Using a Model to Give a Grand Tour of a First Course in Differential Equations

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Abstract

Formulating a versatile population growth model affords an opportunity to survey some of the important concepts that are presented during a first course in ordinary differential equations. While the full model is nonlinear with respect to the dependent variable, it can be solved explicitly, yielding an implicit representation of the solution using the separation of variables method, which suggests that an analysis of the long term behavior of all solutions might be challenging. Fortunately, the fact that there are three equilibrium solutions offers a gateway to the geometrical theory of differential equations: the phase line, the stability of the equilibria, and the long term behavior of the entire family of solutions. For many students, this experience offers a **compelling** answer to their query concerning the utility of calculus itself.

Outline

- Curriculum, course syllabus, and NSF grant
- Progress through case studies:
  - Model Building
  - Separation of variables
  - Power series solutions
  - Equilibria and the phase line
  - Linearization
  - Bifurcation

Curriculum and course syllabus

Objectives: The principal course objectives are to provide an introduction to the techniques used for the solution of elementary differential equations and the qualitative analysis of their solutions. Furthermore, a primary goal is to offer a research experience which conveys the essentials of mathematical research, at least at an elementary level. The traditional first year calculus sequence (MAT 151-MAT 152) serves as the course prerequisite.

Curriculum and course syllabus

Learning Outcome: At the end of the semester you should be able to: 1) **Classify** a differential equation according to its type and order; 2) **Solve** elementary (ordinary) first order and (ordinary) linear second order initial value problems; 3) **Analyze** the phase plane behavior of the solution set of a (non-linear) differential equation. Considerations will include the location of equilibrium and periodic solutions as well as describing the long term behavior of solutions; 4) **Relate** the behavior of a mathematical model to real time situations; and 5) **Formulate and analyze** a simple research problem. A detailed **report** describing the research, pertinent conclusions, and next steps will be the outcome of the process. Performing a related **literature search** and the development of a **bibliography** are expected components of the report.

Model Building 1: The Malthusian Law

$$\frac{dp}{dt} = rp$$

The Malthusian law is based upon the asumption that the growth rate of an isolated population is proportional to the instantaneous population. The principal assumption is that the population is isolated and there is an unlimited supply of resources necessary to sustain it. This also introduces the first Linear Homogeneous Equation in the course and

- Integrating Factor
- General Solutions
- Solutions of the Initial Value Problem

- Linear Homogeneous Equation

$$\frac{dp}{dt} - rp = 0, \tag{1}$$

with the integrating factor e^{-rt}, i.e.

$$e^{-rt}\frac{dp}{dt} - e^{-rt}rp = 0, \tag{2}$$

or

$$\frac{d}{dt}\left(e^{-rt}p\right) = 0, \tag{3}$$

thus

$$p(t) = ce^{rt}, \tag{4}$$

$$p(t) = p(0)e^{rt}. \tag{5}$$

The solution as presented in Equation (4) is the *general solution.*

As we shall see, the initial value problem is an important concept within the context of ordinary differential equations. Given an initial state of a realizable system, the objective is to determine the evolution of the system for future time. Several possibilities are possible. The system may exist for all future time or it may cease to exist after some finite time interval. Such behavior is dependent upon the natural laws being used to formulate the model.

Model Building 2: The Logistic Law

The Logistic Law incorporates an additional factor that takes into account the possibility that a population will eventually outgrow its resources and no longer be sustainable. An additional linear factor

$(c - dp)$, where constants $c > 0$, $d > 0$, so that a new equation

$$\frac{dp}{dt} = rp(c - dp), \tag{6}$$

to help account for this possibility. Indeed, for small values of p the linear factor is positive but once $p > \frac{c}{d}$, then $c - dp < 0$ and the growth rate of the population becomes negative. In other words, once $p > \frac{c}{d}$, the right hand side of the differential equation is negative and thus the population begins to decrease.

Model Building 3: Critical Thresholds

If a population is too small for there to be sufficient procreation, then it will eventually die out. Large predators are an example of this phenomenon. Unless the population exceeds a critical threshold for there to be sufficient social encounters, the population is not viable. Another linear factor can be added to the right hand side of the differential equation to account for this possibility, specifically,

$$a - bp, \; a > 0, \; b > 0$$

where

$$\frac{a}{b} < \frac{c}{d}.$$

Thus, the resulting differential equation is

$$\frac{dp}{dt} = -p(a - bp)(c - dp). \tag{7}$$

The model now has the desired properties. If $p < \frac{a}{b}$, then the right hand side is negative since all three factors are positive and the population decreases. When $\frac{a}{b} < p < \frac{c}{d}$, the right hand side is positive since the first and third factors are positive but the second one is negative. Thus, the population increases when $\frac{a}{b} < p < \frac{c}{d}$. Finally, if $p > \frac{c}{d}$, then the first factor is positive whereas the other two are negative implying that the right hand side is negative. Therefore, the population decreases as it should once the maximum sustainable population is exceeded.

Separation of Variables

The models above also provide examples for the Separation of Variables approach.

Such is the case for the population growth model. In particular, the differential equation may be rewritten as

$$\frac{dp}{dt} = -kp\left(1 - \frac{p}{V}\right)\left(1 - \frac{p}{S}\right) \tag{8}$$

where

$$k = ac,\ V = \frac{a}{b},\ \text{and } S = \frac{c}{d}.$$

Dividing both sides of Equation (8) by the terms on the right hand side finally gives

$$\frac{1}{p\left(1 - \frac{p}{V}\right)\left(1 - \frac{p}{S}\right)}\frac{dp}{dt} = -k. \tag{9}$$

Now the left hand side only depends upon p and the right hand side only depends upon t. In other words, the differential equation takes the form

$$f(p)\frac{dp}{dt} = g(t)$$

and if p is assumed to be an implicit function of t, then an implicit solution may be obtained by integrating each side with respect to t, i.e.

$$\int f(p)\frac{dp}{dt}\,dt = \int g(t)\,dt.$$

Anti-differentiation yields an implicit solution

$$F(p) = G(t) + C, \quad \text{where} \quad F'(t) = f(t),\ G'(t) = g(t).$$

Power Series Solutions

If the coefficients of p and its derivative are complicated functions of the independent variable t, a closed form representation of the solution $y(t)$ may be of little value for analyzing its long term behavior as $t \to \infty$ nor for obtaining numerical approximations. Power series methods are sometime a first attempt to produce numerical approximations to otherwise unwieldly expressions for $y(t)$.

$$p(t) = \sum_{k=0}^{\infty} a_k t^k \tag{10}$$

so that

$$p'(t) = \sum_{k=0}^{\infty} k a_k t^{k-1} = \sum_{k=-1}^{\infty} (k+1) a_{k+1} t^k = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k. \tag{11}$$

$$
\begin{aligned}
\frac{dp}{dt} - rp &= \sum_{k=0}^{\infty}(k+1)a_{k+1}t^k - \sum_{k=0}^{\infty} r a_k t^k = 0 \\
&= \sum_{k=0}^{\infty}\left[(k+1)a_{k+1} - r a_k\right] t^k = 0,
\end{aligned}
$$

and

$$
(k+1)a_{k+1} - r a_k = 0 \text{ or equivalently } a_{k+1} = \frac{r a_k}{k+1}. \tag{12}
$$

Once an *initial condition* is given, say $p(0) = 1$, it is then possible to recursively generate the coefficients of the power series. The initial condition implies $a_0 = 1$. Subsequently,

$$\begin{aligned} a_1 &= \frac{ra_0}{1} = r \\ a_2 &= \frac{ra_1}{2} = \left(\frac{r}{2}\right)(r) = \frac{r^2}{2} \\ \cdots & \qquad \cdots, a_k = \frac{r^k}{k!} \cdots . \end{aligned}$$

Putting everything together, we see that the power series representation for the solution $p(t)$ is

$$p(t) = \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} \frac{r^k}{k!} t^k = e^{rt} \tag{13}$$

Equilibria

The population growth model has three distinguished solutions, namely, $p = 0$, $p = V$ and $p = S$.

These three constant solutions or *equilibrium solutions* as they are called, play a signficant role in understanding the behavior of the solutions of the population growth model and

- Stable vs Unstable Equilibrium
- Sink vs Source vs Node

The Phase Line

The stability properties of the three equilibria are evident if several solutions are plotted on the p-axis using the information that we have about the sign of $p'(t)$. The p-axis is known as the *phase line*. Furthermore, given any differential equation of the form

$$\frac{dp}{dt} = f(p),$$

where $f(p)$ is differentiable, the second derivative of a solution $p(t)$ may be found from the relationship

$$\frac{d^2p}{dt^2} = f(p)f'(p). \tag{14}$$

Consequently, it is possible to obtain a plot of the solutions in the tp-plain using the information afforded by the phase line and the sign of $p''(t)$.

The preceeding analysis of the long term behavior of the solutions of the population growth model is typical of the *geometric thoery* of differential equations which was developed by Henri Poincare during the latter half of the nineteenth century. Its utility derives from its capability to determine the long term behavior of a differential equation's solutions without having explicit analytical expressions for them which is often the case.

Linearization

If a nonlinear autonomous differential equation has several equilibria, it can be a challenge to determine the local behavior of the solutions near each one not withstanding the availability of a closed form representation of the global solution, **especially when one extends the analysis to higher dimensions**. Another approach is to analyze the *linearized* differential equation near each of the equilibria and then endeavor to deduce the global behavior of the solutions from the composite local behaviors. To be specific, we consider the nonlinear autonomous equation-a special case of (6)

$$\frac{dp}{dt} = p(p-1). \tag{15}$$

There are two equilibria, $p = 0$ and $p = 1$.

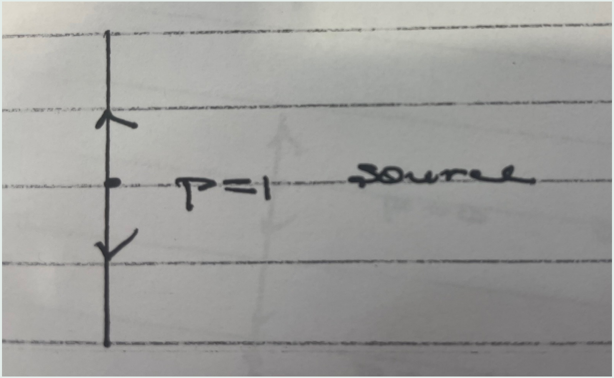
The linearization at $p = 0$ is

$$\frac{dp}{dt} = -p, \text{ with the phase line as shown right,} \tag{16}$$

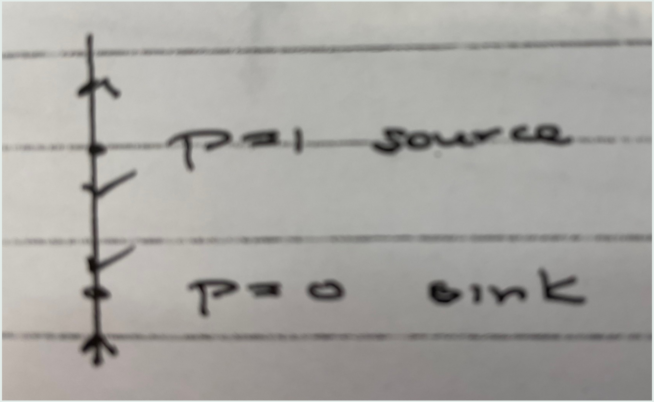
$p = 0$ sink

Linearization of the differential equation near the equilibrium $p = 1$ with $u = p - 1$ is

$$\frac{du}{dt} = u, \text{with the phase line,} \tag{17}$$



Combining both local behaviors on the entire phase line produces the phase portrait shown below



Bifurcaton

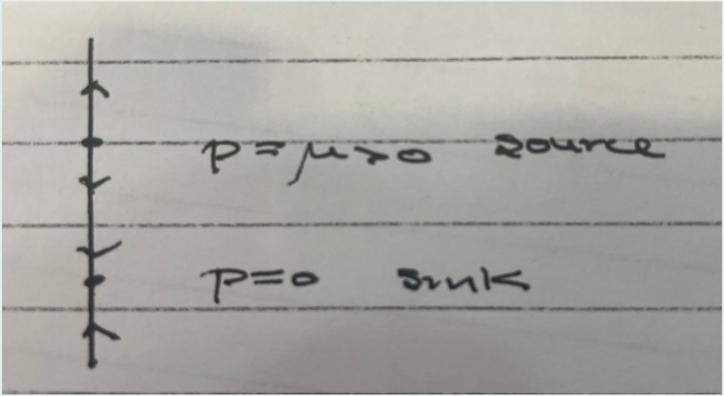
Lastly, we consider the situation when the right hand side of an autonomous differential equation also depends upon a parameter. Specifically, we generalize Equation (15) to include a real parameter μ-another special case of (6)

$$\frac{dp}{dt} = p(p - \mu). \tag{18}$$

There are three cases to consider, $\mu > 0,\ \mu = 0,$ and $\mu < 0$.

1. $\mu > 0$

   When $\mu > 0$ there are two critical points, namely, $p = 0$ and $p = \mu$. The same linearization analysis that was applied to Equation (15) demonstrates that $p = \mu$ is a source whereas $p = 0$ is a sink. The phase line is depicted in Figure 4.



2. $\mu = 0$

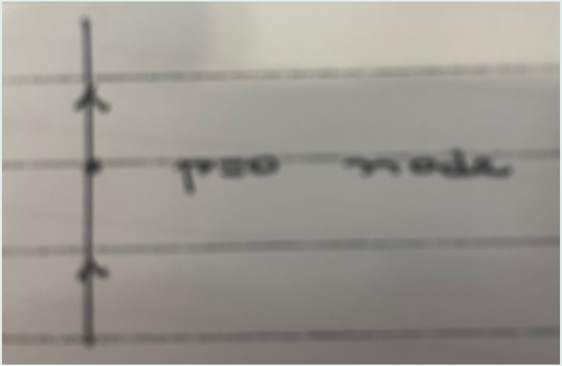
In this case, the differential equation takes the form

$$\frac{dp}{dt} = p^2. \tag{19}$$

Now there is only one equilibrium, $p = 0$. However, the linearized differential equation is degenerate, that is,

$$\frac{dp}{dt} = 0 \tag{20}$$

and there are only constant solutions $p(t) = p(0) = p_0$. The right hand side of the nonlinear Equation (19) is positive so that solutions are always increasing. Thus, $p = 0$ is a *node* and the phase line is as illustrated in Figure 5.



3. $\mu < 0$

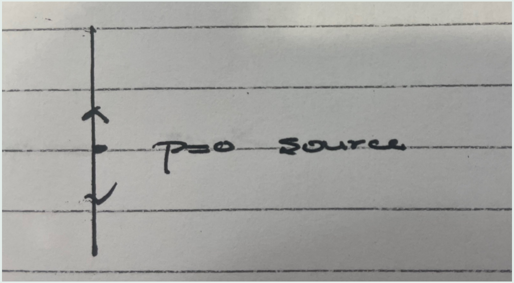
For convenience, we set $\mu = -\sigma$ so that $\sigma > 0$ and the differential equation (18) becomes

$$\frac{dp}{dt} = p(p + \sigma) = p^2 + \sigma p. \tag{21}$$

Plainly, there are two critical points, $p = 0$ and $p = -\sigma$. At $p = 0$, the linearized differential equation is

$$\frac{dp}{dt} = \sigma p \tag{22}$$

which implies that $p = 0$ is a source cf. Figure below,

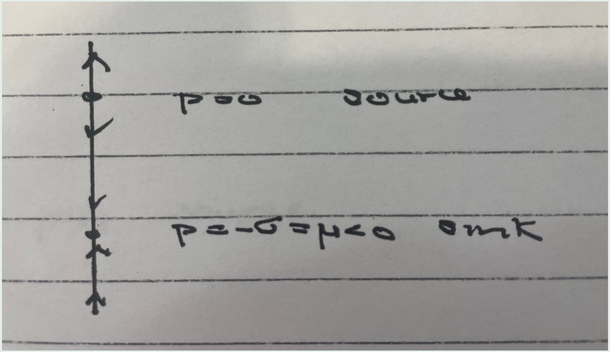


The linearized differential equation at $p = -\sigma$ is found by first obtaining the Taylor series expansion of the right hand side of Equation (21) $G(p)$ about $p = -\sigma$ so that the linearized differential equation is

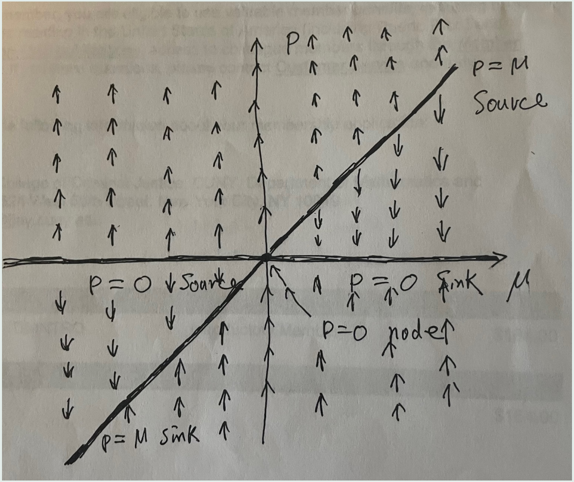
$$\frac{du}{dt} = -\sigma u, \ \sigma > 0 \tag{23}$$

from which it follows that $u = 0$ is a sink or equivalently,

$p = -\sigma = \mu < 0$ is a sink. The combined phase portrait is presented in Figure below



And the bifurcation diagram having μ as the horizontal axis is given below:


P
p = M
Source
μ
p = 0 Source
p = 0 Sink
p = 0 node
p = M sink

Thanks for your attention!

Appendix A - Solving Equation (9)

The solution of the population growth model differential equation

$$\frac{1}{p\left(1-\frac{p}{V}\right)\left(1-\frac{p}{S}\right)}\frac{dp}{dt}=-k$$

is achieved by using the separation of variables technique. Anti-differentiate of the left hand side requires the use of the separation of variables technique, that is, we let

$$\frac{1}{p\left(1-\frac{p}{V}\right)\left(1-\frac{p}{S}\right)}=\frac{A}{p}+\frac{B}{\left(1-\frac{1}{V}p\right)}+\frac{C}{\left(1-\frac{1}{S}p\right)}$$

where A , B , and C are constants to be determined. Combining the terms on the right hand side into a single fraction yields

$$\frac{1}{p\left(1-\frac{p}{V}\right)\left(1-\frac{p}{S}\right)}=$$

$$\frac{A\left(1-\frac{1}{V}p\right)\left(1-\frac{1}{S}p\right)+Bp\left(1-\frac{1}{S}p\right)+Cp\left(1-\frac{1}{V}p\right)}{p\left(1-\frac{p}{V}\right)\left(1-\frac{p}{S}\right)}.$$

Expanding the numerator of the right hand side gives

$$A\left[1-\left(\frac{1}{V}+\frac{1}{S}\right)p+\frac{1}{VS}p^2\right]+B\left[p-\frac{1}{S}p^2\right]+C\left[p-\frac{1}{V}p^2\right]$$

$$=A+\left[B+C-A\left(\frac{1}{V}+\frac{1}{S}\right)\right]p+\left[A\frac{1}{VS}-\left(\frac{C}{V}+\frac{B}{S}\right)\right]p^2$$

and then equating coefficients of like powers of p affords a system of equations for the constants $A,\ B$ and, C,

$$\begin{array}{rcrcrcl} A & & & & & = & 1 \\ -\left(\frac{1}{V}+\frac{1}{S}\right)A & + & B & + & C & = & 0 \\ \frac{1}{VS}A & - & \frac{1}{S}B & - & \frac{1}{V}C & = & 0 \end{array}.$$

Using the fact that $A = 1$, the former system is equivalent to the reduced system

$$\begin{array}{rcrcrcl} -\left(\frac{1}{V}+\frac{1}{S}\right) & + & B & + & C & = & 0 \\ \frac{1}{VS} & - & \frac{1}{S}B & - & \frac{1}{V}C & = & 0 \end{array},$$

that is,

$$\boldsymbol{B} \;+\; \boldsymbol{C} = \left(\frac{1}{\boldsymbol{V}} + \frac{1}{\boldsymbol{S}}\right)$$

$$\frac{1}{\boldsymbol{S}}\boldsymbol{B} \;+\; \frac{1}{\boldsymbol{V}}\boldsymbol{C} = \frac{1}{\boldsymbol{VS}}.$$

If

$$\boldsymbol{\Delta} = \det\begin{pmatrix} 1 & 1 \\ \frac{1}{\boldsymbol{S}} & \frac{1}{\boldsymbol{V}} \end{pmatrix} = \frac{1}{\boldsymbol{V}} - \frac{1}{\boldsymbol{S}},$$

then

$$B = \frac{1}{\Delta} \det \begin{pmatrix} \frac{1}{V} + \frac{1}{S} & 1 \\ \\ \frac{1}{VS} & \frac{1}{V} \end{pmatrix} = \frac{1}{\Delta} \left[\left(\frac{1}{V} + \frac{1}{S} \right) \frac{1}{V} - \frac{1}{VS} \right] = \frac{1}{V^2 \Delta}.$$

But

$$V^2 \Delta = \left(\frac{1}{V} - \frac{1}{S} \right) V^2 = V - \frac{V^2}{S}$$

implying

$$B = \frac{1}{V - \frac{V^2}{S}} = \frac{1}{V\left(1 - \frac{V}{S}\right)}.$$

Similarly,

$$C = \frac{1}{\Delta} \det \begin{pmatrix} 1 & \frac{1}{V} + \frac{1}{S} \\ \\ \frac{1}{S} & \frac{1}{VS} \end{pmatrix} = \frac{1}{\Delta} \left[\frac{1}{VS} - \frac{1}{S} \left(\frac{1}{V} + \frac{1}{S} \right) \right] = -\frac{1}{S^2 \Delta}$$

Now

$$S^2 \Delta = S^2 \left(\frac{1}{V} - \frac{1}{S} \right) = \frac{S^2}{V} - S$$

implying

$$C = \frac{1}{S\left(1 - \frac{S}{V}\right)}.$$

Putting everything together, we finally have the desired partial fraction expansion

$$\frac{1}{p\left(1-\frac{p}{V}\right)\left(1-\frac{p}{S}\right)}=\frac{1}{p}+\frac{1}{V\left(1-\frac{V}{S}\right)}\cdot\frac{1}{\left(1-\frac{p}{V}\right)}+\frac{1}{S\left(1-\frac{S}{V}\right)}\cdot$$

If $p=p(t)$, then anti-differentiation of the right hand side yields

$$\int\frac{1}{p}\frac{dp}{dt}\,dt+\int\frac{1}{V\left(1-\frac{V}{S}\right)}\frac{1}{\left(1-\frac{p}{V}\right)}\frac{dp}{dt}\,dt+\int\frac{1}{S\left(1-\frac{S}{V}\right)}\cdot\frac{1}{\left(1-\frac{p}{S}\right)}$$

or

$$\int\frac{1}{p}\frac{dp}{dt}\,dt-\int\frac{1}{\left(1-\frac{V}{S}\right)}\cdot\frac{(-1/V)}{\left(1-\frac{p}{V}\right)}\frac{dp}{dt}\,dt-\int\frac{1}{\left(1-\frac{S}{V}\right)}\cdot\frac{(-1/S)}{\left(1-\frac{p}{S}\right)}$$

so that

$$\ln p - \frac{1}{\left(1 - \frac{V}{S}\right)} \cdot \ln\left(1 - \frac{p}{V}\right) - \frac{1}{\left(1 - \frac{S}{V}\right)} \cdot \ln\left(1 - \frac{p}{S}\right) = -kt.$$

Finally, the laws of logarithms imply

$$\begin{aligned}
\ln p + \ln\left(1-\frac{p}{V}\right)^{-(1-V/S)} + \ln\left(1-\frac{p}{S}\right)^{-(1-S/V)} &= -kt \\
\ln p\left(1-\frac{p}{V}\right)^{-\left(1-\frac{V}{S}\right)}\left(1-\frac{p}{S}\right)^{-\left(1-\frac{S}{V}\right)} &= -kt \\
p\left(1-\frac{p}{V}\right)^{-\left(1-\frac{V}{S}\right)}\left(1-\frac{p}{S}\right)^{-\left(1-\frac{S}{V}\right)} &= e^{-kt} \\
p\left(1-\frac{p}{V}\right)^{\left(\frac{V}{S}-1\right)}\left(1-\frac{p}{S}\right)^{\left(\frac{S}{V}-1\right)} &= e^{-kt}
\end{aligned}$$

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Please note that solving for p explicitly in terms of t is out of reach.

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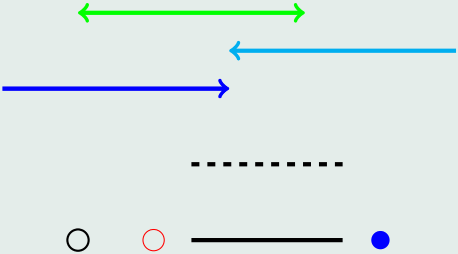

- $p = 0$ sink

The linearization at $p = 0$ is

$$\frac{dp}{dt} = -p, \text{ with the phase line as shown right,} \tag{24}$$