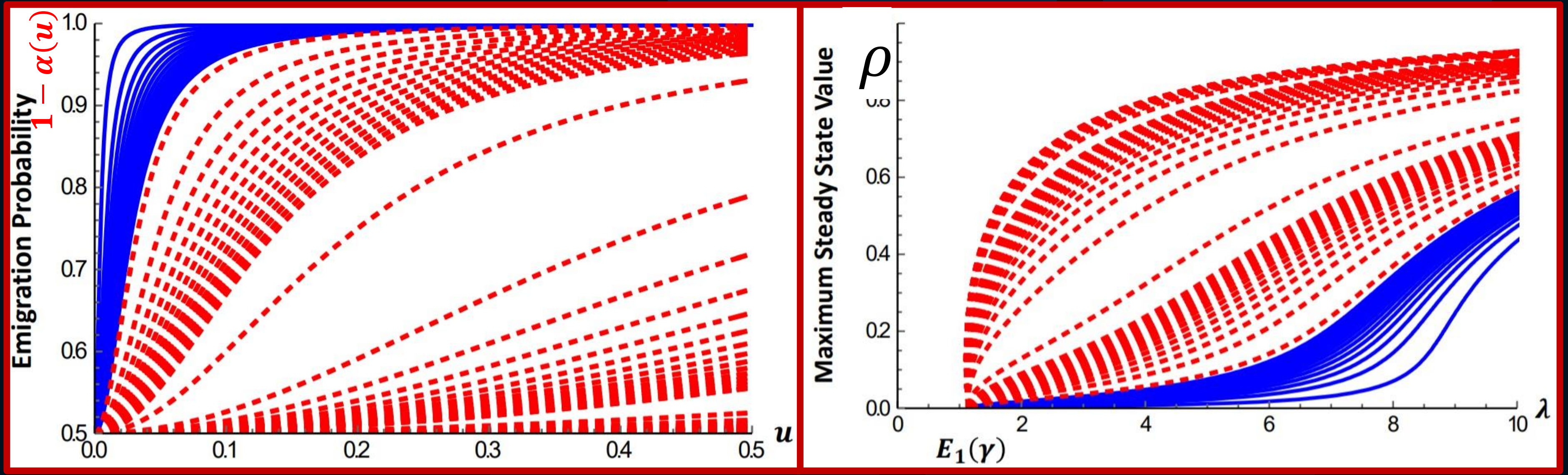


Yes

DDE Form	$\alpha(u)$	$1-\alpha(u)$	Restrictions
DIE	$\alpha(u) \equiv 0.5$	$1-\alpha(u) \equiv 0.5$	none
+DDE	$\alpha(u) = \frac{M_1}{2M_1+u^2}$	$1-\alpha(u) = \frac{M_1+u^2}{2M_1+u^2}$	none
-DDE	$\alpha(u) = \frac{M_1+u^2}{2M_1+u^2}$	$1-\alpha(u) = \frac{M_1}{2M_1+u^2}$	none
UDDE	$\alpha(u) = \frac{M_1}{2M_1+u^2-2M_2u}$	$1-\alpha(u) = \frac{M_1+u^2-2M_2u}{2M_1+u^2-2M_2u}$	$M_1 > M_2^2$
hDDE	$\alpha(u) = \frac{M_1+u^2-2M_2u}{2M_1+u^2-2M_2u}$	$1-\alpha(u) = \frac{M_1}{2M_1+u^2-2M_2u}$	$M_1 > M_2^2$

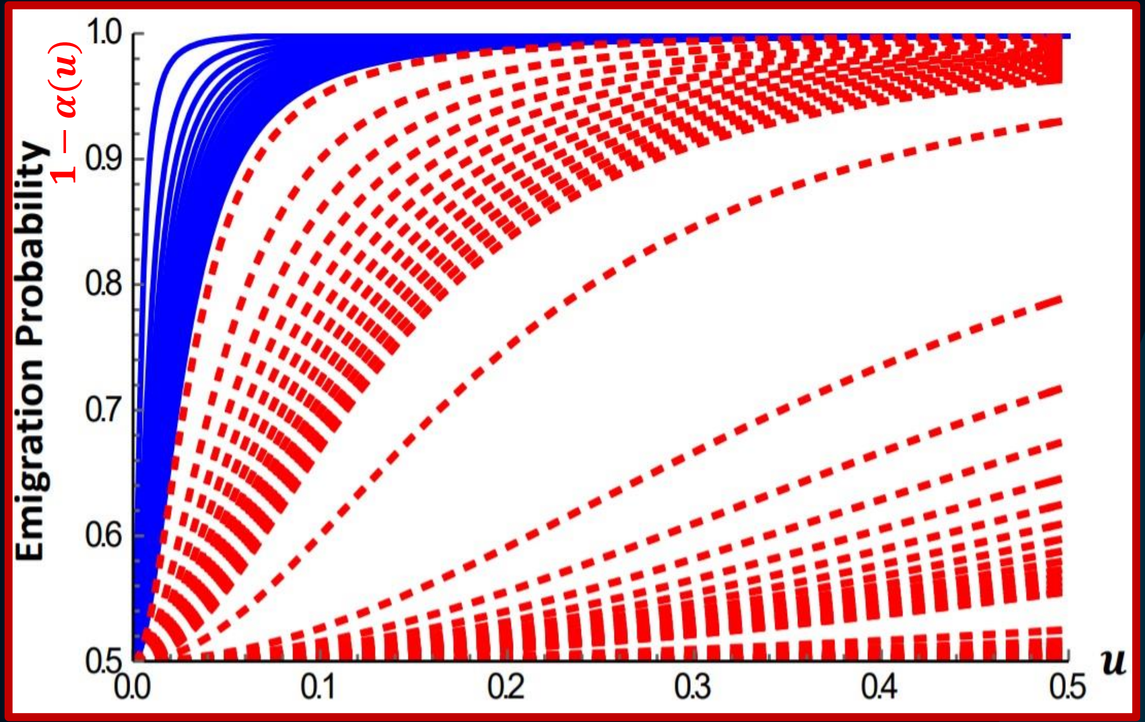
A +DDE can counteract a patch-level Allee effect

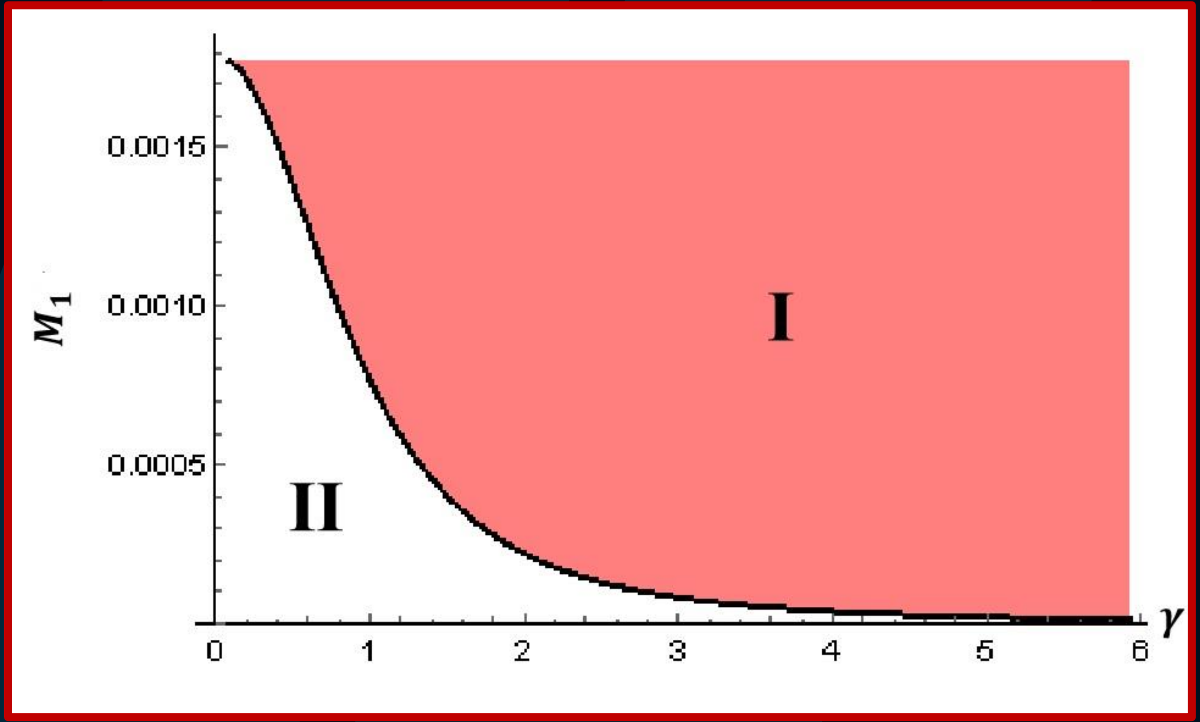
$$1-\alpha(u) = \frac{M_1+u^2}{2M_1+u^2}$$

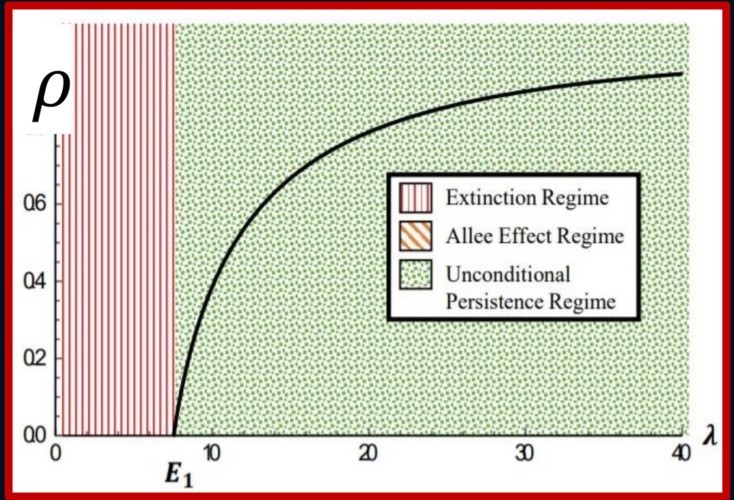


$\gamma = 0.6$ and $a = 0.75$

Allee effect region on the $\gamma - M_1$ plane







$$1 - \alpha(u) = \frac{M_1 + u^2}{2M_1 + u^2}$$

Stability Results for (1)

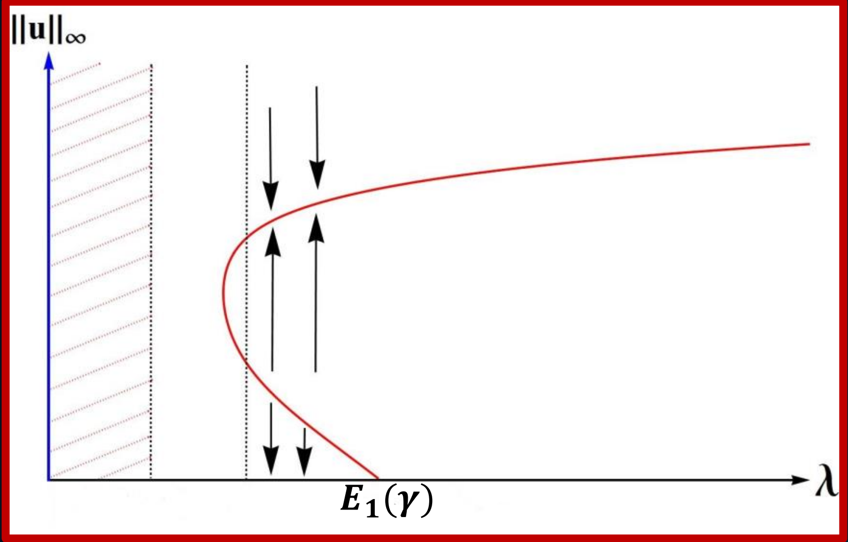
$E_1(\gamma)$ is the P.E.V. of:

$$\begin{cases} -v'' = Ev; (0,1) \\ -v'(0) + \gamma\sqrt{E}g(0)v(0) = 0 \\ v'(1) + \gamma\sqrt{E}g(0)v(1) = 0 \end{cases}$$

(1) $$\begin{cases} -u'' = \lambda f(u); (0,1) \\ -u'(0) + \gamma\sqrt{\lambda}g(u(0))u(0) = 0 \\ u'(1) + \gamma\sqrt{\lambda}g(u(1))u(1) = 0 \end{cases}$$

$$g(u) = \frac{[1-\alpha(u)]}{\alpha(u)}$$

Theorem 2: The trivial solution of (1) is asymptotically stable if $\lambda < E_1(\gamma)$ and it is unstable if $\lambda > E_1(\gamma)$.



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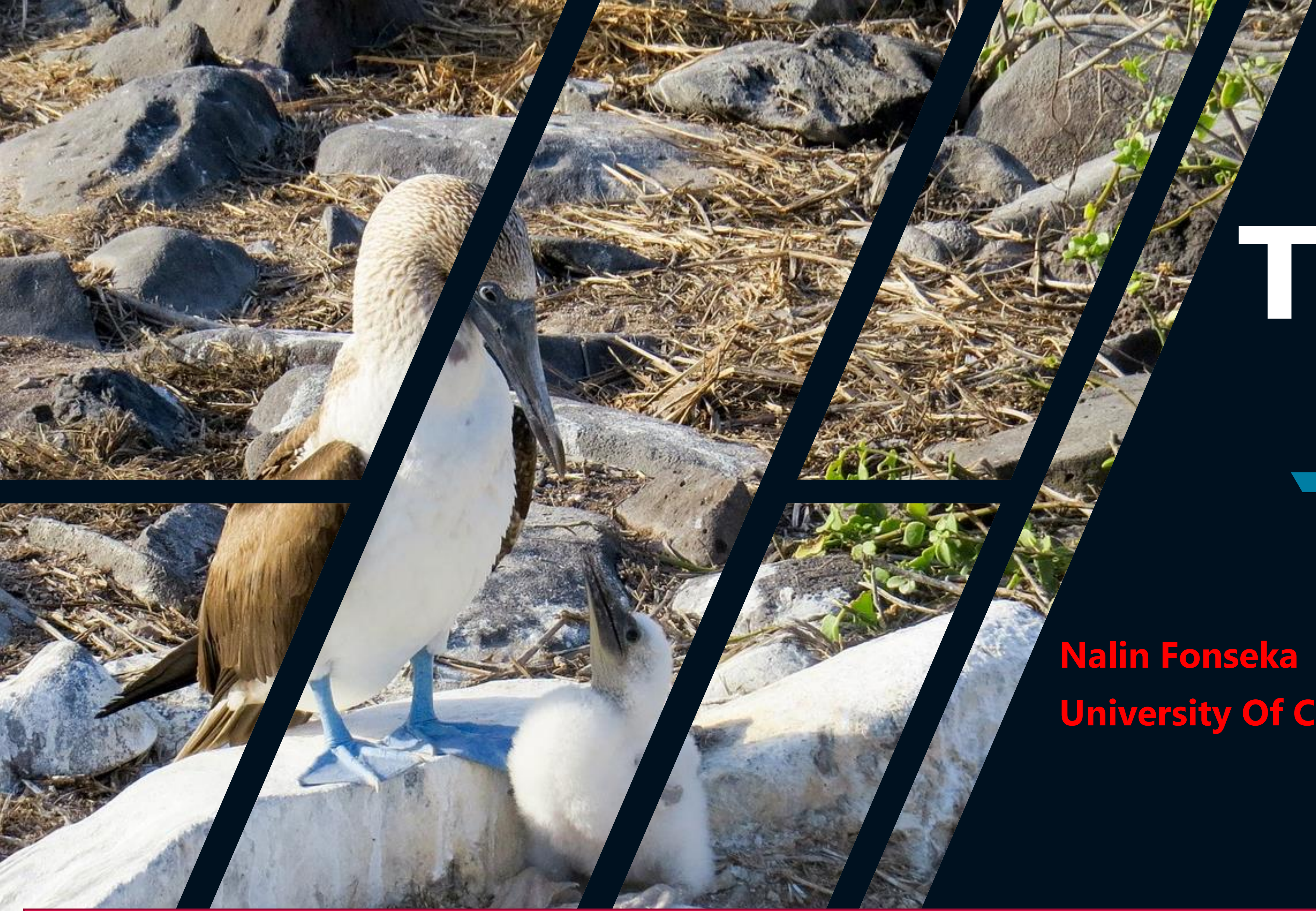

THANK YOU!

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