Falling Water: A Modeling Story

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Dordt University

Building on a Foundation

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Resources: Modeling Scenarios

- 1-014-T-Draining Containers

  We examine the question, "Given two rectangular circular cylinders of water with the same volume, but different radii, with a small bore hole of same radius on the center of the bottom through which water exits the cylinder, which empties faster?"
- 1-140-T-Leaky Bucket

  We seek to model the height of water in a cylindrical tank (bucket) in which water flows out the bottom of the tank through a small bore hole while we are pouring water into the tank at the top of the tank at a constant (or varying) rate.
- 1-064-T-Torricelli Box

  The time it takes a column of water to empty and the time it takes the same volume of column of water with a box (various sizes) submerged in the column of water are compared through modeling with Torricelli's Law.

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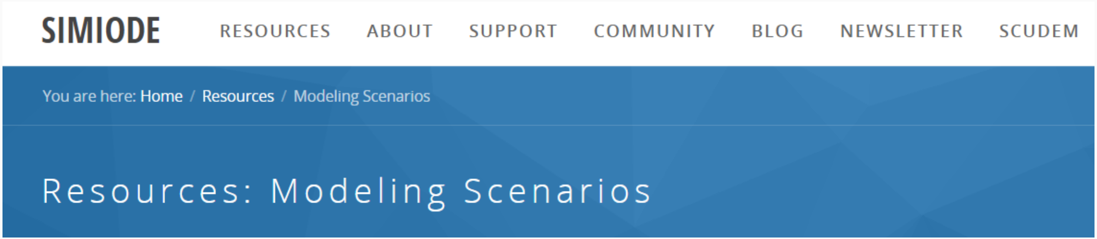


- 1-058-T-Water Clocks

  We apply Torricelli's Law to the task of building a water clock in which the height of the water in a container falls at a constant rate when the container has a hole in the bottom to let the water flow out. First, we review the principles and derivation of the applicable physics in Torricelli's Law. Second, we determine the shape of the container, given a constant spigot size for the exit hole, so that the water height falls at a constant rate.

- 1-074-T-Bottle Water Flow

  We offer an experiment in which data is collected to ascertain a parameter in the differential equation formulation of Torricelli's Law for water flow from a container.

What More is Needed?

- Modeling other tank geometries.
- Data and videos available for classroom use.
- Models growing in complexity with solving techniques.

Torricelli's Law

- The velocity at which water flows out the bottom of a container is proportional to $\sqrt{h}$ and thus we can model change in volume as:

$$\frac{dV}{dt} = -A_h\sqrt{2gh} = -k\sqrt{h}.$$

- If the volume of the container can be written as $V(h)$ then we can differentiate with respect to t and produce $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$.
- This yields a model for $\frac{dh}{dt}$ which can be solved to produce $h(t)$ as desired.

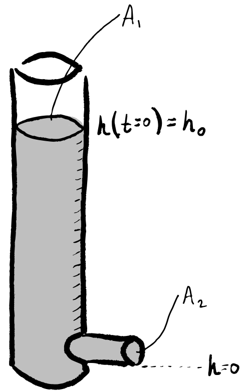
Cylindrical Tank Modeling

Modeling a Draining Cylindrical Tank

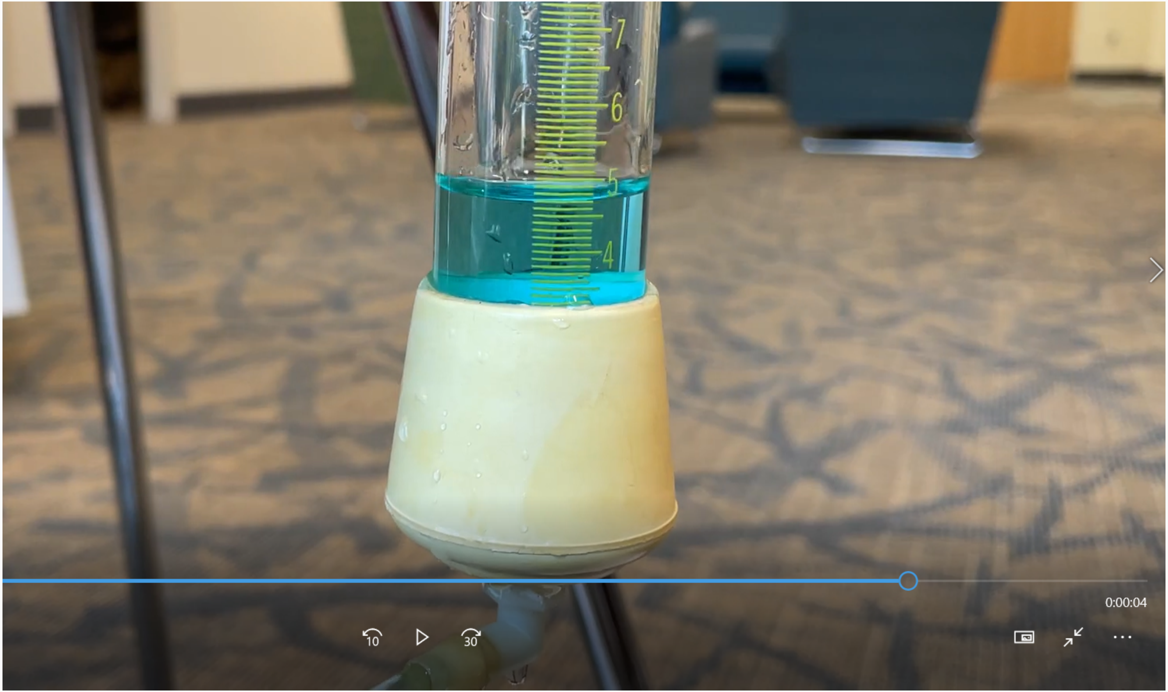
- If the container is a cylinder with cross-sectional area A then $V(h) = Ah$ and thus $\frac{dV}{dt} = A\frac{dh}{dt}$.
- Equating the equation for $\dfrac{dV}{dt} = -k\sqrt{h}$ yields the separable differential equation:

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h}.$$

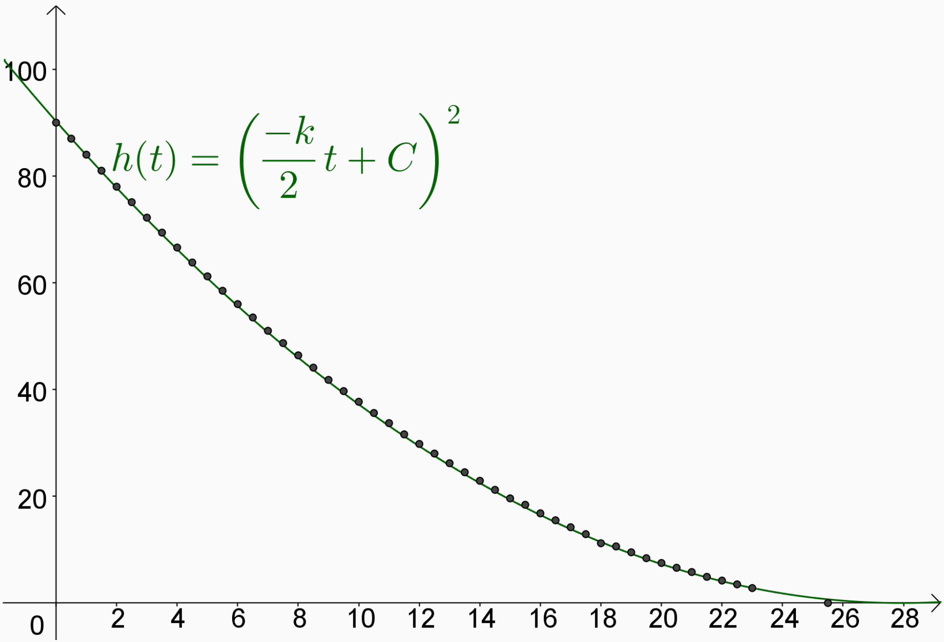
Draining a Cylinder is Cool



Carefully Measured Data - Cylinder



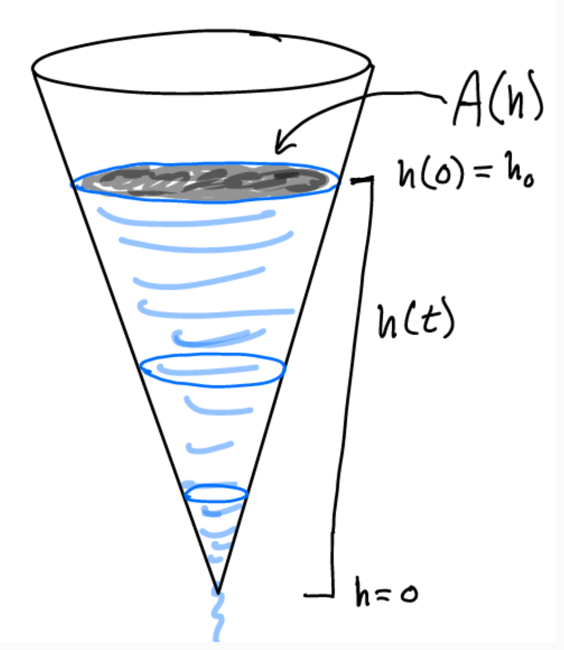
Carefully Measured Data - Cylinder



$$h(t) = \left(\frac{-k}{2}t + C\right)^2$$

Conical Tank Modeling

Draining a Conical Tank



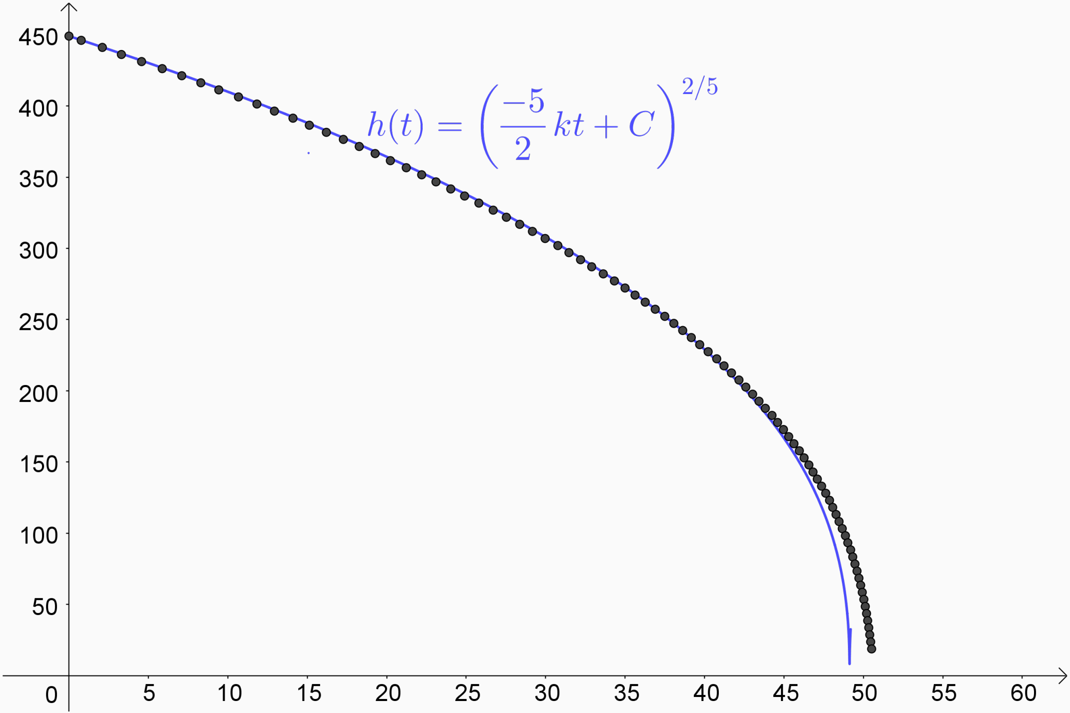
Modeling a Draining Conical Tank

- If the container is conical with height-dependent cross-sectional area $A(h)$ then we can write $V(h) = \frac{1}{3}A(h) \cdot h$.
- It follows that $\dfrac{dV}{dt} = \pi \tan^2(\theta) h^2 \dfrac{dh}{dt}$.
- We then derive the separable differential equation:

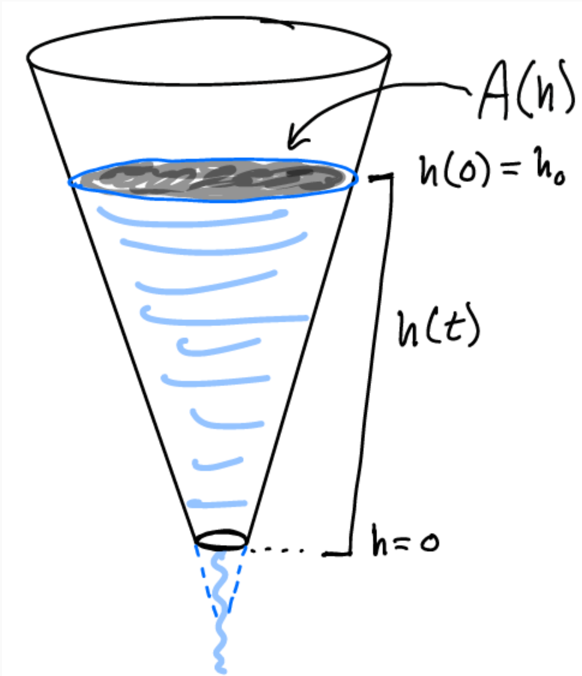
$$\frac{dh}{dt} = -\frac{k}{\pi \tan^2(\theta) h^2}\sqrt{h} = -\frac{k_1}{h^{3/2}}$$

Carefully Measured Data - Cone

Carefully Measured Data - Cone



Draining a Conical Tank - Upgraded

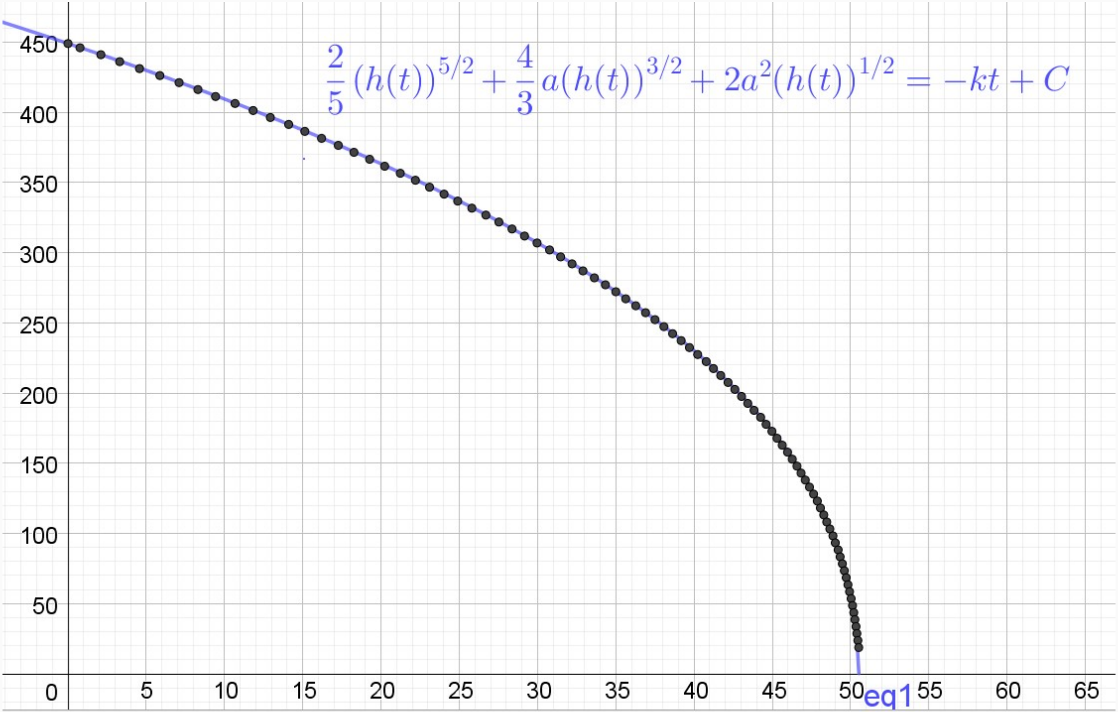


Modeling the Upgraded Conical Tank

- The previous model failed to account for the truncation of the cone by lenght T.
- Replace the h term in the denominator with $h + T$.
- This yields the following corrected model:

$$\frac{dh}{dt} = -\frac{k}{\pi \tan^2(\theta)(h+T)^2}\sqrt{h} = -\frac{k_1\sqrt{h}}{(h+T)^2}. \quad (1)$$

Carefully Measured Data - Cone - Upgraded Model




3 SEPARABLE DE MODELS
BUT WAIT THERE'S MORE
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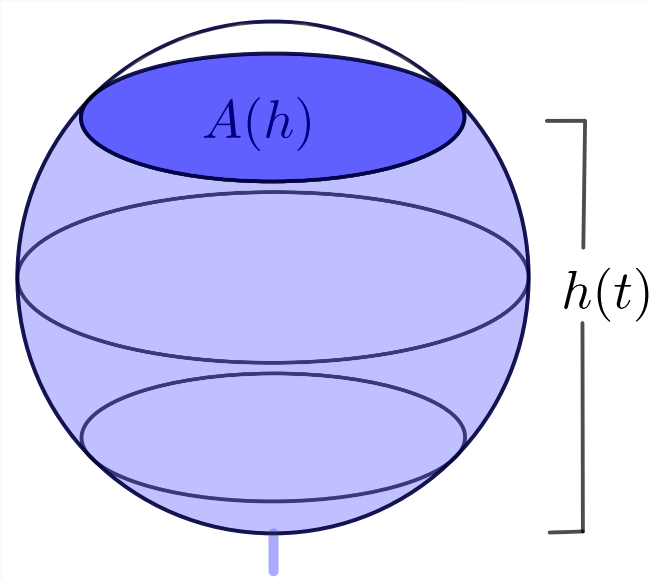
Spherical Tank Modeling

Draining a Spherical Tank



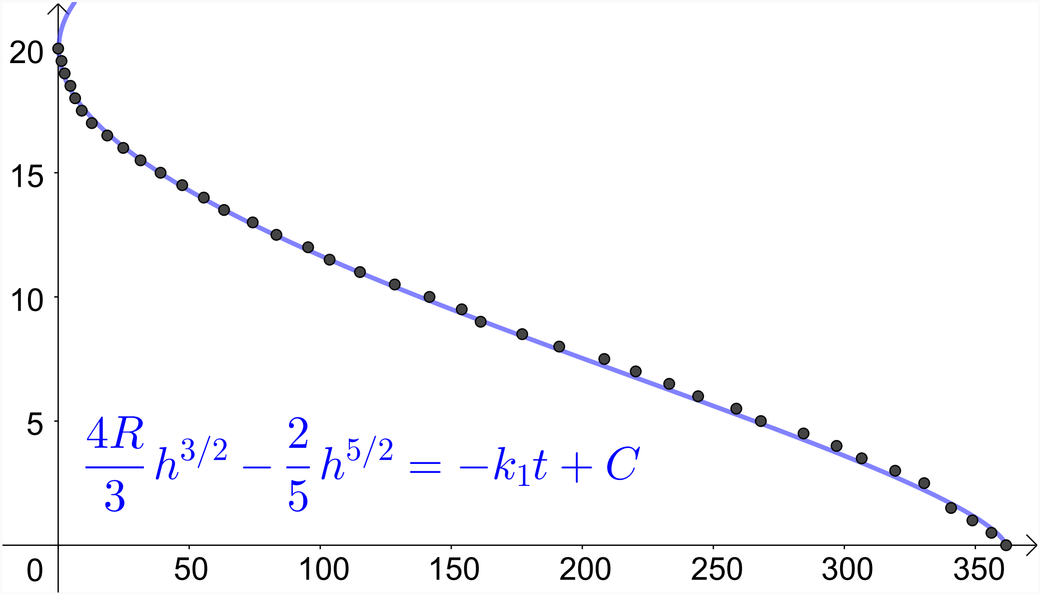
Modeling a Draining Spherical Tank

- If the container is spherical then the volume $V(h)$ which is given by $V(h) = \pi R h^2 - \frac{1}{3}\pi h^3$, where R is the radius of the sphere.
- Differentiating yields $\dfrac{dV}{dt} = \pi(2Rh - h^2)\dfrac{dh}{dt}$.
- Equating with $\dfrac{dV}{dt} = -k\sqrt{h}$ as before we obtain the model:

$$\frac{dh}{dt} = -\frac{k}{\pi(2Rh - h^2)}\sqrt{h} = -\frac{k_1}{2Rh^{1/2} - h^{3/2}}.$$

Carefully Measured Data - Sphere

Carefully Measured Data - Sphere



Model Progression with Theory Progression

Various Separable Models and Their Benefits

- Cylinder
  - Develop Torricelli's Law.
  - Easy to find a cylinder and gather data.
  - Data matches solution very well.
  - Can even use in Calculus class.
- Cone Basic
  - Adds geometry to Torricelli's Law.
  - Interesting to have students predict whether $h(t)$ will be convex or concave.
  - Data doesn't quite match solution.
- Cone Upgraded
  - Adds more complex geometry to Torricelli's Law.
  - Data matches **implicit** solution beautifully.
- Sphere
  - Adds another layer of geometry.
  - Data fits well with **implicit** solution.

Student Feedback

Data + Modeling is Fantastic

- Students love being able to see that the mathematical “answer” matches with the data.
- Students love to see that modeling allows them to answer questions they wouldn’t otherwise be able to.

Student Survey Questions

Table 1: Question 1: How important to you personally did you think the content of Differential Equations would be at the beginning of the course?

Not at all important	Not so important	Somewhat important	Very important	Extremely Important
0%	21.4%	60.7%	17.8%	0%

Table 2: Question 2: How important to you personally do you think the content of Differential Equations is now?

Not at all important	Not so important	Somewhat important	Very important	Extremely Important
0%	3.5%	25.0%	50.0%	21.4%

Student Survey Questions

Table 3: Paired shifts of individuals from question 1 to 2, treating categories as integers 1 - 5.

-1	0	1	2
3.6%	32.1%	42.9%	21.4%

Student Survey Questions

Table 4: Because of the "real-world" modeling problems you were motivated to learn...

Much less.	Somewhat less.	The same.	Somewhat more.	Much more.
0%	3.6%	25.0%	53.6%	17.9%

Table 5: Because of the "real-world" modeling problems you learned the standard/typical topics...than/as without them.

much less	somewhat less	the same	somewhat more	much more
0%	3.6%	25.0%	53.6%	17.9%

Thanks For Your Attention!

References

1. Thomas J. Clark (2023) Falling Water: A Modeling Story, PRIMUS, 34:2, 168-181.
2. Thomas J. Clark (2019) Weighing Fog: Modeling on Day 1 of Differential Equations, PRIMUS, 29:7, 648-661.
3. Sania Qureshi; Brian Winkel (2016), "1-058-T-WaterClocks," https://www.simiode.org/resources/3144.
4. Brian Winkel (2017), "1-014-T-DrainingContainers," https://www.simiode.org/resources/3368.
5. Brian Winkel (2016), "1-140-T-LeakyBucket," https://www.simiode.org/resources/3131.
6. Brian Winkel (2017), "1-064-T-TorricelliBox," https://www.simiode.org/resources/3618.

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Any Questions?

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