Student Solutions

For Version 2.0 (HTML)

Section 1.4

Exercise Solution 1.4.1. *General solution* $u(t) = t^2/2 + C$*, particular solution* $u(t) = t^2/2 + 3$.

Exercise Solution 1.4.3. *General solution* $u(t) = e^t + C$*, particular solution* $u(t) = e^t + 3$.

Exercise Solution 1.4.5. *General solution* $u(t) = \sin(t) + C$*, particular solution* $u(t) = \sin(t) + 1$.

Exercise Solution 1.4.7. *General solution* $v(t) = gt$*, particular solution* $v(t) = gt + v_0$.

Exercise Solution 1.4.9. *General solution* $u(t) = t^3/6 + C_1t + C_2$*, particular solution* $u(t) = t^3/6 + 3t + 1$.

Exercise Solution 1.4.11. *General solution* $y(t) = -gt^2/2 + C_1t + C_2$*, particular solution* $y(t) = -gt^2/2 + 10$.

Exercise Solution 1.4.13. *The input salt rate to the tank is* $5\frac{\text{liter}}{\text{min}} \times 50\frac{\text{grams}}{\text{liter}} = 250\frac{\text{grams}}{\text{minute}}$. *The outflow rate of salt is* $5\frac{\text{liter}}{\text{min}} \times \frac{u(t)}{100}\frac{\text{grams}}{\text{liter}} = \frac{u(t)}{20}\frac{\text{grams}}{\text{minute}}$. *The ODE is*

$$u'(t) = 250 - \frac{u(t)}{20}$$

with initial condition $u(0) = 0$. *The solution is* $u(t) = 5000 - 5000e^{-t/20}$ *grams. The solution rises from* $u(0) = 0$ *and asymptotically approaches* $u = 5000$ *grams of salt in the tank. The limiting concentration is* $5000/100 = 50$ *grams per liter, the same as the incoming salt solution.*

Section 1.5

Exercise Solution 1.5.1.

(a) *Momentum is mass times velocity, so has dimension* MLT^{-1}.

(b) *Angular velocity is measured in radians per unit time, so has dimension* T^{-1}.

(c) *From force times distance we have* $[Fd] = [F][d] = MLT^{-2}L = ML^2T^{-2}$.

(d) *Pressure is force per area, so has dimension* $MLT^{-2}L^{-2} = ML^{-1}T^{-2}$.

Exercise Solution 1.5.3. *From* $v' = P - kv$ *we see that we need* $[v'] = [kv]$, *or* $LT^{-2} = [k]LT^{-1}$, *so* $[k] = T^{-1}$.

Exercise Solution 1.5.5. *The function* $u(t)$ *has dimension* M *(mass), so* $[u'(t)] = MT^{-1}$. *Also,* $[r] = L^3T^{-1}$ *(volume per time) and* $[c_1] = ML^{-3}$ *(mass per volume). Also* $[V] = L^3$. *Then* $[rc_1] = L^3T^{-1}ML^{-3} = MT^{-1}$ *and* $[ru/V] = L^3T^{-1}ML^{-3} = MT^{-1}$. *Thus each of* u', rc_1, *and* ru/V *has dimension* MT^{-1} *and the ODE is dimensionally consistent.*

In the solution $u(t) = c_1V(1-e^{-rt/V})$ *we find that* $[-rt/V] = L^3T^{-1}TL^{-3} = 1$, *so the argument to the exponential is dimensionless, and hence so is the quantity* $(1 - e^{-rt/V})$. *The quantity* $[c_1V] = ML^{-3}L^3 = M$ *has dimension mass, and this is consistent with* $[u] = M$.

Exercise Solution 1.5.7. *We have* $[P] = T$, $[2\pi] = 1$, $[r] = L$, $[G] = M^{-1}L^3T^{-2}$, *and* $[m] = M$. *Then*

$$[2\pi\sqrt{r^3/(Gm)}] = (1)L^{3/2}M^{1/2}L^{-3/2}T^1M^{-1/2} = T$$

which is $[P]$, *so this is dimensionally consistent.*

Exercise Solution 1.5.9. *We have* $[P] = T$, $[\ell] = L$, $[m] = M$, *and* $[g] = LT^{-2}$. *A formula of the form* $P = \ell^a m^b g^c$ *requires* $T = L^aM^bL^cT^{-2c}$, *which leads to* $b = 0, a + c = 0, -2c = 1$, *so* $a = 1/2, b = 0, c = -1/2$, *and then*

$$P = K\sqrt{\ell/g}$$

for some dimensionless constant K *For the "linearized pendulum" this is correct, with* $K = 2\pi$*; for the general nonlinear pendulum this is also correct, but* K *depends on the initial angle of the pendulum.*

Exercise Solution 1.5.11. *We have* $[f] = T^{-1}$, $[\lambda] = ML^{-1}$, $[\tau] = MLT^{-2}$, *and* $[\ell] = L$. *Then* $f = \lambda^a \tau^b \ell^c$ *forces* $T^{-1} = M^a L^{-a} M^b L^b T^{-2b} L^c$ *or*

$$a + b = 0, \quad , -a + b + c = 0, \quad -2b = -1$$

with solution $a = -1/2, b = 1/2$, *and* $c = -1$. *Then*

$$f = \frac{K}{\ell}\sqrt{\tau/\lambda}$$

for some dimensionless constant K *(which turns out as* $K = 1/2$ *in ideal situations.)*

Section 2.1

Exercise Solution 2.1.1. *Integrating factor* e^{-t}*, general solution* $u(t) = Ce^t - 3$*, specific solution is* $u(t) = 6e^t - 3$*.*

Exercise Solution 2.1.3. *Integrating factor* e^{3t}*, general solution* $u(t) = Ce^{-3t} + 1$*, specific solution is* $u(t) = 4e^{-3t} + 1$*.*

Exercise Solution 2.1.5. *Integrating factor* e^{-t}*, general solution* $u(t) = Ce^t - \sin(t) - \cos(t)$*, specific solution is* $u(t) = 2e^t - \sin(t) - \cos(t)$*.*

Exercise Solution 2.1.7. *Integrating factor* $e^{-t^2/2}$*, general solution* $u(t) = Ce^{t^2/2} - 1$*, specific solution is* $u(t) = 3e^{t^2/2} - 1$*.*

Exercise Solution 2.1.9. *Integrating factor* $e^{-\cos(t)}$*, general solution* $u(t) = Ce^{-\cos(t)} - 1$*, specific solution is* $u(t) = 5e^1 e^{-\cos(t)} - 1 = 5e^{1-\cos(t)} - 1$*.*

Exercise Solution 2.1.12.

(a) $[k] = T^{-1}$.

(b) Write the ODE as $u'(t) + ku(t) = 0$ *and use integrating factor* e^{kt} *to find* $u(t) = Ce^{-kt}$*, Then* $u(0) = u_0$ *implies* $C = u_0$*, so* $u(t) = u_0 e^{-kt}$*. Since* k *is positive the exponential decays to zero as* t *increases to infinity.*

(c) The equation $u(t + \Delta t) = u(t)/2$ *becomes* $u_0 e^{-k(t+\Delta t)} = u_0 e^{-kt}/2$*, which simplifies to* $e^{-k\Delta t} = 1/2$*. Solve for* $\Delta t = \ln(2)/k$*. This does not depend on the variable* t *itself.*

Exercise Solution 2.1.14. *Write the ODE as* $x'(t) + x(t)/100 = 0.2$ *and use integrating factor* $e^{t/100}$ *to find* $d(e^{t/100}x(t))/dt = 0.2e^{t/100}$*. Integrate to find* $e^{t/100}x(t) = 20e^{t/100} + C$ *and so* $x(t) = 20 + Ce^{-t/100}$ *is the general solution. Then* $x(0) = 3$ *yields* $20 + C = 3$*, so* $C = -17$ *and* $x(t) = 20 - 17e^{-t/100}$*.*

Exercise Solution 2.1.16. *The rate in is* $(0.2)(4) = 0.8$ *kg per minute, and the rate out is* $(x(t)/400)(4) = x(t)/100$ *kg per minute. The ODE is* $x'(t) = 0.8 - x(t)/100$ *with* $x(0) = 0$*. The solution is* $x(t) = 80 - 80e^{-t/100}$*. The amount of salt limits to* 80 *kg.*

Exercise Solution 2.1.19.

(a) *Write the ODE as* $q'(t) + q(t)/RC = V_0/R$ *and use integrating factor* $e^{t/RC}$ *to obtain*

$$\frac{d}{dt}(q(t)e^{t/RC}) = (V_0/R)e^{t/RC}.$$

Integrate to find

$$e^{t/RC}q(t) = V_0Ce^{t/RC} + A$$

for some arbitrary constant of integration A. *The general solution is then* $q(t) = V_0C + Ae^{-t/RC}$. *If* $q(0) = 0$ *then* $A = -V_0C$ *and the solution is* $q(t) = V_0C(1 - e^{-t/RC})$.

(b) *As* $t \to \infty$ *we find* $q(t) \to V_0C$.

(c) *With* $[C] = [q]/[V] = M^{-1}L^{-2}T^2Q^2$ *and* $[R] = ML^2T^{-1}Q^{-2}$ *we find* $[RC] = [R][C] = T$.

(d) *This occurs when* $e{-}t/RC = 1/100$, *which leads to* $t = RC\ln(100) \approx 4.6RC$.

Section 2.2

Exercise Solution 2.2.1. *General solution* $u(t) = Ce^t - 3$*, specific solution is* $u(t) = 6e^t - 3$.

Exercise Solution 2.2.3. *General solution* $u(t) = Ce^{-3t} + 1$*, specific solution is* $u(t) = 4e^{-3t} + 1$.

Exercise Solution 2.2.5. *General solution* $u(t) = Ce^{-\cos(t)} - 1$*, specific solution is* $u(t) = 5e^1 e^{-\cos(t)} - 1 = 5e^{1-\cos(t)} - 1$.

Exercise Solution 2.2.7. *General solution* $u(t) = Ce^{-\cos(t)}$*, specific solution is* $u(t) = e^1 e^{-\cos(t)} = e^{1-\cos(t)}$.

Exercise Solution 2.2.9. *General solution* $u(t) = e^{e^t}$*, specific solution is* $u(t) = 3e^{e^t - 1}$.

Exercise Solution 2.2.11. *Separate variables as* $dv/(P - kv) = dt$ *and integrate to find* $-\frac{1}{k}\ln|P - kv| = t + C$*. Then* $\ln|P - kv| = -kt + C$ *and so* $P - kv = Ce^{-kt}$ *(*$C \neq 0$*, but again,* $C = 0$ *is permissible, it corresponds to* $v(t) = P/k$*). Solve for* $v = P/k + Ce^{-kt}$ *and then* $v(0) = 0$ *implies* $C = -P/k$*, so* $v(t) = \frac{P}{k}(1 - e^{-kt})$.

Exercise Solution 2.2.13. *It's much easier to take the hint. With* $\tilde{r} = r - h$ *and* $\tilde{K} = ((1 - h/r)K$ *we find that*

$$u' = \tilde{r}u(1 - u/\tilde{K}) = (r-h)u(1 - ru/K(r-h)) = (r-h)u - ru/K = ru(1-u/K) - hu$$

which is the harvested logistic equation. The solution to the "standard" logistic equation $u' = \tilde{r}u(1 - u/\tilde{K})$ *is*

$$\begin{aligned} u(t) &= \frac{\tilde{K}}{1 + e^{-\tilde{r}t}(\tilde{K}/u_0 - 1)} \\ &= \frac{(1 - h/r)K}{1 + e^{-(r-h)t}(\frac{K}{u_0}(1 - h/r) - 1)}. \end{aligned}$$

Exercise Solution 2.2.15. *Separate as* $dx/(0.2 - x/100) = dt$ *and integrate to find* $-100\ln|0.2 - x/100| = t + C$*. Solve for* x *as* $x = 20 - Ce^{-t/100}$*. Then* $x = 3$ *when* $t = 0$ *yields* $C = 17$*, so* $x(t) = 20 - 17e^{-t/100}$.

Section 2.3

Exercise Solution 2.3.1. *The ODE is* $u' = f(t, u)$ *with* $f(t, u) = u - 2t$*. Then* $f(0, 0) = 0, f(0, 1) = 1, f(1, 0) = -2, f(1, 1) = -1$*. Crude slope field shown in Figure 2.1.*

Exercise Solution 2.3.3. *The ODE is* $u' = f(t, u)$ *with* $f(t, u) = -u$*. Then* $f(0, 1) = -1, f(0, 2) = -2, f(1, 1) = -1, f(1, 3) = -3$*. Crude slope field shown in Figure 2.2.*

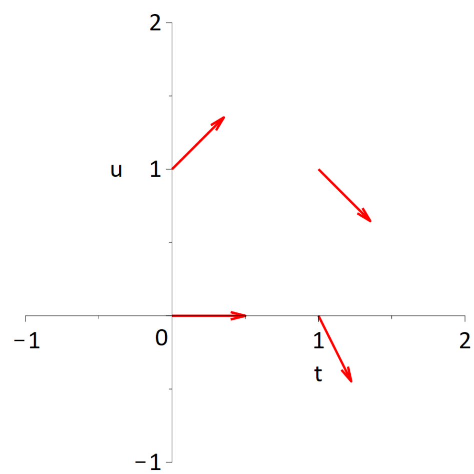

Figure 2.1: Slope field for Exercise 2.3.1.

Exercise Solution 2.3.5. *Slope field shown in Figure 2.3.*

Exercise Solution 2.3.7. *Slope field shown in Figure 2.4. In this case* $u = 0$ *is an equilibrium solution.*

Exercise Solution 2.3.9. *Slope field shown in Figure 2.5. In this case* $u = 0$ *and* $u = 3$ *are equilibrium solutions.*

Exercise Solution 2.3.11. *Slope field shown in Figure 2.6. In this case* $u = 0$ *and* $u = 3$ *are equilibrium solutions.*

Exercise Solution 2.3.13. *The phase portrait is in Figure 2.7, solutions with* $u(0) = 2$ *and* $u(0) = -2$ *in the right panel.*

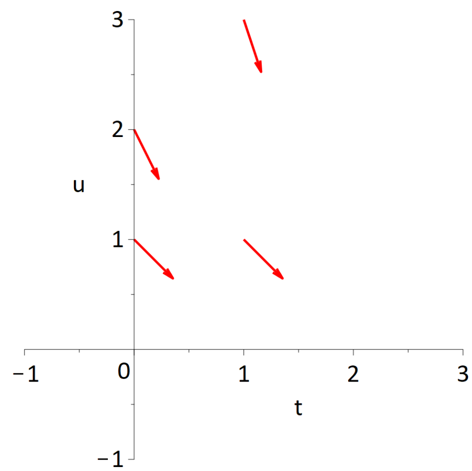

Figure 2.2: Slope field for Exercise 2.3.3.

Exercise Solution 2.3.15. *The phase portrait is in Figure 2.8, solutions with* $v(0) = 0$ *and* $v(0) = 15/k$ *in the right panel.*

Exercise Solution 2.3.17. *The phase portrait is in Figure 2.9, solutions with* $u(0) = 1/2$, $u(0) = 3/2$ *in the right panel.*

Exercise Solution 2.3.19. *See Figure 2.10. Solution with* $u(0) = 0$ *increases asymptotically to equilibrium at* $u = c_1V$*, solution with* $u(0) = 2c_1V$ *decreases asymptotically to equilibrium at* $u = c_1V$*.*

Exercise Solution 2.3.21. *Take* $u' = (u-1)(u-3)$ *(the right side can be multiplied by any positive constant).*

Exercise Solution 2.3.23. *Take* $u' = -(u-1)^2(u-3)$ *(the right side can be multiplied by any positive constant).*

Exercise Solution 2.3.25. *The ODE is* $u' = f(u)$ *with* $f(u) = hu - u^2$*. Here* $u = 0$ *and* $u = h$ *are always the only fixed points. We have* $f'(u) = h - 2u$*. For* $h > 0$ *the fixed point at* 0 *is unstable (*$f'(0) = h$*) and the fixed point at* $u = h$ *is stable (*$f'(h) = -h$*). For* $h < 0$ *the stability is reversed. A bifurcation occurs at* $h = 0$*. See Figure 2.11 for the bifurcation diagram.*

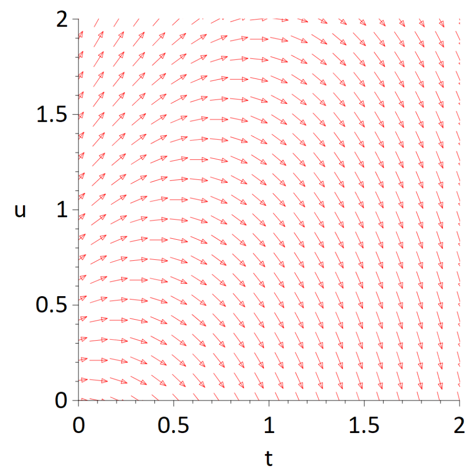

Figure 2.3: Slope field for Exercise 2.3.5.

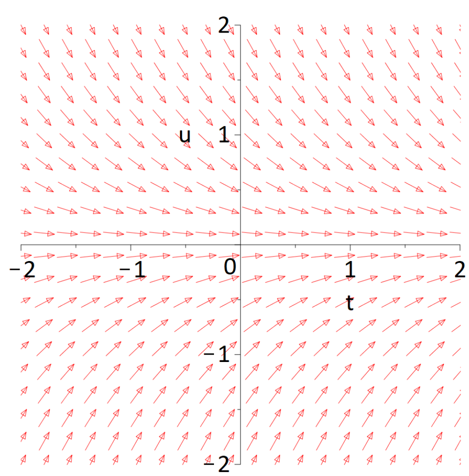

Figure 2.4: Slope field for Exercise 2.3.7.

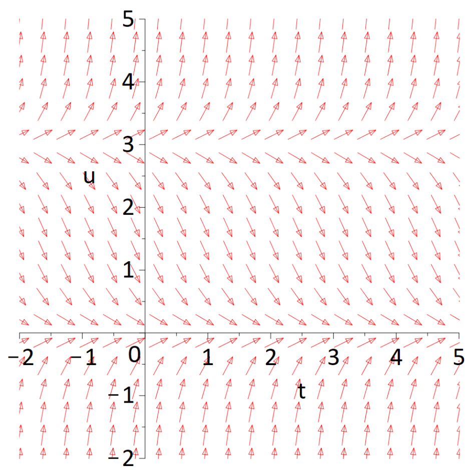

Figure 2.5: Slope field for Exercise 2.3.9.

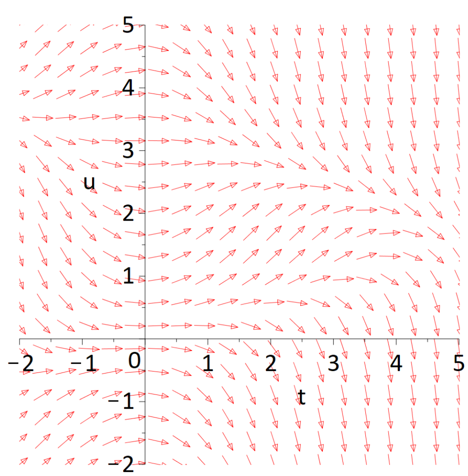

Figure 2.6: Slope field for Exercise 2.3.11.

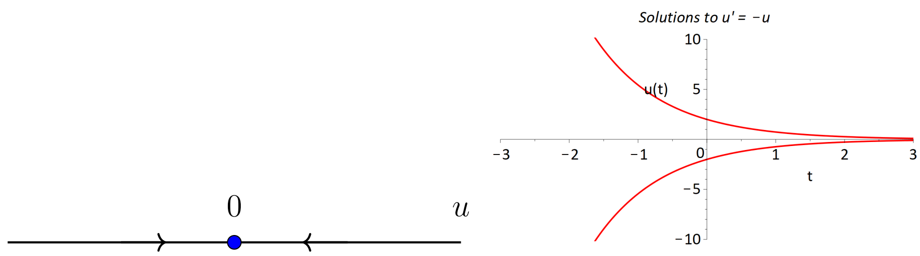

Figure 2.7: Phase portrait for $u' = -u$ (left) and some solutions (right).

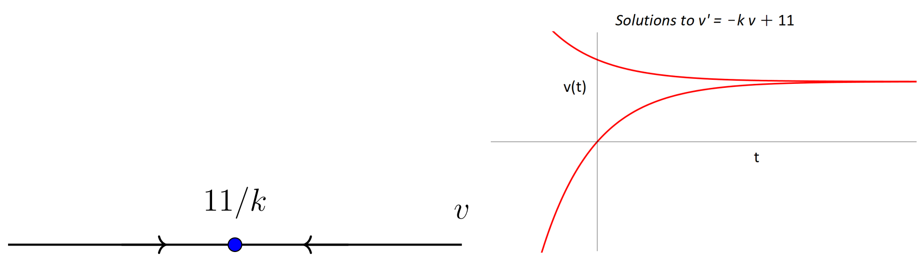

Figure 2.8: Phase portrait for $v' = 11 - kv$ (left) and some solutions (right).

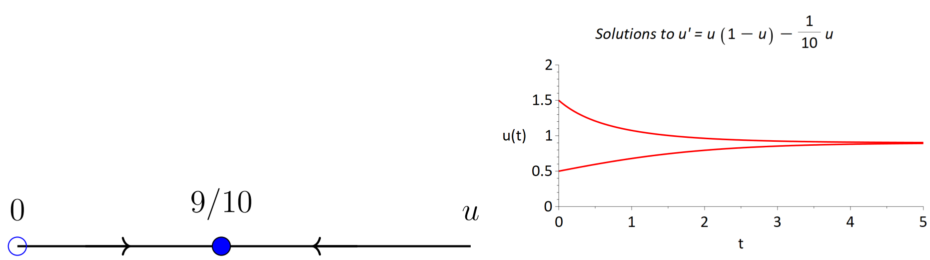

Figure 2.9: Phase portrait for $u'(t) = u(t)(1 - u(t)) - u(t)/10$ (left) and some solutions (right).

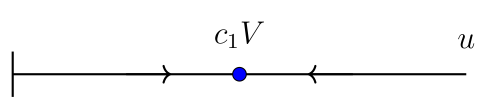

Figure 2.10: Phase portrait for $u'(t) = rc_1 - ru(t)/V$.

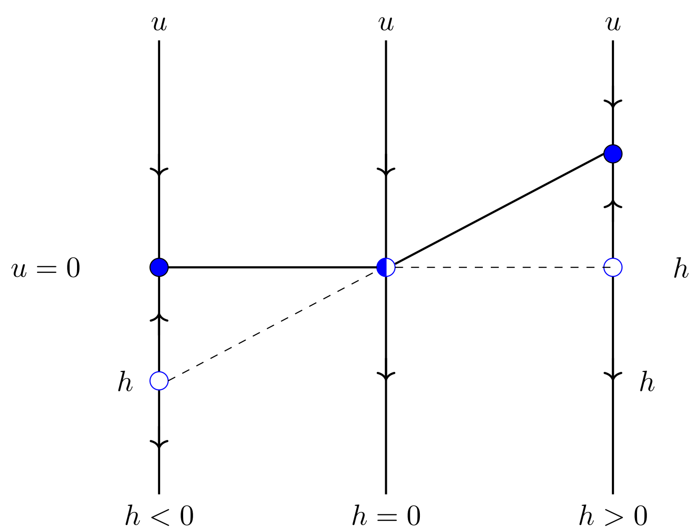

Figure 2.11: Bifurcation diagram for $u' = hu - u^2$.

Section 2.4

Exercise Solution 2.4.1. *Here $f(t,u) = u + 3$, which is continuous for all u and t. Also $\frac{\partial f}{\partial u} = 1$, also continuous everywhere.*

Exercise Solution 2.4.3. *Here $f(t,u) = 1/u$, which is continuous near $u = 2$ (everywhere except $u = 0$). Also $\frac{\partial f}{\partial u} = 1/u^2$, which is continuous near $u = 2$.*

Exercise Solution 2.4.6. *Solution is $u(t) = 2$, maximum domain $-\infty < t < \infty$.*

Exercise Solution 2.4.8. *Solution is $u(t) = -\ln(1-t)$, maximum domain $-\infty < t < 1$.*

Section 3.1

Exercise Solution 3.1.1. *Find* $u_2 = 6.0$*, true solution is* $u(t) = 4e^t - 3$ *with* $u(1) \approx 7.873$*.*

Exercise Solution 3.1.3. *Find* $u_4 = 2.460$*, true solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) \approx 2.449$*.*

Exercise Solution 3.1.5. *True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) \approx 2.811124397$*. With* $h = 1, 0.1, 0.01$ *Euler estimates are* $2.8683, 2.8164, 2.8116$*, errors*
$0.0572, 0.005291, 0.000525$*, roughly. This is consistent with first order accuracy.*

Exercise Solution 3.1.7. *True solution is* $u(t) = 2/(1-2t)$*, which has an asymptote at* $t = 1/2$*. With* $h = 0.5, 0.1, 0.01, 0.001$ *the Euler estimates are* $4, 8.2182, 36.257, 217.64$*. It's clear the Euler's method is reproducing the asymptotic blow-up.*

Exercise Solution 3.1.11. *The true solution is* $u(t) = 1/(1-t)$*, but the maximum domain of this solution is* $(-\infty, 1)$ *(given that we started at* $t = 0$*). Euler's Method with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates for* $u(1)$ *equal to* $2, 6.13, 30.39$*, and* 193.1*. For* $u(2)$ *we obtain* $6, 5.65 \times 10^{103}, \infty, \infty$ *(the last two are really floating point overflow.) All Euler estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

Section 3.2

Exercise Solution 3.2.1. *Find* $u_1 = 3.5, u_2 = 7.5625$. *True solution is* $u(t) = 4e^t - 3$ *with* $u(1) \approx 7.873$.

Exercise Solution 3.2.3. *Find* $u_1 = 2.12132, u_2 = 2.23607, u_3 = 2.34521, u_4 = 2.44950$. *True solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) = \sqrt{6} \approx 2.44950$.

Exercise Solution 3.2.5. *For* $h = 1$ *we find approximation* 2.8035*; for* $h = 0.1$*,* 2.81106*; for* $h = 0.01$*,* 2.81112. *True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) = 3e^{-5/3} \approx 2.81112$.

Exercise Solution 3.2.7. *For* $h = 0.5$ *we find approximation* 7.0*; for* $h = 0.1$*,* 23.76*; for* $h = 0.01$*,* 211.2*; for* $h = 0.001$*,* 2086. *True solution is* $u(t) = \frac{1}{1/2-t}$ *and* $u(0.5)$ *is undefined (*u *limits to* ∞ *as* $t \to 1/2$ *from the left). Clearly the improved Euler iterates try to track this.*

Exercise Solution 3.2.10. *The true solution is* $u(t) = 1/(1-t)$*, but the maximum domain of this solution is* $(-\infty, 1)$ *(given that we started at* $t = 0$*). The improved Euler method with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates for* $u(2)$ *equal to* $133.65, \infty, \infty, \infty$ *(the last three are really floating point overflow.) All improved Euler estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

Section 3.3

Exercise Solution 3.3.1. *Find* $u_2 = 7.8694$*, true solution is* $u(t) = 4e^t - 3$ *with* $u(1) = 4e - 3 \approx 7.8731$.

Exercise Solution 3.3.3. *Find* $u_4 = 2.44949$*, true solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) = \sqrt{6} \approx 2.44949$.

Exercise Solution 3.3.5. *For* $h = 1$ *we find approximation* 2.81108*; for* $h = 0.1$*,* 2.81112*; for* $h = 0.01$*,* 2.81112*. True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) = 3e^{-5/3} \approx 2.81112$.

Exercise Solution 3.3.7. *For* $h = 0.5$ *we find approximation* 16.98*; for* $h = 0.1$*,* 82.03*; for* $h = 0.01$*,* 819.9*; for* $h = 0.001$*,* 8199.1*. True solution is* $u(t) = \frac{1}{1/2-t}$ *and* $u(0.5)$ *is undefined (*u *limits to* ∞ *as* $t \to 1/2$ *from the left). Clearly RK4 tries to track this.*

Exercise Solution 3.3.10. *The true solution is* $u(t) = 1/(1-t)$*, but the maximum domain of this solution is* $(-\infty, 1)$ *(given that we started at* $t = 0$*). The RK4 method with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates for* $u(2)$ *equal to* $1.67 \times 10^{11}, \infty, \infty, \infty$ *(the last three are really floating point overflow.) All RK4 estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

Section 3.4

Exercise Solution 3.4.1.

(a) *The sum of squares function is*

$$S(a) = (0.1a - 0.11)^2 + (0.6a - 0.5)^2 + (1.1a - 0.6)^2 + (1.4a - 0.5)^2.$$

Setting $S'(a) = 0$ *yields minimizer* $a \approx 0.472$*, easily confirmed with a graph of* $S(a)$*. The residual is* 0.0833*. The fit to the data is shown in Figure 3.12, left panel.*

(b) *The sum of squares function is*

$$S(a, b) = (0.1a + b - 0.11)^2 + (0.6a + b - 0.5)^2 + (1.1a + b - 0.6)^2 + (1.4a + b - 0.5)^2.$$

Setting $\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$ *and solving for* a *and* b *yields minimizer* $a \approx 0.309, b \approx 0.180$*, easily confirmed with a graph of* $S(a, b)$*. The residual is* 0.0474*. Of course this residual is smaller since throwing* b *into the computation gives us "more to work with" when fitting the data (informally). The fit to the data is shown in Figure 3.12, right panel.*

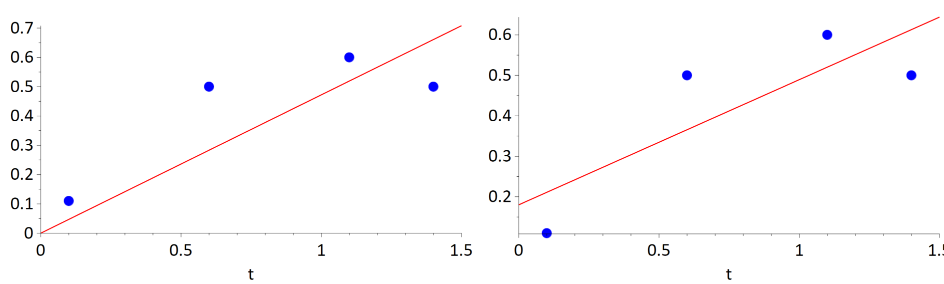

Figure 3.12: Best fit to data for Exercise 3.4.1, $u(t) = at$ (left panel) and $u(a, b, t) = at + b$ (right panel).

Exercise Solution 3.4.3. *Forming an appropriate sum of squares* $S(k, P)$ *and minimizing by solving* $\frac{\partial S}{\partial k} = 0, \frac{\partial S}{\partial P} = 0$ *yields minimizer* $P \approx 8.5997, k \approx 0.8072$*. A plot of the Hill-Keller solution with these parameters and the data is shown in Figure 3.13.*

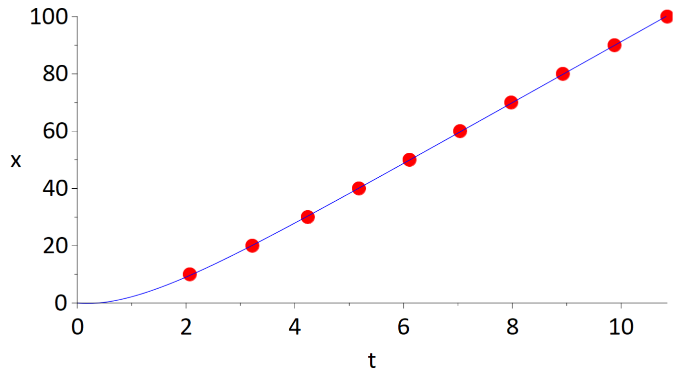

Figure 3.13: Position $x(t)$ from Hill-Keller solution with $P = 8.5997, k = 0.8072$ (blue) and data from Tori Bowie's 2017 race (red).

Exercise Solution 3.4.5. *From the hint it's easy to see that*

$$S''(m) = 2\sum_{j=1}^{n} x_j^2.$$

If any x_j is nonzero then this quantity is positive. Also, given that $S(m)$ is of the form $Am^2 + Bm + C$ where $A > 0$, it's clear that $S(m)$ limits to infinity as $m \to \pm\infty$.

Section 4.1

Exercise Solution 4.1.1. *Suppose the mass is at position $u(t)$ at time t. In this position the spring on the left exerts force $-k_1 u$ (pulling the mass back to the left if $u > 0$, pushing it right if $u < 0$) and the spring on the right exerts a similar force $-k_2 u$. If $u' > 0$ (mass moving to the right) then the dashpot on the left exerts force $-c_1 u'$, and the dashpot on the right exerts force $-c_2 u'$. The total force on the mass is thus $-(k_1 + k_2)u - (c_1 + c_2)u'$, and Newton's Second Law yields $mu'' = -(k_1 + k_2)u - (c_1 + c_2)u'$ or*

$$mu'' + (c_1 + c_2)u' + (k_1 + k_2)u = 0.$$

Exercise Solution 4.1.3.

(a) *The ODE is*

$$5000u''(t) + (2 \times 10^4)u'(t) + (5 \times 10^5)u = 0.$$

(b) *Compute*

$$\begin{aligned} u(t) &= \frac{\sqrt{6}e^{-2t}}{1200}\sin(4\sqrt{6}t) + \frac{e^{-2t}}{100}\cos(4\sqrt{6}t) \\ u'(t) &= -\frac{\sqrt{6}}{24}e^{-2t}\sin(4\sqrt{6}t) \\ u''(t) &= \frac{\sqrt{6}e^{-2t}}{12}\sin(4\sqrt{6}t) - e^{-2t}\cos(4\sqrt{6}t). \end{aligned}$$

Simple algebra shows that the ODE is satisfied (write the ODE as $5000(u''(t) + 4u'(t) + 100u(t)) = 0$). A plot of the solution is shown in the left panel of Figure 4.14.

(c) *The building goes through a full oscillation in P seconds where $4\sqrt{6}P = 2\pi$, so $P = \pi/(2\sqrt{6}) \approx 0.64$ seconds.*

(d) *The acceleration $u''(t)$ is graphed in the middle panel of Figure 4.14. Maximum occurs initially, 1 meter per second squared, about $1/9.8 \approx 0.102$ g's.*

(e) *The ODE is now*

$$5000u''(t) + (5 \times 10^5)u = 0.$$

A solution of the form $u(t) = u_0\cos(\omega)$ exists if $\omega = 10$, and taking $u_0 = 0.01$ yields the initial data. The solution is graphed in the right panel of Figure 4.14.

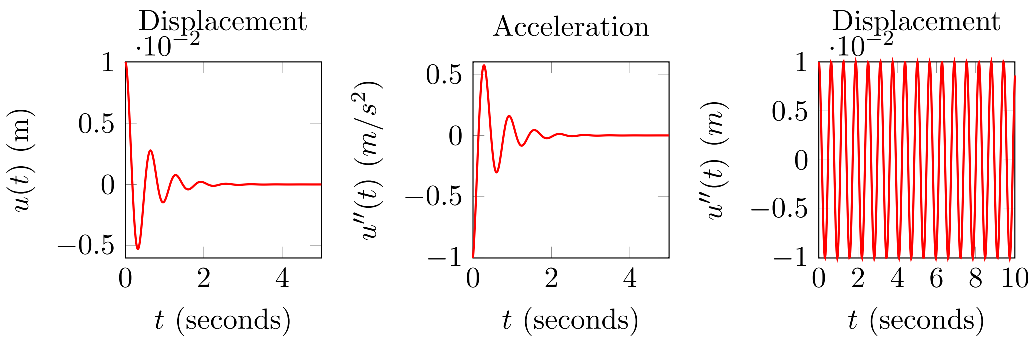

Figure 4.14: Solution $u(t) = \frac{\sqrt{6}e^{-2t}}{1200}\sin(4\sqrt{6}t) + \frac{e^{-2t}}{100}\cos(4\sqrt{6}t)$ (left panel) and $u''(t)$ (middle panel), undamped displacement (right panel).

Exercise Solution 4.1.5. *The ODE is*

$$10^{-3}q''(t) + 10q'(t) + 10^4 q(t) = 3.$$

And equilibrium solution $q(t) = q^*$ *occurs when* $10^4 q^* = 3$ *(since* $q'' = q' = 0$*) and so* $q^* = 3\times 10^{-4}$ *coulombs. The current in the circuit is* $I(t) = q'(t) = 0$.

Section 4.2

Exercise Solution 4.2.1. *ODE is* $3u''(t) + 24u'(t) + 60u(t) = 0$*, characteristic equation* $3r^2 + 24r + 60 = 0$*, roots* $-4 \pm 2i$*, underdamped.*

Exercise Solution 4.2.3. *ODE is* $2u''(t) + 12u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 12r + 10 = 0$*, roots* $-1, -5$*, overdamped.*

Exercise Solution 4.2.5. *ODE is* $2u''(t) + 4u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 4r + 10 = 0$*, roots* $-1 \pm 2i$*, underdamped.*

Exercise Solution 4.2.7. *ODE is* $2u''(t) + 12u'(t) + 18u(t) = 0$*, characteristic equation* $2r^2 + 12r + 18 = 0$*, double root* -3*, critically damped.*

Exercise Solution 4.2.9. *ODE is* $2u''(t)+8u'(t)+6u(t) = 0$*, characteristic equation* $2r^2 + 8r + 6 = 0$*, roots* $-1, -3$*, overdamped.*

Exercise Solution 4.2.11. *ODE is* $u''(t)+6u'(t)+8u(t) = 0$*, characteristic equation* $r^2+6r+8 = 0$*, roots* $-2, -4$*, general solution* $u(t) = c_1e^{-2t}+c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-2t}/2 - 7e^{-4t}/2$*.*

Exercise Solution 4.2.13. *ODE is* $2u''(t) + 10u'(t) + 12u(t) = 0$*, characteristic equation* $2r^2 + 10r + 12 = 0$*, roots* $-2, -3$*, general solution* $u(t) = c_1e^{-2t} + c_2e^{-3t}$*. Specific solution is* $u(t) = 9e^{-2t} - 7e^{-3t}$*.*

Exercise Solution 4.2.15. *ODE is* $2u''(t) + 10u'(t) + 8u(t) = 0$*, characteristic equation* $2r^2 + 10r + 86 = 0$*, roots* $-1, -4$*, general solution* $u(t) = c_1e^{-t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-t}/3 - 5e^{-4t}/3$*.*

Exercise Solution 4.2.17. *ODE is* $3u''(t) + 18u'(t) + 24u(t) = 0$*, characteristic equation* $3r^2 + 18r + 24 = 0$*, roots* $-2, -4$*, general solution* $u(t) = c_1e^{-2t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-2t}/2 - 7e^{-4t}/2$*.*

Exercise Solution 4.2.19. *ODE is* $u''(t)+4u'(t)+5u(t) = 0$*, characteristic equation* $r^2 + 4r + 5 = 0$*, roots* $-2 \pm i$*, general solution* $u(t) = c_1e^{(-2+i)t} + c_2e^{(-2-i)t}$*. Specific solution is* $u(t) = (1 - 4i)e^{(-2+i)t} + (1 + 4i)e^{(-2-i)t}$*. The real-valued general solution is* $u(t) = d_1e^{-2t}\cos(t) + d_2e^{-2t}\sin(t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(t)+8e^{-2t}\sin(t)$*.*

Exercise Solution 4.2.21. *ODE is* $2u''(t) + 16u'(t) + 64u(t) = 0$*, characteristic equation* $2r^2 + 16r + 64 = 0$*, roots* $-4 \pm 4i$*, general solution* $u(t) = c_1e^{(-4+4i)t} + c_2e^{(-4-4i)t}$*. Specific solution is* $u(t) = (1 - 3i/2)e^{(-4+4i)t} + (1 + 3i/2)e^{(-4-4i)t}$*. The real-valued general solution is* $u(t) = d_1e^{-4t}\cos(4t) + d_2e^{-4t}\sin(4t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-4t}\cos(4t) + 3e^{-4t}\sin(4t)$*.*

Exercise Solution 4.2.23. *ODE is* $2u''(t) + 8u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 8r + 10 = 0$*, roots* $-2 \pm i$*, general solution* $u(t) = c_1 e^{(-2+i)t} + c_2 e^{(-2-i)t}$*. Specific solution is* $u(t) = (1-4i)e^{(-2+i)t} + (1+4i)e^{(-2-i)t}$*. The real-valued general solution is* $u(t) = d_1 e^{-2t}\cos(t) + d_2 e^{-2t}\sin(t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(t) + 8e^{-2t}\sin(t)$*.*

Exercise Solution 4.2.25. *ODE is* $2u''(t) + 16u'(t) + 50u(t) = 0$*, characteristic equation* $2r^2 + 16r + 50 = 0$*, roots* $-4 \pm 3i$*, general solution* $u(t) = c_1 e^{(-4+3i)t} + c_2 e^{(-4-3i)t}$*. Specific solution is* $u(t) = (1-2i)e^{(-4+3i)t} + (1+2i)e^{(-4-3i)t}$*. The real-valued general solution is* $u(t) = d_1 e^{-4t}\cos(3t) + d_2 e^{-4t}\sin(3t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-4t}\cos(3t) + 4e^{-4t}\sin(3t)$*.*

Exercise Solution 4.2.27. *ODE is* $u''(t) + 4u'(t) + 4u(t) = 0$*, characteristic equation* $r^2 + 4r + 4 = 0$*, double root* -2*, general solution* $u(t) = c_1 e^{-2t} + c_2 t e^{-2t}$*. Specific solution is* $u(t) = 2e^{-2t} + 8te^{-2t}$*.*

Exercise Solution 4.2.29. *ODE is* $2u''(t) + 8u'(t) + 8u(t) = 0$*, characteristic equation* $2r^2 + 8r + 8 = 0$*, double root* -2*, general solution* $u(t) = c_1 e^{-2t} + c_2 t e^{-2t}$*. Specific solution is* $u(t) = 2e^{-2t} + 8te^{-2t}$*.*

Exercise Solution 4.2.31.

(a) *The ODE is* $20000u''(t) + 80000u'(t) + 60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 4r + 3) = 20000(r+1)(r+3) = 0$*, roots* $r = -1, -3$*. The general solution to the ODE is* $u(t) = c_1 e^{-t} + c_2 e^{-3t}$ *and the initial data requires* $c_1 + c_2 = 0, -c_1 - 3c_2 = 0.1$*, solution* $c_1 = 0.05, c_2 = -0.05$*. The solution is thus* $u(t) = 0.05e^{-t} - 0.05e^{-3t}$*. This system is overdamped. A plot of* $u(t)$ *is shown in the left panel of Figure 4.15.*

(b) *The ODE is* $20000u''(t) + 40000u'(t) + 60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 2r + 3) = 0$*, roots* $r = -1 \pm i\sqrt{2}$*. The general solution to the ODE is* $u(t) = c_1 e^{(-1+i\sqrt{2})t} + c_2 e^{(-1-i\sqrt{2})t}$ *and the initial data requires* $c_1 + c_2 = 0, (-1 + i\sqrt{2})c_1 + (-1 - i\sqrt{2})c_2 = 0.1$*, solution* $c_1 = -i\sqrt{2}/40 \approx -0.0353i, c_2 = i\sqrt{2}/40 \approx 0.0353i$*. The real-valued version of the solution is* $u(t) = \sqrt{2}e^{-t}\sin(t\sqrt{2})/20$*. This system is underdamped. A plot of* $u(t)$ *is shown in the right panel of Figure 4.15.*

(c) *The ODE is* $20000u''(t) + 60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 3) = 0$*, roots* $r = \pm i\sqrt{3}$*.*

The general solution to the ODE is $u(t) = c_1 e^{it\sqrt{3}} + c_2 e^{-it\sqrt{3}}$ *and the initial data requires* $c_1 + c_2 = 0, i\sqrt{3}c_1 - i\sqrt{3}c_2 = 0.1$*, solution* $c_1 = -i\sqrt{3}/60 \approx -0.0289i, c_2 = i\sqrt{6}/60 \approx 0.0289i$*. The real-valued version of the solution is* $u(t) = \sqrt{3}\sin(t\sqrt{3})/30$*. This system is underdamped. A plot of* $u(t)$ *is shown in the left panel of Figure 4.16.*

(d) *The choice* $c = 40000\sqrt{3} \approx 69282$ *yields a critically damped system. The ODE is* $20000u''(t) + 40000\sqrt{3}u'(t) + 60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 2\sqrt{3}r + 3) = 0$*, double root* $r = -\sqrt{3}$*. The general solution to the ODE is* $u(t) = c_1 e^{-t\sqrt{3}} + c_2 t e^{-t\sqrt{3}}$ *and the initial data requires* $c_1 = 0$ *and* $c_2 = 1/10$*. The solution is* $u(t) = te^{-t\sqrt{3}}/10$*. A plot of* $u(t)$ *is shown in the right panel of Figure 4.16.*

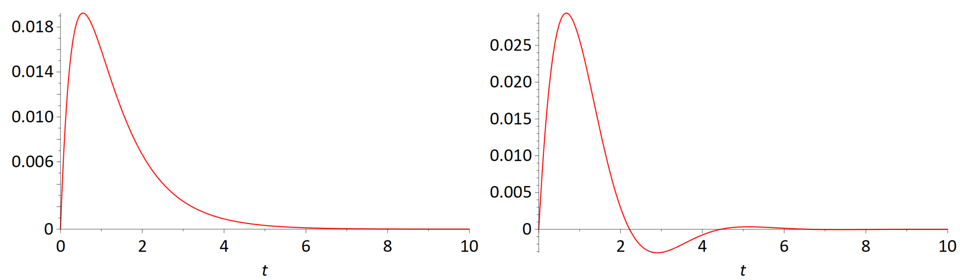

Figure 4.15: Solution to $20000u''(t) + 80000u'(t) + 60000u(t) = 0$ (left) and $20000u''(t) + 40000u'(t) + 60000u(t) = 0$ (right), both with $u(0) = 0, u'(0) = 0.1$.

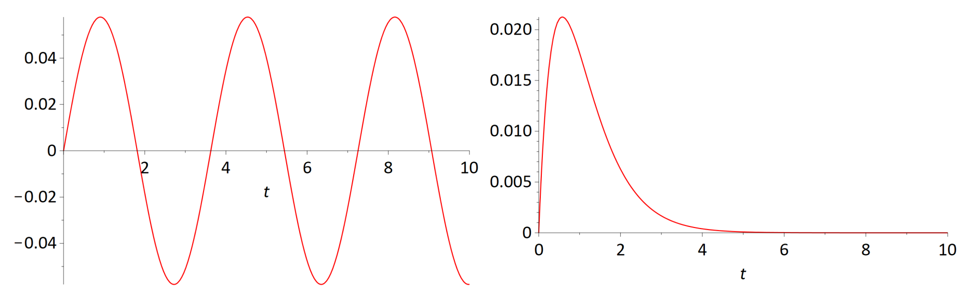

Figure 4.16: Solution to $20000u''(t) + 60000u(t) = 0$ (left) and $20000u''(t) + 40000\sqrt{3}u'(t) + 60000u(t) = 0$ (right), both with $u(0) = 0, u'(0) = 0.1$.

Exercise Solution 4.2.33.

(a) *This system is an undamped spring-mass system.*

(b) *The characteristic equation is* $r^2 + gr/L = 0$ *with roots* $r = \pm i\sqrt{g/L}$*. The general solution will be of the form*

$$\theta(t) = c_1 \cos(t\sqrt{g/L}) + c_2 \sin(t\sqrt{g/L}).$$

(c) *The period is* $P = 2\pi/\sqrt{g/L} = 2\pi\sqrt{L/g}$*. This makes perfect sense: period increases as* L *increases, decreases as* g *decreases. Moreover,* $[g] = LT^{-2}$*,* $[L] = L$*, and so* $[P] = T$*.*

Exercise Solution 4.2.35.

(a) *The identity* $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ *with* $x = \omega t$ *and* $y = \phi$ *becomes (after multiplying by* C*)*

$$C\sin(\omega t + \phi) = C\sin(\omega t)\cos(\phi) + C\cos(\omega t)\sin(\phi).$$

Comparison of the right side above to $A\cos(\omega t) + B\sin(\omega t)$ *shows they will be identical as functions of* t *is* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$*.*

(b) *Squaring each side of each of* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$ *and adding yields* $C^2 = A^2 + B^2$*, so* $C = \sqrt{A^2 + B^2}$*.*

(c) *Take the quotient of the left and right sides of* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$ *to obtain* $\tan(\phi) = A/B$ *or* $\phi = \arctan(A/B)$ *if* $B > 0$*. If* $B < 0, A > 0$ *then* $\phi = \arctan(A/B) + \pi$*, while if* $B < 0, A < 0$ *then* $\phi = \arctan(A/B) - \pi$*.*

Section 4.3

Exercise Solution 4.3.1. $u_h(t) = c_1e^{-4t} + c_2e^{-5t}$, $u_p(t) = e^{-3t}$. *General solution* $u(t) = e^{-3t}+c_1e^{-4t}+c_2e^{-5t}$, *specific solution* $u(t) = e^{-3t}+11e^{-4t}-10e^{-5t}$.

Exercise Solution 4.3.3. $u_h(t) = c_1e^{-4t}\cos(4t)+c_2e^{-4t}\sin(4t)$, $u_p(t) = 1$. *General solution* $u(t) = 1 + c_1e^{-4t}\cos(4t) + c_2e^{-4t}\sin(4t)$, *specific solution* $u(t) = 1 + e^{-4t}\cos(4t) + 7e^{-4t}\sin(4t)/4$.

Exercise Solution 4.3.5. $u_h(t) = c_1e^{-t} + c_2e^{-3t}$, $u_p(t) = 3t - 4$. *General solution* $u(t) = 3t-4+c_1e^{-t}+c_2e^{-3t}$, *specific solution* $u(t) = 3t-4+9e^{-t}-3e^{-3t}$.

Exercise Solution 4.3.7. $u_h(t) = c_1e^{-t} + c_2e^{-4t}$, $u_p(t) = -\cos(3t)/5 - \sin(3t)/15$. *General solution* $u(t) = c_1e^{-t} + c_2e^{-4t} - \cos(3t)/5 - \sin(3t)/15$, *specific solution* $u(t) = 4e^{-t} - 9e^{-4t}/5 - \cos(3t)/5 - \sin(3t)/15$.

Exercise Solution 4.3.9. $u_h(t) = c_1e^{-3t/2} + c_2te^{-3t/2}$, $u_p(t) = t^2/9 - 5t/27 + 4/27$. *General solution* $u(t) = c_1e^{-3t/2} + c_2te^{-3t/2} + t^2/9 - 5t/27 + 4/27$, *specific solution* $u(t) = 50e^{-3t/2}/27 + 161te^{-3t/2}/27 + t^2/9 - 5t/27 + 4/27$.

Exercise Solution 4.3.11. $u_h(t) = c_1e^{-2t} + c_2e^{-5t}$, $u_p(t) = -e^{-3t}(2t^2 + 2t+3)$. *General solution* $u(t) = -e^{-3t}(2t^2+2t+3)+c_1e^{-2t}+c_2e^{-5t}$, *specific solution* $u(t) = -e^{-3t}(2t^2 + 2t + 3) + 7e^{-2t} - 2e^{-5t}$.

Exercise Solution 4.3.13. $u_h(t) = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$, $u_p(t) = e^{-2t}$. *General solution* $u(t) = e^{-2t} + c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$, *specific solution* $u(t) = e^{-2t} + e^{-t}\cos(3t) + 2e^{-t}\sin(3t)$.

Exercise Solution 4.3.15. $u_h(t) = c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t)$, $u_p(t) = te^{-2t}$. *General solution* $u(t) = te^{-2t}+c_1e^{-2t}\cos(3t)+c_2e^{-2t}\sin(3t)$, *specific solution* $u(t) = te^{-2t} + 2e^{-2t}\cos(3t) + 2e^{-2t}\sin(3t)$.

Exercise Solution 4.3.17. $u_h(t) = c_1e^{-t} + c_2e^{-4t}$, $u_p(t) = -\cos(2t)$. *General solution* $u(t) = -\cos(2t) + c_1e^{-t} + c_2e^{-4t}$, *specific solution* $u(t) = -\cos(2t) + 5e^{-t} - 2e^{-4t}$.

Exercise Solution 4.3.19. $u_h(t) = c_1e^{-2t} + c_2e^{-5t}$, $u_p(t) = 5t/2 - 1/4$. *General solution* $u(t) = 5t/2-1/4+c_1e^{-2t}+c_2e^{-5t}$, *specific solution* $u(t) = 5t/2 - 1/4 + 47e^{-2t}/12 - 5e^{-5t}/3$.

Exercise Solution 4.3.21. $u_h(t) = c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$, $u_p(t) = (5t-2)\cos(t) + (10t - 14)\sin(t)$. *General solution* $u(t) = (5t - 2)\cos(t) + (10t - 14)\sin(t)+c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$, *specific solution* $u(t) = (5t-2)\cos(t)+(10t - 14)\sin(t) + 4e^{-t}\cos(t) + 16e^{-t}\sin(t)$.

Exercise Solution 4.3.23. $u_h(t) = c_1\cos(t) + c_2\sin(t)$, $u_p(t) = t$, *general solution* $u(t) = t + c_1\cos(t) + c_2\sin(t)$, *specific solution* $u(t) = t + 2\cos(t) + 2\sin(t)$.

Exercise Solution 4.3.24. $u_h(t) = c_1e^{-4t}+c_2e^{-5t}$, $u_p(t) = 2te^{-4t}$, *general solution* $u(t) = 2te^{-4t} + c_1e^{-4t} + c_2e^{-5t}$, *specific solution* $u(t) = 2te^{-4t} + 11e^{-4t} - 9e^{-5t}$.

Exercise Solution 4.3.26. $u_h(t) = c_1e^{-t}+c_2e^{-3t}$, $u_p(t) = -te^{-3t}$, *general solution* $u(t) = -te^{-3t} + c_1e^{-t} + c_2e^{-3t}$, *specific solution* $u(t) = -te^{-3t} + 5e^{-t} - 3e^{-3t}$.

Exercise Solution 4.3.28. $u_h(t) = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$, $u_p(t) = -te^{-t}\cos(t)$, *general solution* $u(t) = -te^{-t}\cos(t)+c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$, *specific solution* $u(t) = -te^{-t}\cos(t) + 2e^{-t}\cos(t) + 6e^{-t}\sin(t)$.

Exercise Solution 4.3.30. $u_h(t) = c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t)$, $u_p(t) = 4te^{-2t}\sin(2t)$, *general solution* $u(t) = 4te^{-2t}\sin(2t)+c_1e^{-2t}\cos(2t)+c_2e^{-2t}\sin(2t)$, *specific solution* $u(t) = 4te^{-2t}\sin(2t) + 2e^{-2t}\cos(2t) + 7e^{-2t}\sin(2t)/2$.

Exercise Solution 4.3.32. $u_h(t) = c_1\cos(t)+c_2\sin(t)$, $u_p(t) = -t\cos(t)/2$, *general solution* $u(t) = -t\cos(t)/2 + c_1\cos(t) + c_2\sin(t)$, *specific solution* $u(t) = -t\cos(t)/2 + 2\cos(t) + 7\sin(t)/2$.

Exercise Solution 4.3.33. *Substituting* $u_p(t) = Ae^{at}$ *into* $mu''(t)+cu'(t)+ku(t) = e^{at}$ *produces* $A(ma^2+ca+k)e^{at} = e^{at}$, *so that* $A(ma^2+ca+k) = 1$. *Since* a *is not a root of the characteristic equation,* $ma^2+ca+k \neq 0$ *and so we can solve uniquely for* A *as* $A = 1/(ma^2 + ca + k)$.

Exercise Solution 4.3.35.

(a) *The solution is now*

$$u(t) \approx -0.03 + 0.005e^{-1.51t} + 0.0251e^{-215.9t}.$$

The graph is shown in the left panel of Figure 4.17. The maximum deflection is now -0.03*, but the solution is much more "abrupt" near* $t = 0$*, e.g., subjects the rider to a much higher acceleration.*

(b) The solution is now

$$u(t) \approx -0.03 - 0.403e^{-13.04t}\sin(12.49t) + 0.03e^{-13.04t}\cos(12.49t).$$

The graph is shown in the right panel of Figure 4.17. The maximum deflection is now -0.146 *(which would actually bottom out the shock at a 140mm travel). A significantly underdamped system would feel unpleasantly "bouncy."*

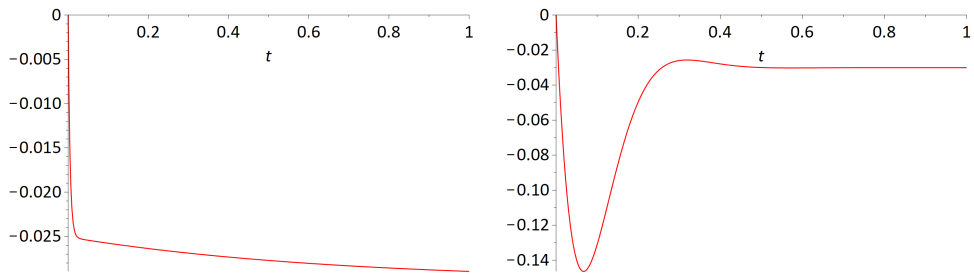

Figure 4.17: Solution to shock absorber ODE with $c = 10^4$ (left) and $c = 1000$ (right).

Section 4.4

Exercise Solution 4.4.1. $G(\omega) = 1/\sqrt{(2\omega^2-8)^2+\omega^2}$. *Resonance occurs at* $\omega = \sqrt{62}/4 \approx 1.969$. *A plot is shown in Figure 4.18. Periodic response is* $-\frac{9\,\sin(4\,t)}{74} - \frac{3\,\cos(4\,t)}{148}$ *with amplitude* $3\sqrt{37}/148 \approx 0.123$.

Exercise Solution 4.4.3. $G(\omega) = 1/2\sqrt{\omega^4-16\omega^2+100}$. *Resonance occurs at* $\omega = 2\sqrt{2} \approx 2.828$. *A plot is shown Figure 4.19. Periodic response is* $\frac{5\,\sin(2\,t)}{26} + \frac{15\,\cos(2\,t)}{52}$ *with amplitude* $5\sqrt{13}/52 \approx 0.347$.

Exercise Solution 4.4.5. *The gain is the same as part (d),* $G(\omega) = 1/2\sqrt{100\omega^4-999\omega^2+2500}$, *and again resonance occurs at* $\omega = 3\sqrt{222}/20 \approx 2.235$. *A plot is shown in Figure 4.20. Periodic response is* $-(5.26\times10^{-4})\sin(10t)-(5.54\times10^{-6})\cos(10t)$, *amplitude* 5.26×10^{-4}. *Much smaller than (d), even though the amplitude of the driving force is the same.*

Exercise Solution 4.4.7. $G(\omega) = 1/\sqrt{(\omega^2-1)^2+100\omega^2}$. *Resonance does not occur here. A plot is shown in Figure 4.21. Periodic response is* $-\frac{6\,\cos(2\,t)}{409} + \frac{40\,\sin(2\,t)}{409} \approx (-0.0147\,\cos(2.0\,t) + 0.0978\,\sin(2.0\,t))$ *with amplitude* $2/\sqrt{409} \approx 0.0989$.

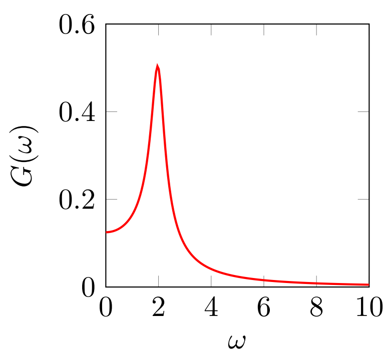

Figure 4.18: Gain function for Exercise 4.4.1.

Exercise Solution 4.4.10. *The gain function is*

$$G(\omega) = \frac{1}{\sqrt{(L\omega^2-1/C)^2+R^2\omega^2}}.$$

If resonance occurs for $\omega > 0$ *then* $G'(\omega) = 0$ *at that frequency, which leads to*

$$G'(\omega) = -\frac{\omega(2CL^2\omega^2+CR^2-2L)}{C((L\omega^2-1/C)^2+R^2\omega^2)^{3/2}} = 0.$$

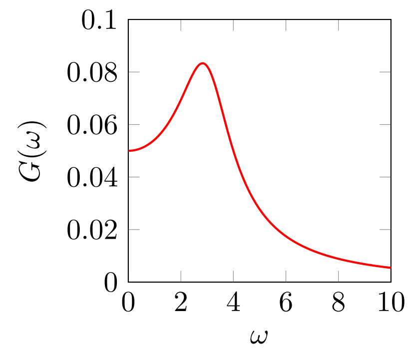

Figure 4.19: Gain function for Exercise 4.4.3.

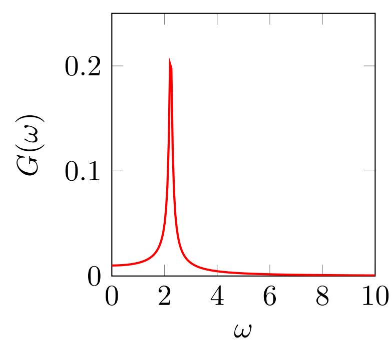

Figure 4.20: Gain function for Exercise 4.4.5.

The numerator is zero for $\omega > 0$ *when* $2CL^2\omega^2 + R^2C - 2L = 0$, *which yields*

$$\omega = \frac{\sqrt{4L/C - 2R^2}}{2L}.$$

Exercise Solution 4.4.12. *The gain function is*

$$G(\omega) = \frac{1}{(m\omega^2 - k)^2 + c^2\omega^2}.$$

Resonance occurs at $\omega_{res} = \sqrt{k/m - (c/m)^2/2}$. *Then* $(m\omega_{res}^2 - k)^2 = c^4/4m^2$ *while* $c^2\omega_{res}^2 = c^4/2m^2 + kc^2/m$. *Then*

$$(m\omega_{res}^2 - k)^2 + c^2\omega_{res}^2 = kc^2/m - c^4/4m^2 = c^2(k/m - c^2/4m^2).$$

Then $\sqrt{(m\omega_{res}^2 - k)^2 + c^2\omega_{res}^2} = c\sqrt{k/m - c^2/4m^2} = c\omega_{nat}$ *so that the peak gain at resonance is*

$$G(\omega_{res}) = \frac{1}{c\omega_{nat}}.$$

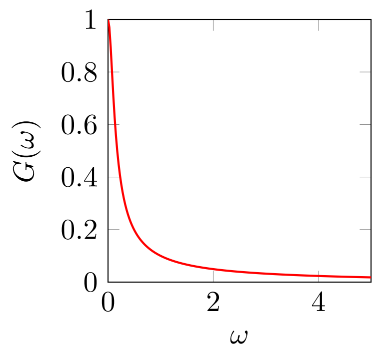

Figure 4.21: Gain function for Exercise 4.4.7.

Exercise Solution 4.4.14.

(a) *Here* $\omega_{res} \approx 0.98, \omega_- \approx 0.748, \omega_+ \approx 1.166,$ *and* $Q \approx 2.345$.

(c) *Here* $\omega_{res} \approx 3.162, \omega_- \approx 3.137, \omega_+ \approx 3.187,$ *and* $Q \approx 63.24$.

(e) *In this case no real computation is needed—it's clear the we should take "*$Q = \infty$*".*

Note that in (b)-(d) the quantity Q *scales in proportion to* $1/c$.

Exercise Solution 4.4.16.

(a) *Here the solution is* $u(t) \approx -5.263\cos(t) + 5.263\cos(0.9t)$ *with* $\omega_0 = 1$, $\omega = 0.9$, *and* $\delta = 0.1$. *The period of the beats is* $20\pi \approx 62.8$. *See Figure 4.22*

(c) *Here the solution is* $u(t) \approx -2.564\cos(2t)+2.564\cos(1.9t)$ *with* $\omega_0 = 2$, $\omega = 1.9$, *and* $\delta = 0.1$. *The period of the beats is* $20\pi \approx 62.8$. *See Figure 4.23*

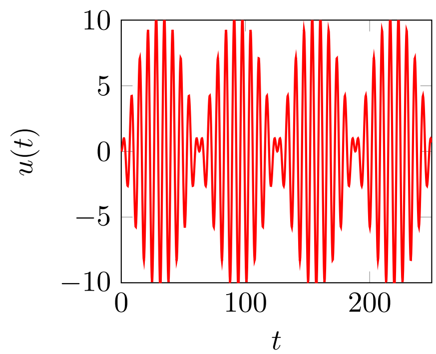

Figure 4.22: Solution $u(t)$ for part (a) of Exercise 4.4.16.

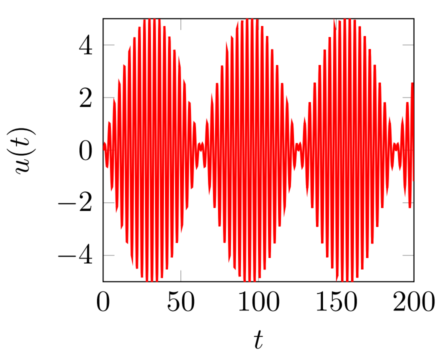

Figure 4.23: Solution $u(t)$ for part (c) of Exercise 4.4.16.

Section 4.5

Exercise Solution 4.5.1. *We find* $[k] = T^{-1}$. *If* $t_c = k^\alpha u_0^\beta$ *then taking the dimension of each side yields* $T = T^{-\alpha} M^\beta$ *which forces* $\alpha = -1, \beta = 0$, *and so* $t_c = k^{-1}$. *Since* $[u_0] = M$, *any characteristic mass scale of the form* $u_c = k^\alpha u_0^\beta$ *has* $M = T^{-\alpha} M^\beta$, *so* $\alpha = 0, \beta = 1$, *and* $u_c = u_0$. *With* $\tau = t/t_c = kt$ *or* $t = \tau/k$ *and* $u(t) = u_c \bar{u}(\tau) = u_0 \bar{u}(kt)$ *we find* $du/dt = k u_0 \frac{d\bar{u}}{d\tau}$ *and the ODE* $du/dt = -ku$ *becomes* $k u_0 \frac{d\bar{u}}{d\tau} = -k u_0 \bar{u}$ *or* $d\bar{u}/dt = -\bar{u}$ *with initial data* $\bar{u}(0) = u_0/u_0 = 1$.

Exercise Solution 4.5.3. *We find* $[u'] = \Theta T^{-1}$, *and since* $[u] = [A] = \Theta$ *we must have* $k = T^{-1}$. *We try a characteristic time scale of the form*

$$t_c = k^\alpha A^\beta.$$

This leads to $M^0 L^0 T^1 \Theta^0 = M^0 T^{-\alpha} L^0 \Theta^\beta$ *with solution* $\alpha = -1$, $\beta = 0$. *The only characteristic scale of this form is* $t_c = 1/k$. *Similarly consider a characteristic scale for* u *of the form*

$$u_c = k^\alpha A^\beta.$$

This leads to $M^0 L^0 T^0 \Theta^1 = M^0 T^{-\alpha} L^0 \Theta^\beta$ *with solution* $\alpha = 0$, $\beta = 1$. *The only characteristic scale of this form is* $u_c = A$.

Take $\tau = t/t_c = kt$ *(so* $t = \tau/k$*) and* $\bar{u} = u/u_c = u/A$ *(so* $u(t) = A\bar{u}(\tau)$*). Then* $du/dt = \frac{A}{t_c} d\bar{u}/d\tau = kAd\bar{u}/\tau$. *The Newton cooling ODE* $du/dt = -k(u - A)$ *becomes* $kAd\bar{u}/d\tau = -k(A\bar{u} - A)$ *or*

$$\frac{d\bar{u}}{d\tau} = -(\bar{u} - 1).$$

The initial condition $u(0) = u_0$ *becomes* $\bar{u}(0) = u_0/A$. *The characteristic scale* $u_c = A$ *is exactly the ambient temperature to which all solutions decay.*

Exercise Solution 4.5.5. *We have* $[u] = M$ *and so* $[u'] = MT^{-1}$. *Also* $[V] = L^3$, $[r] = L^3 T^{-1}$ *and* $[c_1] = ML^{-3}$. *A characteristic time scale is of the form*

$$t_c = V^\alpha r^\beta c_1^\gamma$$

which leads to $M^0 L^0 T^1 = M^\gamma L^{3\alpha + 3\beta - 3\gamma} T^{-\beta}$. *We conclude that* $\gamma = 0, 3(\alpha + \beta - \gamma) = 0, -\beta = 1$, *with solution* $\alpha = 1, \beta = -1, \gamma = 0$. *That is,* $t_c = V/r$.

A characteristic mass scale u_c *for* u *is of the form*

$$u_c = V^\alpha r^\beta c_1^\gamma$$

which leads to $M^1L^0T^0 = M^\gamma L^{3\alpha+3\beta-3\gamma}T^{-\beta}$*. We conclude that* $\gamma = 1, 3(\alpha+\beta-\gamma) = 0, -\beta = 0$*, with solution* $\alpha = 1, \beta = 0, \gamma = 1$*. That is,* $u_c = c_1V$*.*

We then have $\tau = t/t_c = rt/V$ *or* $t = V\tau/r$*. Also,* $\bar{u}(\tau) = u(t)/u_c = u(t)/(c_1V)$ *or* $u(t) = c_1V\bar{u}(\tau)$*. Then* $du/dt = c_1V\frac{d\bar{u}}{d\tau}\frac{d\tau}{dt} = rc_1 d\bar{u}d\tau$*. The original ODE* $du/dt = rc_1 - ru/V$ *becomes, after cancellations,*

$$\frac{d\bar{u}}{d\tau} = 1 - \bar{u}(\tau).$$

Section 5.1

Exercise Solution 5.1.1.

(a) *The solution is* $u_1(t) \approx 5.78 - 0.78e^{-kt}$ *for* $0 < t < 12$.

(b) *The initial data for* $u_2(t)$ *is* $u_2(12) = u_1(12) \approx 5.683$ *mg. Then* $u_2(t) \approx 8.67 - 2.99e^{-k(t-12)}$. *This can also be expressed as* $u_2(t) \approx 8.67 - 23.82e^{-kt}$.

(c) *The function* $u_3(t)$ *will satisfy* $u_3(18) = u_2(18) + 5 \approx 7.61$ *mg, with* $u_3' = -ku_3 + 1$ *for* $t > 18$. *The solution is* $u_3(t) \approx 5.78 + 6.83e^{-k(t-18)}$ *or alternatively, as* $u_3(t) \approx 5.78 + 153.79e^{-kt}$.

(d) *The solution is plotted in Figure 5.24.*

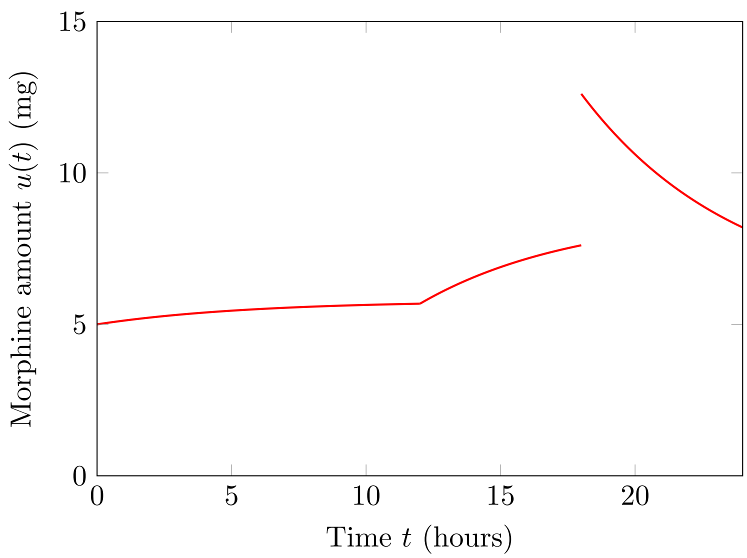

Figure 5.24: Amount of morphine (mg) in patient's system.

Exercise Solution 5.1.5. *The relevant ODE for* $0 < t < 0.003$ *is* $10q'(t) + 10^4 q(t) = 2$ *with initial condition* $q(0) = 0$. *The solution is* $q = q_1$ *where* $q_1(t) = (1 - e^{-1000t})/5000$. *For* $t > 0.003$ *the ODE becomes* $10q'(t) + 10^4 q(t) = 5$ *with initial condition* $q(0.003) = q_1(0.003) \approx 0.00019$. *The solution to this ODE is* $q = q_2$ *with* $q_2(t) \approx 5 \times 10^{-4} - (6.226 \times 10^{-3})e^{-1000t} \approx 5 \times 10^{-4} - (3.1 \times 10^{-4})e^{-1000(t-0.003)}$. *At* $t = 0.005$ *the charge is* $q_2(0.005) \approx 4.58 \times 10^{-4}$.

Section 5.2

Exercise Solution 5.2.1. $F(s) = 6/s^3$.

Exercise Solution 5.2.3. $P(s) = (s+3)/((s+3)^2 + 49)$

Exercise Solution 5.2.6. *Use linearity.* $f(t) = t - 2$

Exercise Solution 5.2.8. *Write* $G(s) = 2\frac{s}{s^2+4} + \frac{2}{s^2+4}$ *so* $g(t) = 2\cos(2t) + \sin(2t)$.

Exercise Solution 5.2.10. *From* $\mathcal{L}^{-1}(2/s^3) = t^2$ *it follows that* $f(t) = t^2 e^{-3t}$.

Exercise Solution 5.2.11. *The poles of* $F(s)$ *are at* $s = -1$ *and* $s = -2$ *(both multiplicity* 1*), so* $f(t)$ *is a linear combination of* e^{-t} *and* e^{-2t}.

Exercise Solution 5.2.13. *The poles of* $F(s)$ *are at* $s = i$ *and* $s = -i$*, both of multiplicity* 1*, so* $f(t)$ *is a linear combination of* e^{it} *and* e^{-it}*, or* $\sin(t)$ *and* $\cos(t)$.

Exercise Solution 5.2.15. $F(s)$ *has a pole at* $s = 1$ *of multiplicity* 3 *and poles at* $s = -1 \pm i$ *of multiplicity* 1*, so* $f(t)$ *will contain terms* e^t, te^t, t^2e^t*, and* $e^{(-1+i)t}$, $e^{(-1-i)t}$*. These last two terms are equivalent to* $e^{-t}\sin(t)$ *and* $e^{-t}\cos(t)$.

Exercise Solution 5.2.18. *Laplace transform both sides of the ODE and fill in the initial data to find* $sU(s) - 6 = 2U(s)$*, so* $U(s) = 6/(s-2)$ *and* $u(t) = 6e^{2t}$.

Exercise Solution 5.2.21. *Laplace transform both sides of the ODE, fill in the initial data, and collect the* $U(s)$ *terms on the left, all other terms on the right to find* $(s^2 + 3s + 2)U(s) = 6s + 22$*. Then*

$$U(s) = \frac{6s+22}{s^2+3s+2} = \frac{16}{s+1} - \frac{10}{s+2}$$

after a partial fraction decomposition. Then $u(t) = 16e^{-t} - 10e^{-2t}$.

Exercise Solution 5.2.23. *Laplace transform both sides of the ODE, fill in the initial data, and collect the* $U(s)$ *terms on the left, all other terms on the right to find* $(s^2 + 2s + 10)U(s) = s + 4$*. Then*

$$U(s) = \frac{s+4}{s^2+2s+10} = \frac{s+4}{(s+1)^2+3^2}$$

after completing the square in the denominator. This can also be written

$$U(s) = \frac{3}{(s+1)^2 + 3^2} + \frac{s+1}{(s+1)^2 + 3^2}$$

which has inverse transform $u(t) = e^{-t}\sin(3t) + e^{-t}\cos(3t)$.

Exercise Solution 5.2.25. *Laplace transform both sides of the ODE, fill in the initial data, and collect the* $U(s)$ *terms on the left, all other terms on the right to find* $(3s^2 + 6s + 6)U(s) = 3s$. *Then*

$$U(s) = \frac{s}{s^2 + 2s + 2} = \frac{s}{(s+1)^2 + 1}$$

after completing the square in the denominator. This can also be written

$$U(s) = \frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}$$

which has inverse transform $u(t) = e^{-t}\cos(t) - e^{-t}\sin(t)$.

Exercise Solution 5.2.33.

(a) *If* $f(t) = 1$ *then* $F(s) = 1/s$. *Also,* $\lim_{t\to 0^+} f(t) = 1$ *and* $\lim_{s\to\infty} sF(s) = 1$.

(c) *If* $f(t) = e^t$ *then* $F(s) = 1/(s-1)$. *Also,* $\lim_{t\to 0^+} f(t) = 1$ *and* $\lim_{s\to\infty} sF(s) = 1$.

Exercise Solution 5.2.34.

(a) *If* $f(t) = 4$ *then* $F(s) = 4/s$. *Here* F *has a pole at* $s = 0$ *of multiplicity* 1, *so the theorem is applicable. Also,* $\lim_{t\to\infty} f(t) = 4$ *and* $\lim_{s\to 0^+} sF(s) = 4$.

(c) *If* $f(t) = t^4 e^{-t}$ *then* $F(s) = 24/(s+1)^5$. *Here* F *has a pole at* $s = -1$ *so the theorem is applicable. Also,* $\lim_{t\to\infty} f(t) = 0$ *and* $\lim_{s\to 0^+} sF(s) = 0$.

Exercise Solution 5.2.37. *This equation is nonlinear. There is no simple way to relate the transform* $\mathcal{L}(u^2(t))$ *to* $\mathcal{L}(u(t))$.

Exercise Solution 5.2.38.

(a) *From the rule for first derivatives we have*

$$\mathcal{L}(f''') = \mathcal{L}((f'')') = s\mathcal{L}(f'') - f''(0).$$

Using the rule for $\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$ *yields* $\mathcal{L}(f''') = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.

Exercise Solution 5.2.39.

(a) When $k = 1$ the expression is $(-1)(1/t)^2 F'(1/t) = 1/(1+t)^2$ (use $F'(s) = -1/(s+1)^2$.) A plot of $1/(1+t)^2$ and e^{-t} is shown in the left panel of Figure 5.25.

(b) When $k = 2$ the expression is $((-1)^2/2)(2/t)^3 F''(2/t) = 1/(1+t/2)^3$ (use $F''(s) = 2/(s+1)^3$.) A plot of $1/(1+t/2)^3$ and e^{-t} is shown in the right panel of Figure 5.25.

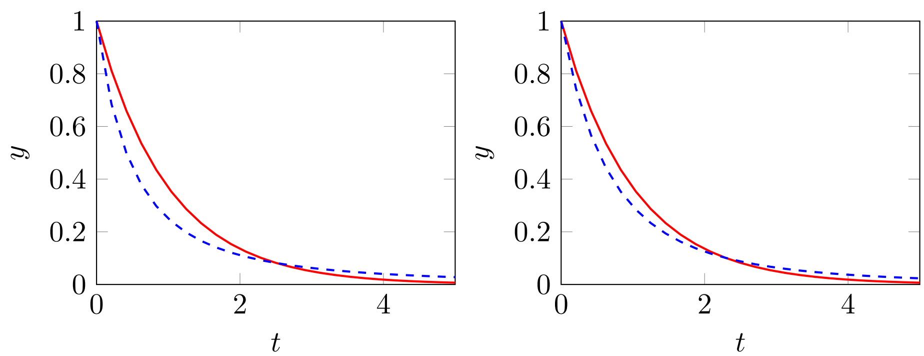

Figure 5.25: Left panel: Graph of e^{-t} (red,solid) and $1/(1+t)^2$ (blue, dashed). Right panel: Graph of e^{-t} (red,solid) and $1/(1+t/2)^3$ (blue, dashed).

Section 5.3

Exercise Solution 5.3.1. $f(t) = 7H(t-5)$.

Exercise Solution 5.3.3. $f(t) = 2(1-H(t-3)) + 5(H(t-3) - H(t-6)) - 3H(t-6) = 2 + 3H(t-3) - 8H(t-6)$.

Exercise Solution 5.3.6. *We find*

- $F(s) = 7e^{-5s}/s$.
- $F(s) = 2/s + 3e^{-3s}/s - 8e^{-6s}/s$.

Exercise Solution 5.3.7. *The inverse transform of* $2/s^2$ *is* $2t$*, so by the second shifting theorem* $f(t) = 2H(t-3)(t-3)$.

Exercise Solution 5.3.9. *The inverse transform of* $(3s+2)/(s^2+4) = 3s/(s^2+4) + 2/(s^2+4)$ *is* $3\cos(2t) + \sin(2t)$ *so* $g(t) = H(t-5)(3\cos(2(t-5)) + \sin(2(t-5)))$.

Exercise Solution 5.3.12. *Transform both sides of the ODE and use the initial data to find* $sU(s) - 1 = -2U(s) + 4e^{-5s}/s$. *Then* $U(s) = 1/(s+2) + 4e^{-5s}/(s(s+2))$. *The inverse transform of* $1/(s+2)$ *is* e^{-2t}. *The inverse transform of* $1/(s(s+2)) = 1/(2s) - 1/(2(s+2))$ *is* $1/2 - e^{-2t}/2$ *so the inverse transform of* $4e^{-5s}/(s(s+2))$ *is* $4H(t-5)(1-e^{-2(t-5)})/2$. *All in all* $u(t) = e^{-2t} + 2H(t-5)(1-e^{-2(t-5)})$. *Graph shown in Figure 5.26.*

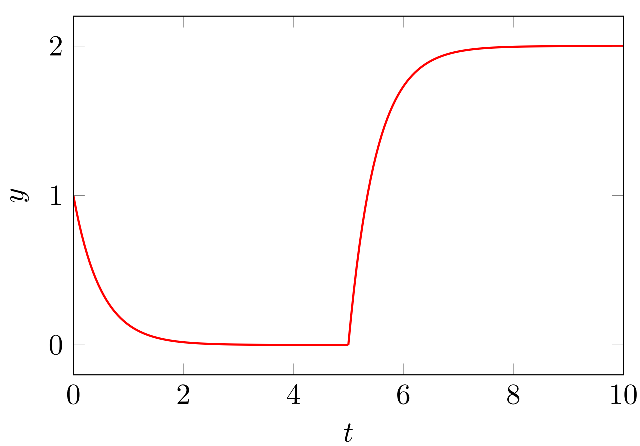

Figure 5.26: Graph of solution for Exercise 5.3.12.

Exercise Solution 5.3.15. *Transforming both sides and using the initial data yields* $s^2U(s) + 4sU(s) + 3U(s) = e^{-s}/s$ *so that* $U(s) = \frac{e^{-s}}{s(s^2+4s+3)} = \frac{e^{-s}}{s(s+1)(s+3)}$. *Then*

$$U(s) = e^{-s}\left(\frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}\right).$$

An inverse transform yields $u(t) = H(t-1)(1/3 - e^{-(t-1)}/2 + e^{-3(t-1)}/6)$. *Graph shown in Figure 5.27.*

Exercise Solution 5.3.17. *Laplace transform and fill in the initial data to find* $(s^2 + 4s + 4)U(s) - s - 6 = 4/s + 8e^{-3s}/s$. *Then*

$$U(s) = \frac{s+6}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{8e^{-3s}}{s(s+2)^2}.$$

A partial fraction decomposition shows

$$\frac{s+6}{(s+2)^2} = \frac{1}{s+2} + \frac{4}{(s+2)^2}.$$

and

$$\frac{4}{s(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}.$$

Use this to find

$$\begin{aligned} u(t) &= e^{-2t} + 4te^{-2t} + 1 - e^{-2t} - 2te^{-2t} \\ &\quad + 2H(t-3)(1 - e^{-2(t-3)} - 2(t-3)e^{-2(t-3)}) \\ &= 1 + 2te^{-2t} + 2H(t-3)(1 - e^{-2(t-3)} - 2(t-3)e^{-2(t-3)}). \end{aligned}$$

Graph shown in Figure 5.28.

Exercise Solution 5.3.19. *The ODE is* $u'(t) = -ku(t) + 1 + 0.5H(t-12)$ *(recall* $k = 0.173$*) with initial condition* $u(0) = 5$. *Laplace transforming, using the initial data, and then solving for* $U(s)$ *yields*

$$U(s) = \frac{5}{s+k} + \frac{1}{s(s+k)} + \frac{e^{-12s}}{2s(s+k)}$$

Inverse transforming yields

$$u(t) = 5e^{-kt} + \frac{1 - e^{-kt}}{k} + H(t-12)\frac{1 - e^{-k(t-12)}}{2k}.$$

A graph is shown in Figure 5.29.

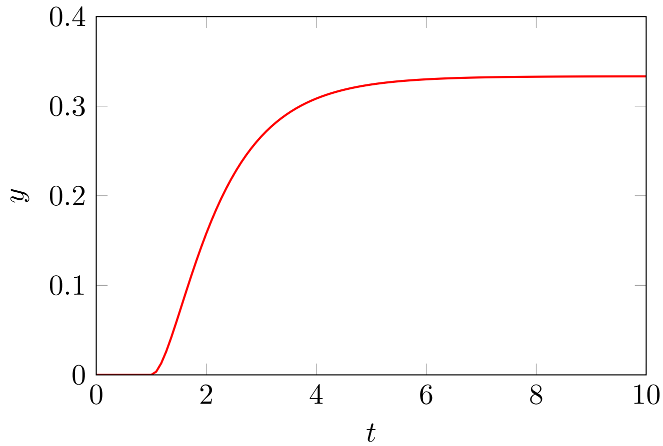

Figure 5.27: Graph of solution to Exercise 5.3.15.

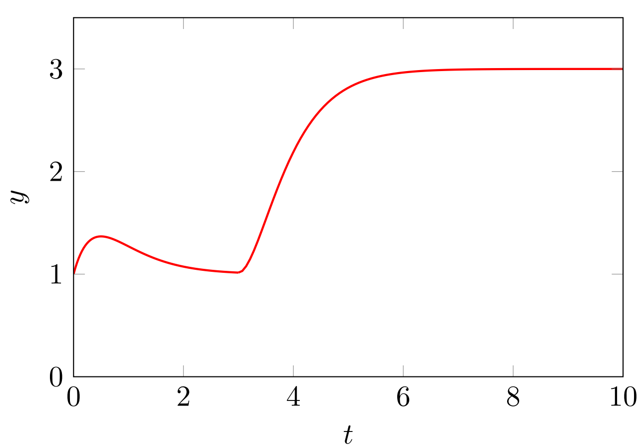

Figure 5.28: Graph of solution to Exercise 5.3.17.

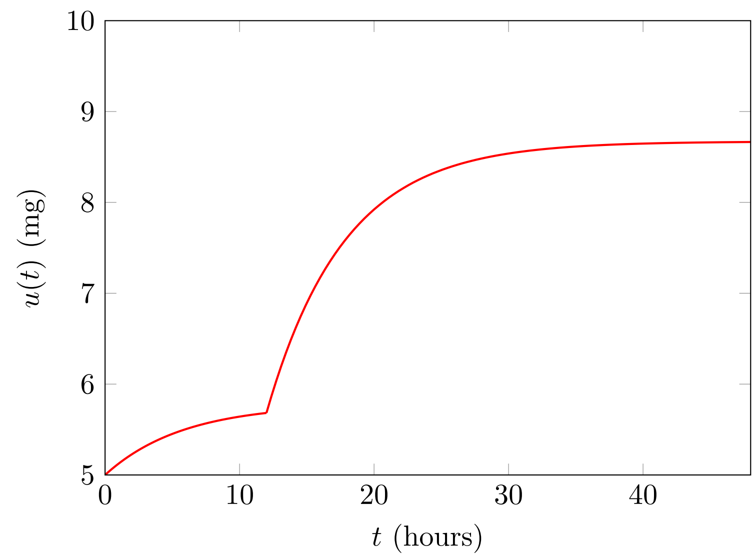

Figure 5.29: Plot of morphine level (mg).

Section 5.4

Exercise Solution 5.4.2. *Transform to find* $sU(s)-1 = -3U(s)+3e^{-3s} - 6e^{-5s}/s$ *so* $U(s) = 1/(s+3) + 3e^{-3s}/(s+3) - 6e^{-5s}/(s(s+3))$ *with inverse transform* $u(t) = e^{-3t} + 3H(t-3)e^{-3(t-3)} - 2H(t-5)(1-e^{-3(t-5)})$*. Graph shown in Figure 5.30.*

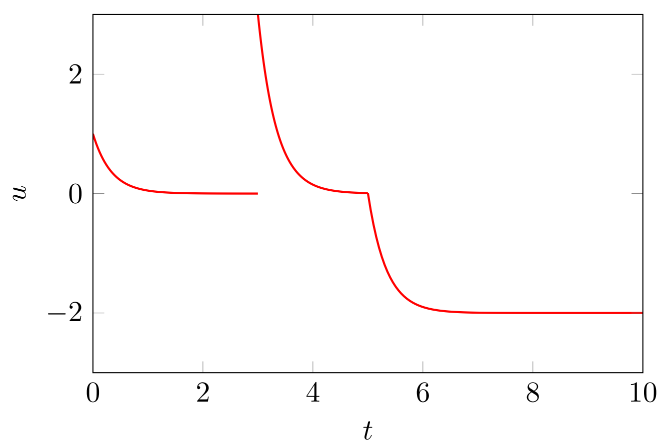

Figure 5.30: Graph of solution to Exercise 5.4.2.

Exercise Solution 5.4.4. *Transform to find* $(s^2+4s+3)U(s) = e^{-s}$*, so* $U(s) = e^{-s}/(s^2+4s+3)$ *and* $u(t) = H(t-1)(e^{-(t-1)} - e^{-3(t-1)})/2$*. Graph in Figure 5.31.*

Exercise Solution 5.4.6. *Transform to find* $(s^2+4s+4)U(s)-s-6 = 1/s+5e^{-2s}$*, so* $U(s) = (s+6)/(s^2+4s+4)+1/(s(s^2+4s+4))+5e^{-2s}/(s^2+4s+4)$*. An inverse transform yields* $u(t) = 1/4 + e^{-2t}(14t+3)/4 + 5H(t-2)(t-2)e^{-2(t-2)}$*. Graph in Figure 5.32.*

Exercise Solution 5.4.9.

(a) The ODE is $4u''(t)+16u'(t)+116u(t) = 20\delta(t-5)$ *with* $u(0) = u'(0) = 0$*, if* $u(t)$ *denotes the mass position.*

(b) Transform both sides to find $(4s^2+16s+116)U(s) = 20e^{-5s}$*, so* $U(s) = 5e^{-5s}/(s^2+4s+29)$*. An inverse transform shows that* $u(t) = H(t-5)e^{-2(t-5)}\sin(5(t-5))$*. The mass remains motionless up until time* $t = 5$*, at which time the blow sets the mass in motion; it oscillates and decays back to position* $u = 0$*.*

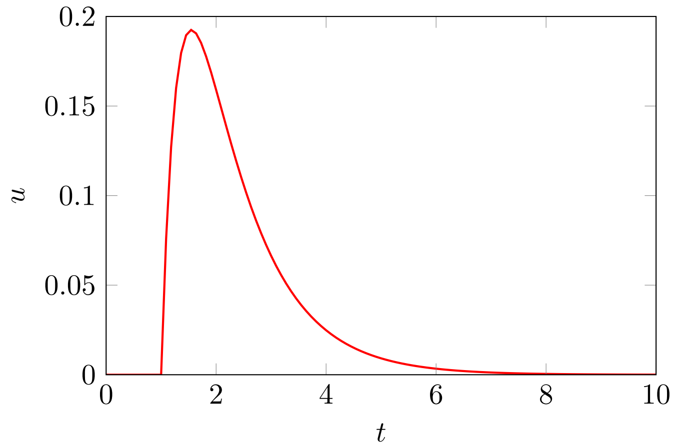

Figure 5.31: Graph of solution to Exercise 5.4.4.

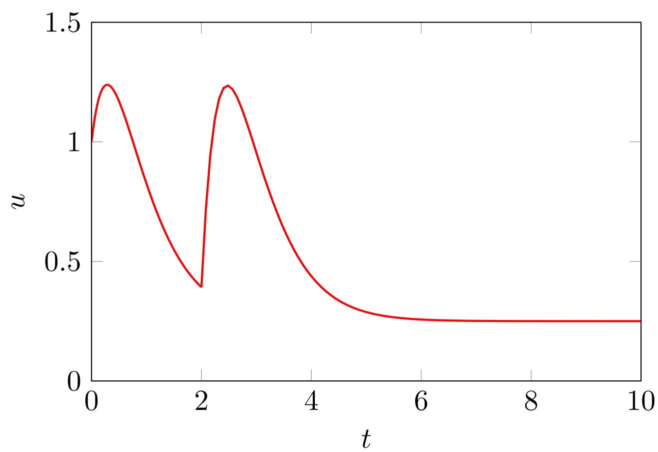

Figure 5.32: Graph of solution to Exercise 5.4.6.

Section 5.5

Exercise Solution 5.5.2. *$F_1(s) = 1/s^2$, $F_2(s) = 1/(s-1)$, $p(t) = e^t - t - 1$, and $P(s) = 1/(s^2(s-1))$.*

Exercise Solution 5.5.4. *$F_1(s) = F_2(s) = 1/(s^2+1)$, $p(t) = (\sin(t) - t\cos(t))/2$, and $P(s) = 1/(s^2+1)^2$.*

Exercise Solution 5.5.6. *$F_1(s) = 1/s^2 + 3/s$, $F_2(s) = e^{-2s}$, $p(t) = H(t-2)(t+1)$, and $P(s) = e^{-2s}/s^2 + 3e^{-2s}/s$.*

Exercise Solution 5.5.7. *Unit impulse response is $\mathcal{L}^{-1}(1/(s+4)) = e^{-4t}$.*

Exercise Solution 5.5.9. *Unit impulse response is $\mathcal{L}^{-1}(1/s) = H(t)$ or 1.*

Exercise Solution 5.5.11. *Unit impulse response is $\mathcal{L}^{-1}(1/(s^2+1)) = \sin(t)$.*

Exercise Solution 5.5.13. *Unit impulse response is $\mathcal{L}^{-1}(1/(s^2+4s+4)) = te^{-2t}$.*

Exercise Solution 5.5.16. *Laplace transform the ODE and use the initial data to find $(as+b)U(s) = F(s)$. We can compute $U(s) = 1/(s(s+5))$ and $F(s) = 1/s$, from which it follows that $(as+b)/(s(s+5)) = 1/s$ or $(as+b)/(s+5) = 1$. We conclude that $a = 1$ and $b = 5$.*

Exercise Solution 5.5.18. *From $U(s) = G(s)F(s) = F(s)/(ms^2+cs+k)$ along with $U(s) = 4e^{-s}((s+1)(s+5))$ and $F(s) = 4e^{-5s}$ we find $G(s) = 1/(ms^2+cs+k) = 1/(s^2+6s+5)$. Then $m = 1, c = 6$, and $k = 5$.*

Exercise Solution 5.5.24. *In each case let's use the convolution theorem (though they can be done directly from the definition of convolution).*

- **Commutativity:** *This is equivalent to the s-domain statement $F_1(s)G(s) = G(s)F_1(s)$, which is clearly true.*
- **Distributivity:** *This is equivalent to the s-domain statement $(aF_1(s) + bF_2(s))G(s) = aF_1(s)G(s) + bF_2(s)G(s)$, also clearly true.*
- **Associativity:** *This is equivalent to the s-domain statement $(F_1(s)F_2(s))G(s) = F_1(s)(F_2(s)G(s))$, also true.*

Section 5.6

Exercise Solution 5.6.1. *Substitute* $u(t) = \frac{r'(t)+kr(t)}{K}$ *into* $y'(t) = -ky(t) + Ku(t)$ *to find ODE*

$$y'(t) = -ky(t) + r'(t) + kr(t).$$

With $y(0) = r(0)$ *it is easy to check that* $y(t) = r(t)$ *is the unique solution to this ODE. If we Laplace transform both sides of* $u(t) = \frac{r'(t)+kr(t)}{K}$ *we obtain* $U(s) = (sR(s) + kR(s))/K = G_c(s)R(s)$. *This corresponds to the s-domain computation.*

Exercise Solution 5.6.3.

(a) *We find* $G_c(s) = K_p$. *With* $G_p(s) = 1/s$ *we then have* $G(s) = G_p(s)G_c(s)/(1 + G_p(s)G_c(s)) = K_p/(s + K_p)$.

Exercise Solution 5.6.4.

(a) *We have* $G_c(s) = K_p + K_i/s + K_d s$. *Given* $G_p(s) = 1/s$ *we find*

$$G(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)} = \frac{K_d s^2 + K_p s + K_i}{(K_d + 1)s^2 + K_p s + K_i}.$$

Section 6.1

Exercise Solution 6.1.1. *Nonlinear (has x_1x_2).*

Exercise Solution 6.1.3. *Nonlinear.*

Exercise Solution 6.1.5. *Nonlinear (x_1/x_2).*

Exercise Solution 6.1.7. *Linear, variable coefficient, homogeneous.*

Exercise Solution 6.1.9. *Linear, constant coefficient, nonhomogeneous.*

Exercise Solution 6.1.11. *Linear, variable coefficient, nonhomogeneous.*

Exercise Solution 6.1.12. *With $x_1 = u$ and $x_2 = u'$*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4x_1/3 - 5x_2/3\end{aligned}$$

with $x_1(0) = 7$ and $x_2(0) = 5$.

Exercise Solution 6.1.14. *With $x_1 = u$ and $x_2 = u'$*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1/2 - \cos(x_2)\end{aligned}$$

with $x_1(0) = 3$ and $x_2(0) = -1$.

Exercise Solution 6.1.16. *With $x_1 = u$, $x_2 = u'$, and $x_3 = u''$,*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -5x_1 - x_2 - 2x_3\end{aligned}$$

with $x_1(0) = 1$, $x_2(0) = 0$, and $x_3(0) = -1$.

Exercise Solution 6.1.18. *Let $x_1 = u_1, x_2 = u_1'$, and $x_3 = u_2$. Then*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 + x_3 + \sin(t)\\ \dot{x}_3 &= -3x_1 + x_3\end{aligned}$$

with $x_1(0) = 1, x_2(0) = 3$, and $x_3(0) = -2$.

Section 6.2

Exercise Solution 6.2.1. *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 7 & -4 \\ 20 & -11 \end{bmatrix}$$

with $\lambda_1 = -1, \lambda_2 = -3$, *and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

A general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

The initial data is obtained with $c_1 = -1, c_2 = 2$.

Exercise Solution 6.2.3. *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$$

with $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$, *and*

$$\mathbf{v}_1 = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}.$$

A complex-valued general solution is

$$\mathbf{x}(t) = c_1 e^{(-1+i)t} \begin{bmatrix} 2+i \\ 5 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 2-i \\ 5 \end{bmatrix}.$$

A real-valued general solution is

$$\mathbf{x}(t) = d_1 e^{-t} \begin{bmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{bmatrix} + d_2 e^{-t} \begin{bmatrix} 2\sin(t) + \cos(t) \\ 5\sin(t) \end{bmatrix}.$$

The initial data is obtained with $d_1 = 2/5, d_2 = -4/5$.

Exercise Solution 6.2.5. *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -6 & 9 & -4 \\ -6 & 11 & -6 \\ -10 & 21 & -12 \end{bmatrix}$$

with $\lambda_1 = -4, \lambda_2 = -2$, $\lambda_3 = -1$, *and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A general solution is

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The initial data is obtained with $c_1 = 1, c_2 = 0, c_3 = -2$.

Exercise Solution 6.2.8. *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$$

with double eigenvalue $\lambda = 1$, *and eigenvector*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

By solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 0, -1 \rangle$ *(or more generally,* $\mathbf{v}_1 = \langle t_1, 2t_1 - 1 \rangle$ *for a free variable* t_1*). We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} t \\ 2t - 1 \end{bmatrix}.$$

The initial data is obtained with $c_1 = 1, c_2 = -1$.

Exercise Solution 6.2.10. *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -10 & -8 \\ 8 & 6 \end{bmatrix}$$

with double eigenvalue $\lambda = -2$, *and eigenvector*

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

By solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 1/8, 0 \rangle$ *(or more generally,* $\mathbf{v}_1 = \langle 1/8 - t_1, t_1 \rangle$ *for a free variable* t_1*). We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -t + 1/8 \\ t \end{bmatrix}.$$

The initial data is obtained with $c_1 = 0, c_2 = 16$.

Exercise Solution 6.2.12.

(a) *The characteristic equation is* $r^2 + 3r + 2 = 0$*, roots* $r_1 = -1, r_2 = -2$*. A general solution is*

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

(b) *The equivalent system is* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -2x_1 - 3x_2$*. The relevant matrix is*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

(c) *The eigenvalues are* $\lambda_1 = -1$ *and* $\lambda_2 = -2$*, with eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The general solution is then

$$\mathbf{x}(t) = c_1 e{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then $x_1(t)$ *is of precisely the same form as* $x(t)$ *in part (a).*

(d) *The equivalent system is* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -kx_1/m - cx_2/m$*. The relevant matrix is*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$$

The eigenvalues are $\lambda_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$ *and* $\lambda_2 = \frac{-c-\sqrt{c^2-4mk}}{2m}$*. These are precisely the roots of the characteristic equation* $mr^2 + cr + k = 0$*. The eigenvectors have the asserted form, namely*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Then general system has a general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Since $r_1 = \lambda_1$ *and* $r_2 = \lambda_2$*,* $x_1(t)$ *is of exactly the same form as* $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$*.*

Section 6.3

Exercise Solution 6.3.1. *Laplace transforming and solving for $X_1(s), X_2(s)$ yields*

$$\begin{aligned} X_1(s) &= \frac{3s+1}{s^2+4s+3} \\ X_2(s) &= \frac{8s+4}{s^2+4s+3}. \end{aligned}$$

An inverse transform shows that $x_1(t) = 4e^{-3t} - e^{-t}$ and $x_2(t) = 10e^{-3t} - 2e^{-t}$.

Exercise Solution 6.3.3. *Laplace transforming and solving for $X_1(s), X_2(s)$ yields*

$$\begin{aligned} X_1(s) &= \frac{s^2-s-6}{s(s+1)(s+3)} \\ X_2(s) &= \frac{2(s^2-3s-9)}{s(s+1)(s+3)}. \end{aligned}$$

An inverse transform shows that $x_1(t) = -2 + 2e^{-t} + e^{-3t}$ and $x_2(t) = -6 + 5e^{-t} + 3e^{-3t}$.

Exercise Solution 6.3.5. *Laplace transforming and solving for $X_1(s), X_2(s)$ yields*

$$\begin{aligned} X_1(s) &= \frac{s(s-3)}{(s+1)(s^2+1)} \\ X_2(s) &= \frac{s(3s-5)}{(s+1)(s^2+1)}. \end{aligned}$$

An inverse transform shows that $x_1(t) = 2e^{-t} - \cos(t) - 2\sin(t)$ and $x_2(t) = 4e^{-t} - \cos(t) - 4\sin(t)$.

Exercise Solution 6.3.7. *Laplace transforming and solving for $X_1(s), X_2(s), X_3(s)$ yields*

$$\begin{aligned} X_1(s) &= \frac{s^3+2s^2+s+6}{s(s+1)(s+2)(s+3)} \\ X_2(s) &= \frac{s+4}{(s+2)(s+3)} \\ X_3(s) &= -\frac{s^2+10s+3}{s(s+1)(s+3)}. \end{aligned}$$

An inverse transform shows that $x_1(t) = 1 + e^{-3t} + 2e^{-2t} - 3e^{-t}$, $x_2(t) = 2e^{-2t} - e^{-3t}$, *and* $x_3(t) = -1 - 3e^{-t} + 3e^{-3t}$.

Exercise Solution 6.3.9.

$$\mathbf{A} = \begin{bmatrix} 7 & -4 \\ 20 & -11 \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t) = e^{-2t}\begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

A guess of the form $\mathbf{x}_p(t) = e^{-2t}\mathbf{v}$ *with* $\mathbf{f}(t) = e^{-2t}\mathbf{w}$ *where* $\mathbf{w} = \langle 3, 7\rangle$ *leads to* $(\mathbf{A} + 2\mathbf{I})\mathbf{v} = -\mathbf{w}$ *and then* $\mathbf{v} = (\mathbf{A} + 2\mathbf{I})^{-1}\mathbf{w} = \langle 1, 3\rangle$. *So*

$$\mathbf{x}_p(t) = e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

A homogeneous general solution is

$$\mathbf{x}_h(t) = c_1 e^{-3t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution to the nonhomogeneous system is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The initial data yields $c_1 = -2$, $c_2 = 5$.

Exercise Solution 6.3.11.

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

A guess of the form $\mathbf{x}_p(t) = \mathbf{v}$ *with* $\mathbf{f}(t) = \mathbf{w}$ *where* $\mathbf{w} = \langle 2, -2\rangle$ *leads to* $\mathbf{A}\mathbf{v} = -\mathbf{w}$ *and then* $\mathbf{v} = (\mathbf{A})^{-1}\mathbf{w} = \langle 8, 13\rangle$. *So*

$$\mathbf{x}_p(t) = \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

A homogeneous general solution is

$$\mathbf{x}_h(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution to the nonhomogeneous system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

The initial data yields $c_1 = 3$, $c_2 = -13$.

Exercise Solution 6.3.13.

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = \cos(t)\begin{bmatrix} 5 \\ 12 \end{bmatrix} + \sin(t)\begin{bmatrix} -3 \\ -12 \end{bmatrix}.$$

Again follow the hints: take a guess of the form $\mathbf{x}_p(t) = \cos(t)\mathbf{v}_1 + \sin(t)\mathbf{v}_2$ *with* $\mathbf{f}(t) = \cos(t)\mathbf{w}_1 + \sin(t)\mathbf{w}_2$ *where* $\mathbf{w}_1 = \langle 5, 12\rangle$ *and* $\mathbf{w}_2 = \langle -3, -12\rangle$*. Then solving the linear system* $(\mathbf{A}^2 + \mathbf{I})\mathbf{v}_1 = -(\mathbf{A}\mathbf{w}_1 + \mathbf{w}_2)$ *yields* $\mathbf{v}_1 = \langle 0, 2\rangle$ *and then* $\mathbf{v}_2 = \mathbf{A}\mathbf{v}_1 + \mathbf{w}_1 = \langle 1, 0\rangle$*. A particular solution is*

$$\mathbf{x}_p(t) = \cos(t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A homogeneous general solution is

$$\mathbf{x}_h(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution to the nonhomogeneous system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The initial data yields $c_1 = 1$*,* $c_2 = -2$*.*

Section 6.4

Exercise Solution 6.4.1. *The eigenvalues and eigenvectors lead to*

$$\mathbf{D} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{P} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} = \begin{bmatrix} -3e^{-2t} + 4e^{-t} & 6e^{-2t} - 6e^{-t} \\ -2e^{-2t} + 2e^{-t} & 4e^{-2t} - 3e^{-t} \end{bmatrix}.$$

For Putzer's algorithm (with $\lambda_1 = -2, \lambda_2 = -1$) we find

$$\begin{aligned} \mathbf{P}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{P}_1 &= \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \\ r_1(t) &= e^{-2t} \\ \mathbf{P}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ r_2(t) &= e^{-t} - e^{-2t}. \end{aligned}$$

Putzer's algorithm yields the same result as diagonalization.

The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \langle 1, 2 \rangle$ is

$$\mathbf{x}(t) = \begin{bmatrix} -8\,\mathrm{e}^{-t} + 9\,\mathrm{e}^{-2\,t} \\ -4\,\mathrm{e}^{-t} + 6\,\mathrm{e}^{-2\,t} \end{bmatrix}.$$

Exercise Solution 6.4.3. *The eigenvalues and eigenvectors lead to*

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

Then

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} = \begin{bmatrix} -2\,\mathrm{e}^{2\,t} + 3\,\mathrm{e}^{-t} & \mathrm{e}^{2\,t} - \mathrm{e}^{-t} \\ -6\,\mathrm{e}^{2\,t} + 6\,\mathrm{e}^{-t} & 3\,\mathrm{e}^{2\,t} - 2\,\mathrm{e}^{-t} \end{bmatrix}.$$

For Putzer's algorithm (with $\lambda_1 = -1, \lambda_2 = 2$) we find

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{P}_1 = \begin{bmatrix} -6 & 3 \\ -18 & 9 \end{bmatrix}$$
$$r_1(t) = e^{-t}$$
$$\mathbf{P}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$r_2(t) = e^{2t}/3 - e^{-t}/3.$$

Putzer's algorithm yields the same result as diagonalization.

The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \langle 0, -2 \rangle$ is

$$\mathbf{x}(t) = \begin{bmatrix} -2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{-t} \\ -6\,\mathrm{e}^{2\,t} + 4\,\mathrm{e}^{-t} \end{bmatrix}.$$

Exercise Solution 6.4.5. *This matrix has one eigenvalue of -2 and a double eigenvalue $\lambda = -1$, defective. With eigenvalues in the order $-2, -1, -1$ and Putzer's algorithm we find*

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{P}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$
$$r_1(t) = e^{-2t}$$
$$\mathbf{P}_2 = \begin{bmatrix} 2 & -2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$
$$r_2(t) = e^{-2t} + e^{-t}$$
$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$r_3(t) = (t-1)e^{-t} + e^{-2t}.$$

Putzer's algorithm yields

$$e^{t\mathbf{A}t} = r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 + r_3(t)\mathbf{P}_2$$
$$= \begin{bmatrix} (2\,t-1)\,\mathrm{e}^{-t}+2\,\mathrm{e}^{-2\,t} & (-2\,t+3)\,\mathrm{e}^{-t}-3\,\mathrm{e}^{-2\,t} & (2\,t-1)\,\mathrm{e}^{-t}+\mathrm{e}^{-2\,t} \\ \mathrm{e}^{-t}t & -(t-1)\,\mathrm{e}^{-t} & \mathrm{e}^{-t}t \\ (-t+2)\,\mathrm{e}^{-t}-2\,\mathrm{e}^{-2\,t} & (t-3)\,\mathrm{e}^{-t}+3\,\mathrm{e}^{-2\,t} & (-t+2)\,\mathrm{e}^{-t}-\mathrm{e}^{-2\,t} \end{bmatrix}.$$

The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with* $\mathbf{x}(0) = \langle 1, 0, -1\rangle$ *is*

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{-2\,t} \\ 0 \\ -\mathrm{e}^{-2\,t} \end{bmatrix}.$$

Section 7.1

Exercise Solution 7.1.1. *The vectors are shown in the left panel of Figure 7.33.*

Exercise Solution 7.1.2. *The vectors are shown in the right panel of Figure 7.33.*

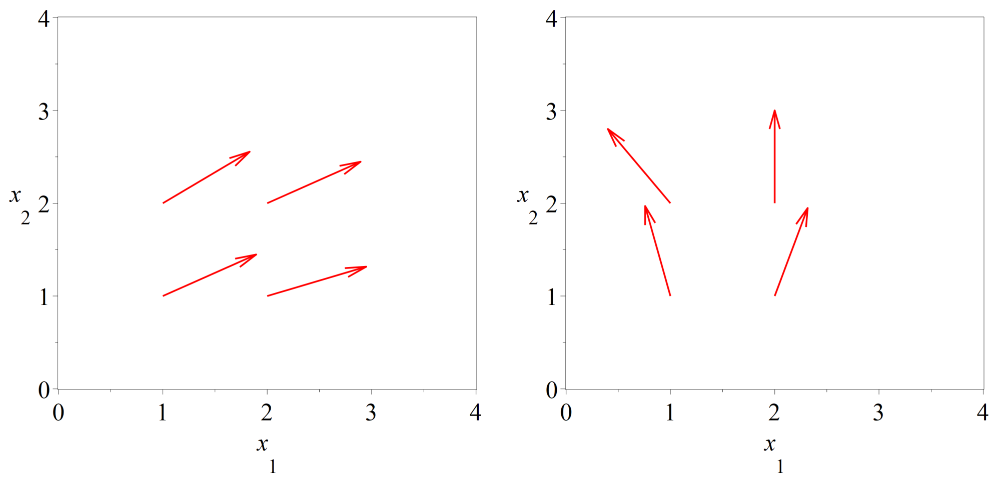

Figure 7.33: Vectors for Exercises 7.1.1 (left panel) and 7.1.2 (right panel).

Exercise Solution 7.1.5. *A direction field and a few solutions are shown in Figure 7.34. Solution converge to either* $(3, 0)$ *or* $(0, 3)$*. It appears that one species must go extinct, the other limits to its carrying capacity.*

Exercise Solution 7.1.8. *A direction field and a few solutions are shown in Figure 7.35. Solutions form closed orbits, indicating that the pendulum never stops moving. This makes perfect sense (no friction).*

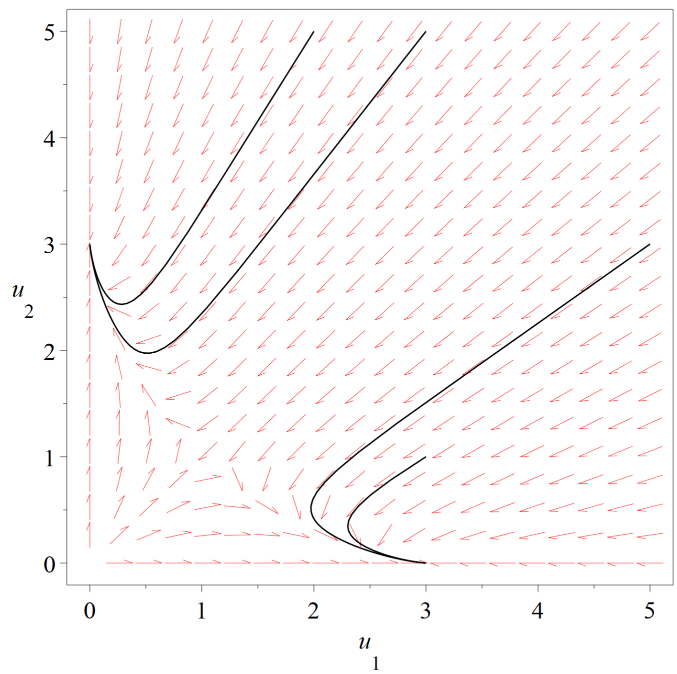

Figure 7.34: Direction field for competing species with $r_1 = 1$, $r_2 = 1$, $K_1 = 3$, $K_2 = 3$, $a = 2$, and $b = 2$, and a few solution trajectories.

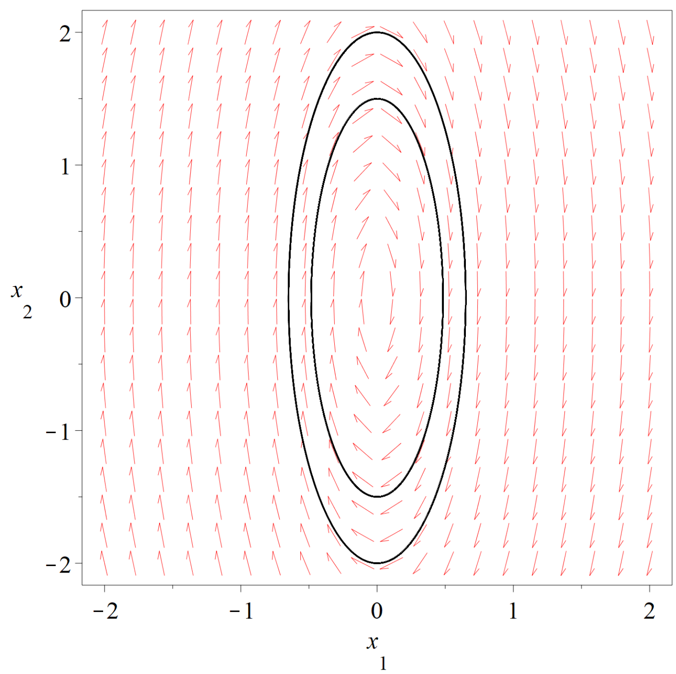

Figure 7.35: Direction field for undamped pendulum equation (as a first order system), with a few solution trajectories.

Section 7.2

Exercise Solution 7.2.1. *See Figure 7.36. Eigenvalues are real,* -2 *and* -4.

Exercise Solution 7.2.3. *See Figure 7.37. Eigenvalues are real,* 2 *and* 4.

Exercise Solution 7.2.5. *SeeFigure 7.38. Eigenvalues are complex,* $-1 \pm 2i$.

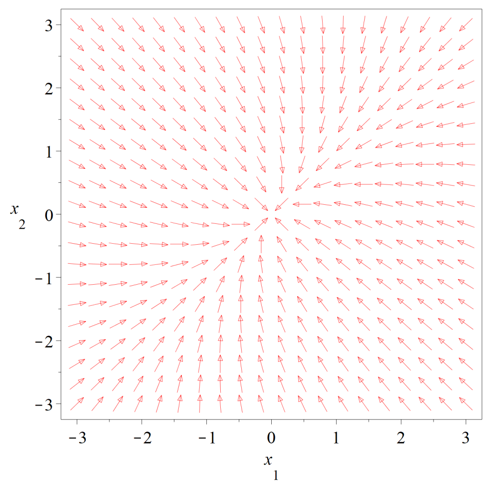

Figure 7.36: Direction field for Exercise 7.2.1.

Exercise Solution 7.2.7. *See Figure 7.39.*

Exercise Solution 7.2.9. *See the left panel in Figure 7.40.*

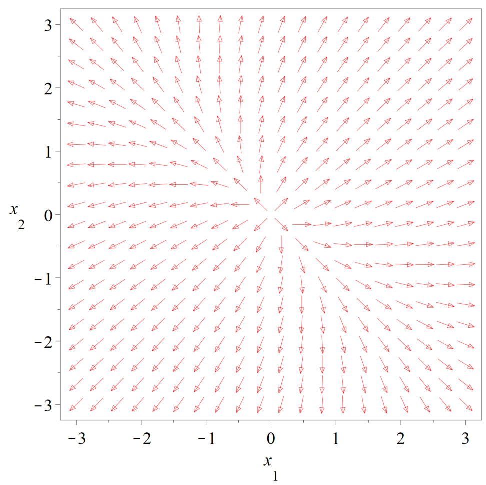

Figure 7.37: Direction field for Exercise 7.2.3.

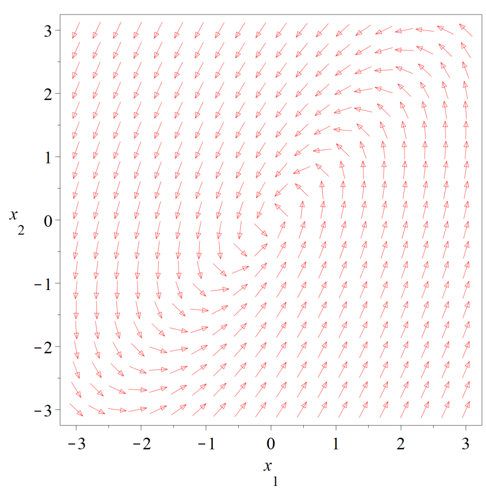

Figure 7.38: Direction field for Exercise 7.2.5.

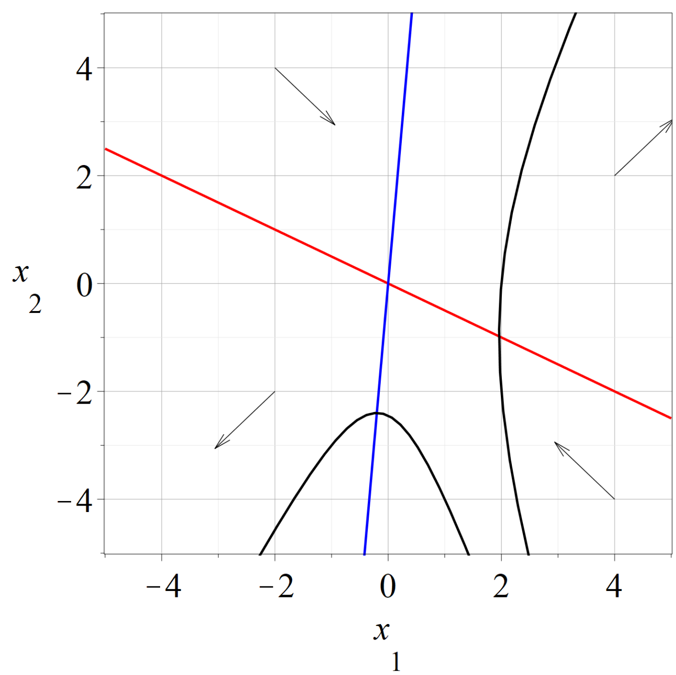

Figure 7.39: Phase portraits and solution curves for Exercise 7.2.7.

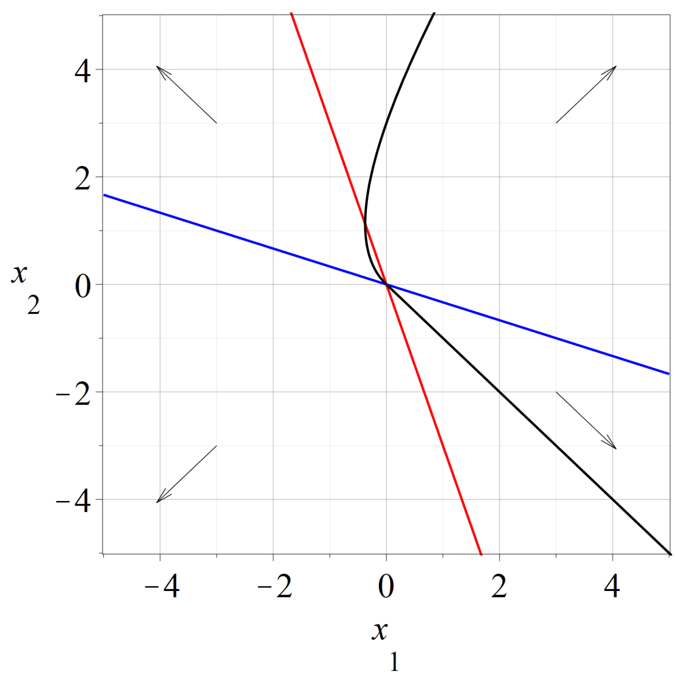

Figure 7.40: Phase portraits and solution curves for Exercise 7.2.9.

Section 7.3

Exercise Solution 7.3.1. *See Figure 7.41 for the phase portrait, Figure 7.42 for solution sketches with the given initial conditions. The solution with initial conditions* $x_1(0) = -1, x_2(0) = 3$ *does not extended past about* $t \approx 1.2$. *The fixed points are* $(-2,-2)$ *and* $(1,1)$. *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} -2x_1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then

$$\mathbf{J}(-2,-2) = \begin{bmatrix} 4 & -1 \\ 1 & -1 \end{bmatrix}.$$

has approximate eigenvalues 3.79 *and* -0.79, *so this is a saddle point. Also*

$$\mathbf{J}(1,1) = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}.$$

has approximate eigenvalues $-1.5 \pm 0.866i$, *so this is an asymptotically stable spiral point.*

Exercise Solution 7.3.3. *See Figure 7.43 for the phase portrait, Figure 7.44 for solution sketches with the given initial conditions. The fixed points are* $(-3,0)$ *and* $(-1,1)$. *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} x_2 & x_1 + 2x_2 \\ 1 & -2 \end{bmatrix}.$$

Then

$$\mathbf{J}(-3,0) = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}.$$

has eigenvalues $-1 \pm i\sqrt{2}$, *so this is an asymptotically stable spiral point. Also*

$$\mathbf{J}(-1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

has approximate eigenvalues 1.3 *and* -2.3, *so this is a saddle point.*

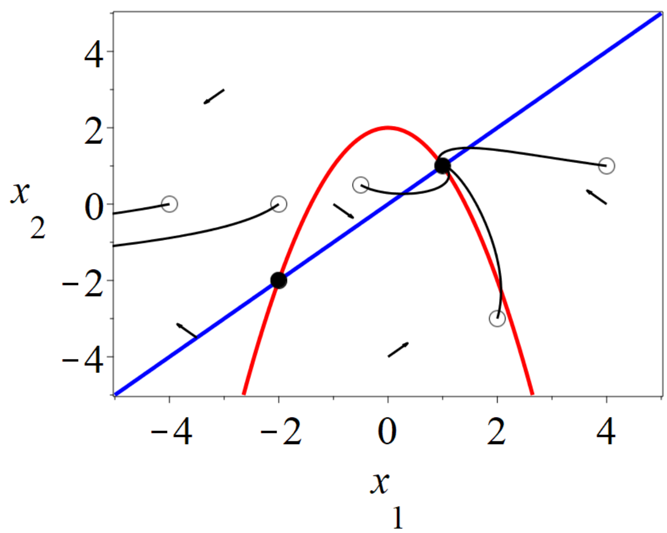

Figure 7.41: Phase portrait for Exercise 7.3.1.

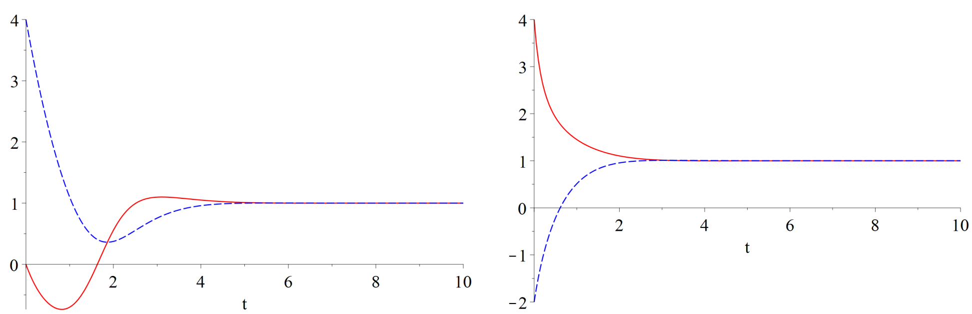

Figure 7.42: Individual solutions components for Exercise 7.3.1, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = 0, x_2(0) = 4$ (left panel) and $x_1(0) = 4, x_2(0) = -2$ (right panel).

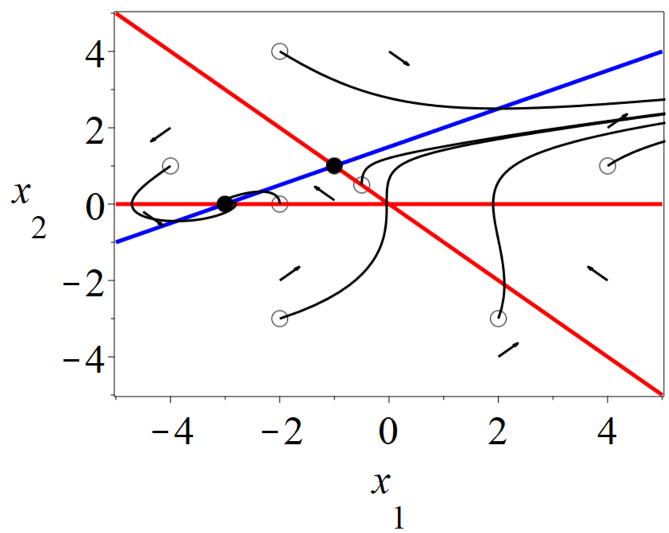

Figure 7.43: Phase portrait for Exercise 7.3.3.

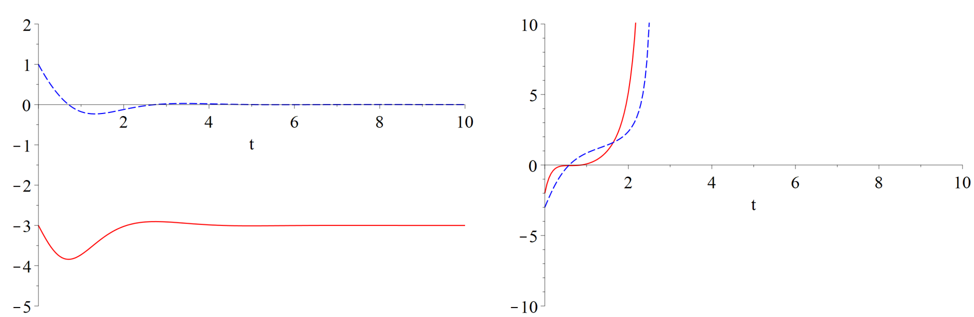

Figure 7.44: Individual solutions components for Exercise 7.3.3, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = -3, x_2(0) = 1$ (left panel) and $x_1(0) = -2, x_2(0) = -3$ (right panel).

Section 7.4

Exercise Solution 7.4.1.

(a) *The equation* $-ax_2+x_2^2 = 0$ *forces* $x_2 = 0$ *or* $x_2 = a$ *and then* $x_1-x_2 = 0$ *yields* $x_1 = 0$ *or* $x_1 = a$. *The fixed points are* $(0,0)$ *and* (a,a).

(b) *The* x_1 *nullcline consists of the horizontal lines* $x_2 = 0$ *and* $x_2 = a$. *For* $x_2 < 0$ *we find* $\dot{x}_1 > 0$ *so solutions move in the direction of increasing* x_1 *(to the right). For* $0 < x_2 < a$ *solutions move to the left, and for* $x_2 > a$ *solutions move to the right. This nullcline is shown in the left panel of Figure 7.45.*

(c) *The* x_2 *nullcline consists of the diagonal line* $x_2 = x_1$. *For* $x_2 < x_1$ *we find* $\dot{x}_2 < 0$ *so solutions move in the direction of decreasing* x_2 *(down). For* $x_2 > x_1$ *solutions upward. This nullcline is shown in the right panel of Figure 7.45.*

(d) *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} = 0 & -a + 2x_2 \\ 1 & -1 \end{bmatrix}.$$

At the fixed point $(0,0)$ *we find*

$$\mathbf{J}(0,0) = \begin{bmatrix} = 0 & -a \\ 1 & -1 \end{bmatrix}.$$

The determinant D *of this matrix equals* a, *which is positive by assumption, so* $(0,0)$ *is always stable. The trace* T *of this matrix is* -1. *If* $0 < a < 1/4$ *(so* $0 < D < T^2/4$*) then* $(0,0)$ *is an asymptotically stable node and if* $a > 1/4$ *then* $(0,0)$ *is an asymptotically stable spiral point.*

At (a,a) *the Jacobian is*

$$\mathbf{J}(a,a) = \begin{bmatrix} 0 & a \\ 1 & -1 \end{bmatrix}.$$

The determinant here is $D = -a$, *so if* $a > 0$ *this is a saddle.*

(e) *See Figure 7.46 for the case* $a > 1/4$ *and Figure 7.47 for the case* $a < 1/4$. *The solutions have the same general behavior, except when* $a < 1/4$ *they do not spiral as they approach the fixed point* $(0,0)$.

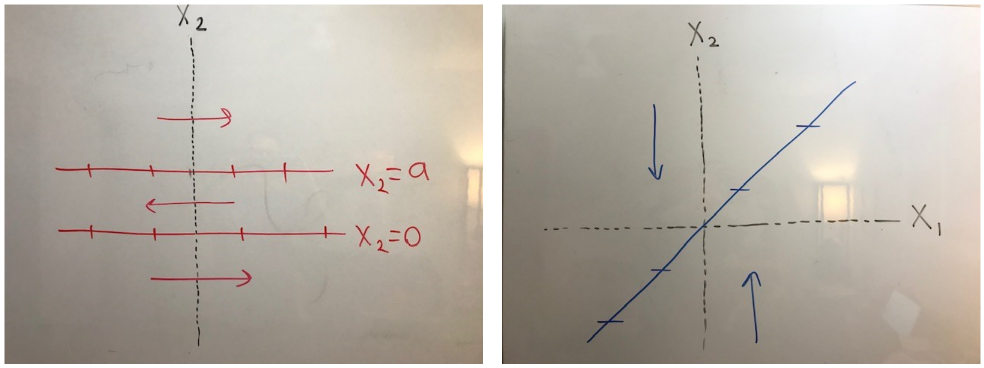

Figure 7.45: Nullclines for x_1 (left) and x_2 (right) for Problem 7.4.1.

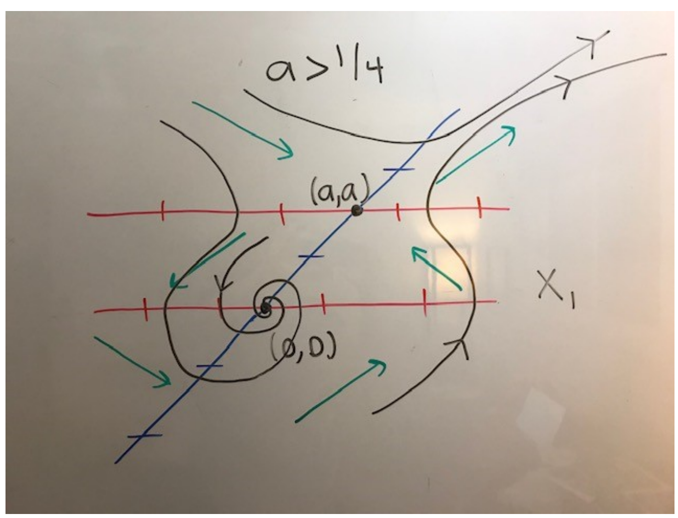

Figure 7.46: Phase portrait for system in Problem 7.4.1, $a > 1/4$.

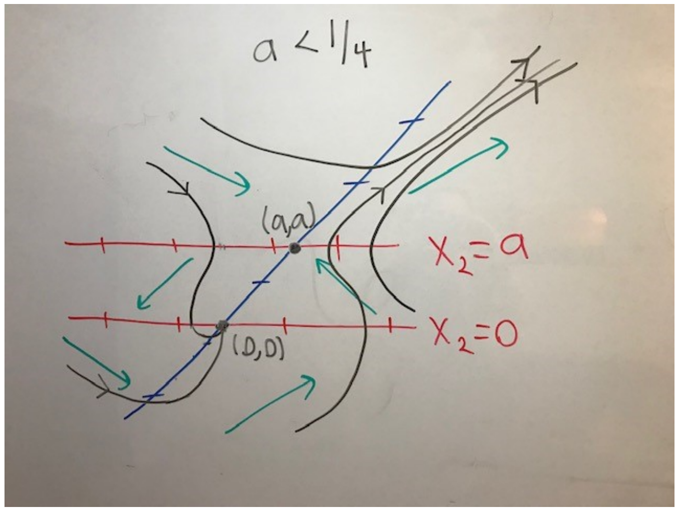

Figure 7.47: Phase portrait for system in Problem 7.4.1, $a < 1/4$.

Exercise Solution 7.4.3. *In each case the Jacobian matrix is*

$$\mathbf{J}(v_1, v_2) = \begin{bmatrix} r_1(1 - 2v_1 - \bar{a}v_2) & -r_1 a v_1 \\ -r_2 b v_2 & r_2(1 - 2v_2 - \bar{b}v_1) \end{bmatrix}.$$

The eigenvalues of $\mathbf{J}(0,0)$ *in every case are* r_1 *and* r_2*, both positive, so the origin is always an unstable node.*

(a) *See Figure 7.48. The fixed points here are* $(0,0), (0,1),$ *and* $(1,0)$*. At* $(0,1)$ *the eigenvalues are* 0 *and* $-r_2$*, so this is not a hyperbolic equilibrium point. At* $(1,0)$ *the eigenvalues are* $-r_1 < 0$ *and* $r_2(1-\bar{b}) > 0$*, so this is a saddle. Although we can't use the Hartman-Grobman Theorem at* $(0,1)$*, it certainly looks stable.*

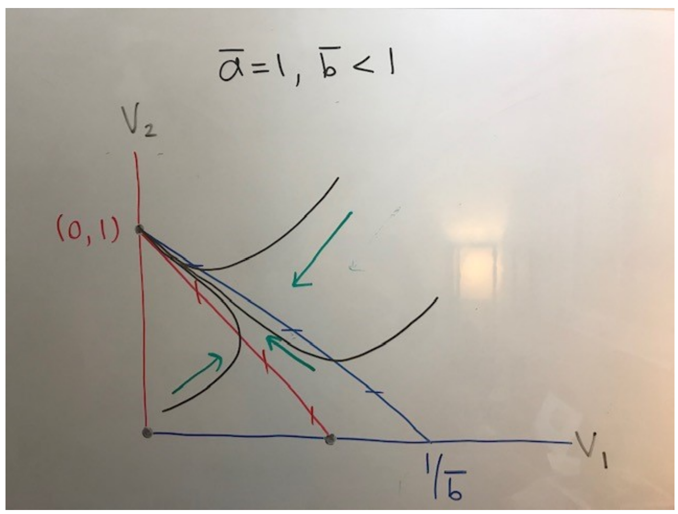

Figure 7.48: Phase portrait for Problem 7.4.3 part (a).

Section 7.5

Exercise Solution 7.5.1.

(a) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 1, 2\rangle$*. Then with* $\mathbf{f}(t, \mathbf{x}) = \langle x_1 - x_2, x_1 + x_2\rangle$ *we have true solution* $\mathbf{x}(t) = \langle e^t(\cos(t) - 2\sin(t)), e^t(2\cos(t) + \sin(t))\rangle$ *with* $\mathbf{x}(1.0) \approx \langle -3.11, 5.22\rangle$*.*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 1, 2\rangle) = \langle 0.5, 3.5\rangle$$

and

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 0.5, 3.5\rangle) = \langle -1, 5.5\rangle.$$

(b) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 1, 2\rangle$*. Then with* $\mathbf{f}(t, \mathbf{x}) = \langle x_1 + x_2, x_1 + x_2\rangle$ *we have true solution* $\mathbf{x}(t) = \langle -1/2 + 3e^{2t}/2, 1/2 + 3e^{2t}/2\rangle$ *with* $\mathbf{x}(1.0) \approx \langle 10.58, 11.58\rangle$*. Also*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 1, 2\rangle) = \langle 2.5, 3.5\rangle$$

and

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 2.5, 3.5\rangle) = \langle 5.5, 6.5\rangle.$$

(c) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 0, 0, 1\rangle$*. Define* $\mathbf{f}(t, \mathbf{x}) = \langle x_1x_2 + 1 - t^3, x_1 + x_2 + t - t^2, x_2x_3 - 1 - t^2 + t^3\rangle$*. Compute*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 0, 0, 1\rangle) = \langle 0.5, 0, 0.5\rangle$$

and

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 0.5, 0, 0.5\rangle) = \langle 0.9375, 0.375, -0.0625\rangle.$$

(d) *The error for each step size is* $0.567, 0.0604$*, and* 0.00607*, approximately proportional to* h*.*

(e) *The error for each step size is* 0.175*,* 0.0196*, and* 0.00199*, approximately proportional to* h*.*

Exercise Solution 7.5.4.

(a) *First, the analytical solution is* $x(t) = e^{-0.25t}$*.*

Set $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $x_0 = 1$*. Then* x_1 *satisfies* $x^1 = (0.5)(-0.25x^1) + 1$*, which leads to* $x^1 \approx 0.889$*. Then* x_2 *satisfies* $x^2 = (0.5)(-0.25x^2) + 0.889$*, which leads to* $x^1 \approx 0.790$*. The true solution value is* $x(1) = e^{-0.25} \approx 0.779$*.*

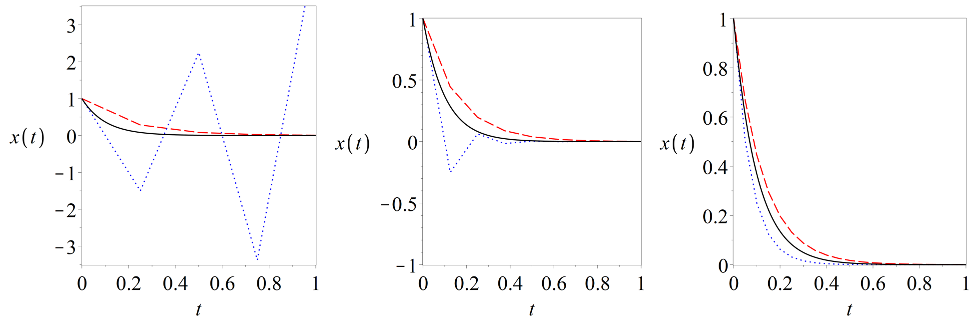

Figure 7.49: Left panel: step size $h = 0.25$ for ODE $x' = -10x$ with $x(0) = 1$ for Euler's method (dotted/blue), the implicit Euler method (dashed/red) and true solution $x(t) = e^{-10t}$ (solid/black). Middle panel: same, step size $h = 0.125$. Right panel: same, step size $h = 0.05$.

(b) *Set* $t_0 = 0, t_1 = 1, t_2 = 2$ *and* $x_0 = 1$. *Then* x_1 *satisfies* $x^1 = 0.5x^1(2 - x^1) + 1$, *which leads to* $x^1 = \sqrt{2} \approx 1.4142$. *Then* x_2 *satisfies* $x^2 = 0.5x^2(2 - x^2) + 1.4142$, *which leads to* $x^2 \approx 1.682$. *The true solution is* $x(t) = 2/(1 + e^{-t})$ *so* $x(2) = 2/(1 + e^{-2}) \approx 1.762$.

(c) *We have* $t_1 = 1, t_2 = 2, t_3 = 3$ *and* $\mathbf{x}^0 = \langle 1, 3\rangle$. *Then* $\mathbf{x}^1 \approx \langle -0.167, 0.167\rangle$, $\mathbf{x}^2 \approx \langle -0.194, -0.139\rangle$, *and* $\mathbf{x}^3 \approx \langle -0.116, -0.106\rangle$. *The true solution is* $\mathbf{x}(t) = \langle 2e^{-5t} - e^{-t}, 4e^{-5t} - e^{-t}\rangle$ *and* $\mathbf{x}(3) \approx \langle -0.0498, -0.0498\rangle$.

(d) *With* $t_0 = 0, t_1 = 0.2, t_2 = 0.4, t_3 = 0.6, t_4 = 0.8, t_5 = 1.0$ *and* $\mathbf{x}^0 = \langle 1, 3\rangle$ *we find iterates*

$$\mathbf{x}^1 \approx \langle 0.589, 2.402\rangle, \mathbf{x}^2 \approx \langle 0.204, 1.968\rangle, \mathbf{x}^3 \approx \langle -0.125, 1.660\rangle,$$
$$\mathbf{x}^4 \approx \langle -0.385, 1.448\rangle, \mathbf{x}^5 \approx \langle -0.579, 1.303\rangle.$$

Exercise Solution 7.5.5.

(a) *See the left panel of Figure 7.49 for step size* $h = 0.25$, *the middle panel for* $h = 0.15$, *and the right panel for* $h = 0.05$. *According to (7.48) (with* $\lambda = 10$*) the iterates here converge to zero when* $h < 0.2$, *which is in accordance with the figure. From Reading Exercise 7.5.4 the iterates should remain positive when* $h < 0.1$, *which again seems correct.*

(b) *The analytical solution is* $x_1(t) = 3e^{-t} - 2e^{-5t}$, $x_2(t) = 3e^{-t} - 4e^{-5t}$. *See Figure 7.50 for parametric plots. When* $h = 1.0$ *the solution goes well outside the view range.*

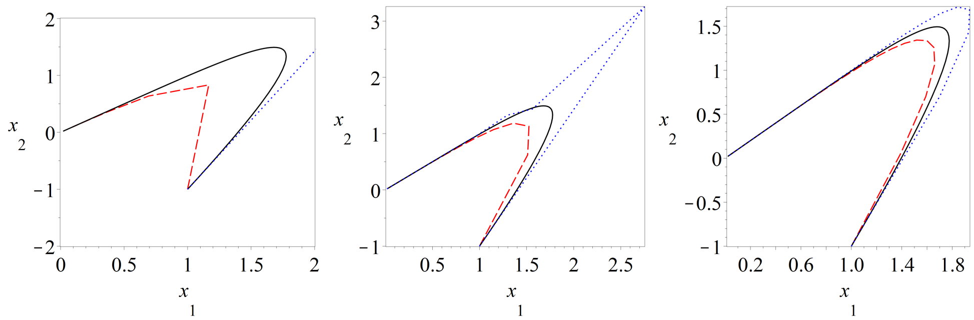

Figure 7.50: Left panel: parametric plot $x_1(t)$ vs $x_2(t)$ for step size $h = 1.0$ for Euler's method (dotted/blue), the implicit Euler method (dashed/red) and true solution $x_1(t) = 3e^{-t} - 2e^{-5t}$, $x_2(t) = 3e^{-t} - 4e^{-5t}$ (solid/black). Middle panel: same, step size $h = 0.25$. Right panel: same, step size $h = 0.1$.

Exercise Solution 7.5.6.

(a) *The true solution is* $x(t) = t - 1 + 2e^{-t}$ *and* $x(1) = 2/e$. *The errors for implicit Euler with step sizes* $h = 0.1, 0.01, 0.001,$ *and* 0.0001 *are* $0.0353276965, 0.0036635421, 0.0003677258, 0.0000367826,$ *respectively.*

(b) *The analytical solution is* $x_1(t) = 6e^{-t} - 52e^{-5t}/25 + 13t/5 - 73/25, x_2(t) = 6e^{-t} - 104e^{-5t}/25 + 11t/5 - 71/25$. *The errors for* $h = 0.1, 0.01,$ *and* 0.001 *are* $0.104268966843117747, 0.0117885987798727332,$ *and* 0.00119297073597383397.

Exercise Solution 7.5.9.

(a) *The system is* $\dot{x}_1 = x_2, \dot{x}_2 = -101x_1 - 2x_2$ *with* $\mathbf{x}(0) = \langle 1, 0 \rangle$.

(b) *The eigenvalues and eigenvectors of* $\mathbf{A}$ *are* $-1 \pm 10i$ *and* $\langle -1 - 10i, 101 \rangle$ *and* $\langle -1 + 10i, 101 \rangle$, *respectively. A real-valued general solution is*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{-t}\sin(10t) \\ e^{-t}(10\cos(10t) - \sin(10t)) \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\cos(10t) \\ -e^{-t}(\cos(10t) + 10\sin(10t)) \end{bmatrix}.$$

With the given initial data the solution is

$$\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos(10t) + \sin(10t)/10 \\ -101\sin(10t)/10 \end{bmatrix}.$$

The solution spirals toward the asymptotically stable fixed point at $\langle 0, 0 \rangle$.

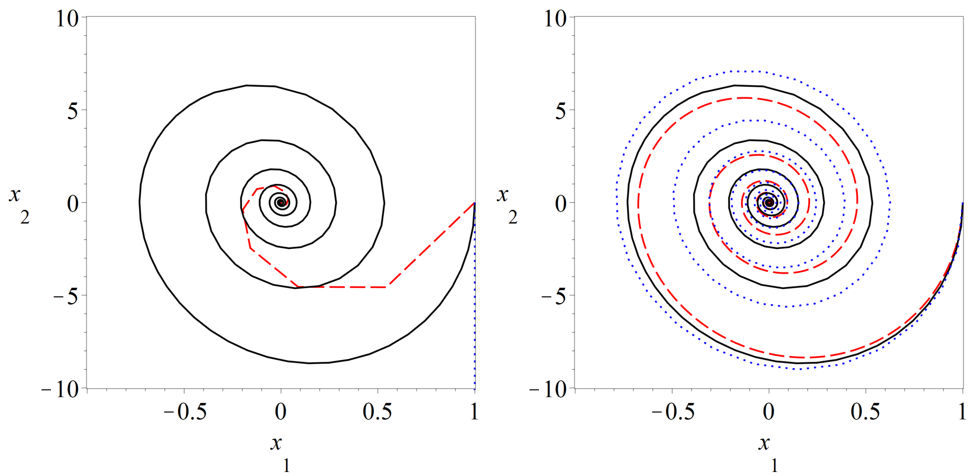

Figure 7.51: Left panel: True solution (solid/black), Euler estimate (dotted/blue) and implicit Euler (dashed/red), step size $h = 0.1$. Right panel: Same, but with $h = 0.005$.

(c) The true solution value is $\mathbf{x}(5) \approx \langle 0.0063, 0.0173\rangle$. *Implicit Euler gives estimate* $\langle 1.52 \times 10^{-9}, 1.79 \times 10^{-9}\rangle$. *Standard Euler's method explodes. A plot is shown in the left panel of Figure 7.51.*

(d) A step size $h \le 0.005$ *tames Euler's method. With* $h = 0.005$ *implicit Euler gives estimate* $\langle 0.00158, 0.0106\rangle$. *Standard Euler's method gives* $\langle 0.0233, 0.0134\rangle$. *A plot is shown in the right panel of Figure 7.51.*

Section 7.6

Exercise Solution 7.6.1.

(b) *Compute $D_1 = a_1, D_2 = a_1a_2 - a_3$, and $D_3 = (a_1a_2 - a_3)a_3$. The roots of the polynomial $p(z) = z^3 + a_1z^2 + a_2z + a_3$ all have negative real part exactly when D_1, D_2, and D_3 are all positive, so $a_1 > 0, a_1a_2 - a_3 > 0$, and $a_3(a_1a_2 - a_3) > 0$. The last condition $a_3(a_1a_2 - a_3) > 0$ can be replaced by $a_1a_2 - a_3 > 0$ when $a_3 > 0$.*

Exercise Solution 7.6.3.

(a) *The system is $\dot{x}_1 = x_2$, $m\dot{x}_2 = 0$ (or just $\dot{x}_2 = 0$, since $m > 0$). Then $\mathbf{f}(\mathbf{x}) = \langle x_2, 0\rangle$.*

(b) *We have $\nabla P = \langle 0, m\rangle$ and then $\nabla P \cdot \mathbf{f} = 0$, so P is a first integral and represents a conserved quantity. The function P is just the momentum $m\dot{x}$ of the particle, so this is conservation of momentum.*

In this very simple setting, in both (b) and (c) here the essential fact is that $\dot{x}$ is constant.

Exercise Solution 7.6.5. *It's easy to check that $x_1 = x_2 = 0$ is an isolated fixed point. A direction field is shown in Figure 7.52, with a few solution curves and the level curves for the function $V(x_1, x_2) = x_1^2 + x_2^2$.*

The linearized system at the origin has Jacobian matrix

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with double eigenvalue 0, which does not allow us to make any conclusion about stability. With $V(x_1, x_2) = x_1^2 + x_2^2$ with have $\nabla V = \langle 2x_1, 2x_2\rangle$ and with $\mathbf{f}(\mathbf{x}) = \langle -x_1^3, -x_2^3$ we find $\nabla V \cdot \mathbf{f} = -2(x_1^4 + x_2^4) < 0$ for $(x_1, x_2) \neq (0,0)$. We conclude that this fixed point is asymptotically stable.

Exercise Solution 7.6.7. *This system has infinitely many fixed points, all along the diagonal line $x_2 = -x_1/2$; see Figure 7.53, in which the direction field is plotted. The fixed points are shown along the dashed blue line, and a few solution trajectories are shown as solid black curves. The Jacobian at each fixed point is*

$$\mathbf{J} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$$

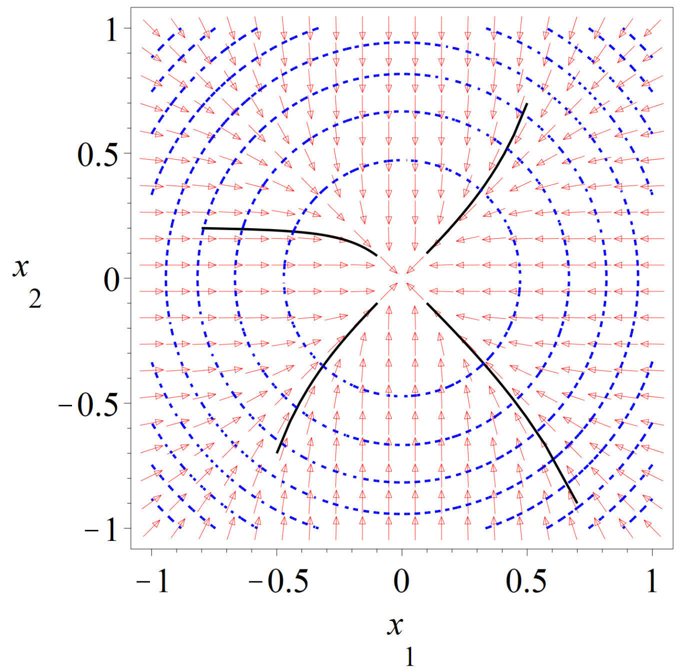

Figure 7.52: Direction field and solution curves (solid black) for system $\dot{x}_1 = -x_1^3, \dot{x}_2 = -x_2^3$, with level curves for $V(x_1, x_2) = x_1^2 + x_2^2$ (dashed blue).

with eigenvalues 0 *and* -5*, which does not (by itself) allow us to make conclusions about the stability of any of these fixed points. For the Lyapunov approach, if we take* $V(x_1, x_2) = x_1^2 + x_2^2$ *as suggested, a straightforward computation shows that* $\nabla V \cdot \mathbf{f} = -2x_1^2 - 8x_1x_2 - 8x_2^2$*. This last expression factors as* $-2(x_1 - 2x_2)^2$*, which is non-positive for all* x_1 *and* x_2*. We can conclude that fixed point at* $(0, 0)$ *(and in fact, any of the fixed points) is stable. We cannot conclude that any given fixed point is asymptotically stable, since they are not isolated. In fact by solving the system analytically we can see that the solution trajectories that start at a point* (a, b) *are straight lines that converge to the fixed point* $((4a - 2b)/5, (-2a + b)/5)$*.*

Exercise Solution 7.6.9. *Straightforward algebra shows that this system has an isolated fixed point at* $x_1 = x_2 = 0$*. The Jacobian at* $(0, 0)$ *is the zero matrix with double eigenvalue* 0*, which does not allow us to make conclusions about the stability of this fixed point. For the Lyapunov approach, if we take*

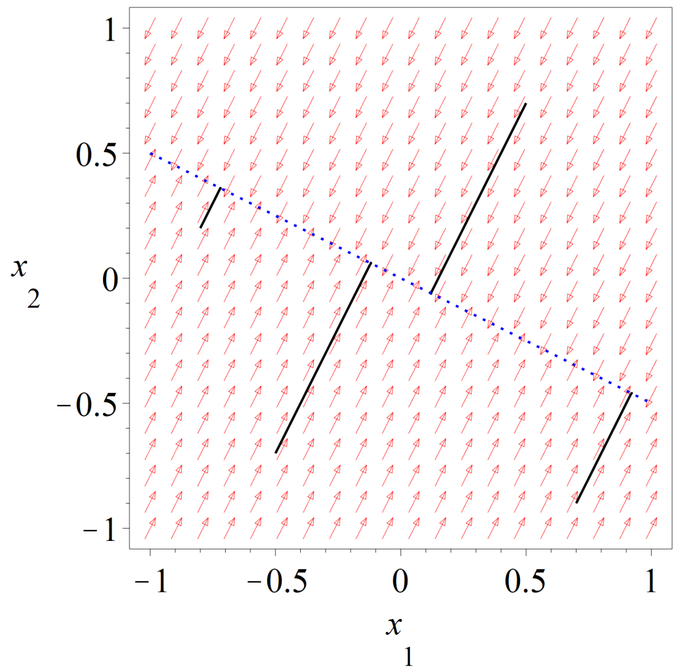

Figure 7.53: Direction field and fixed points (dashed blue line) for system $\dot{x}_1 = -x_1 - 2x_2, \dot{x}_2 = -2x_1 - 4x_2$, with solution trajectories (solid black).

$V(x_1, x_2) = x_1^4 + x_2^4$ as suggested, a straightforward computation shows that

$$\nabla V \cdot \mathbf{f} = 0$$

Thus this is a stable fixed point, but we cannot assert asymptotic stability. In fact, the solutions form closed orbits.

Exercise Solution 7.6.10. *A bit of easy algebra shows that $x_1 = x_2 = x_3 = 0$ is the only fixed point for this system. With $V(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2$ we obtain*

$$\nabla V \cdot \mathbf{f} = -4\,a{x_1}^2{x_2}^4 - 8\,b{x_1}^2{x_2}^4 - 4\,a{x_1}^2{x_3}^2 - 4\,c{x_3}^4 - 4\,b{x_2}^2 - 4\,c{x_3}^2$$

which is easily seen to be non-positive for any choice of a, b, c all positive (which also makes V itself positive definite). Thus the origin is stable, but no choice for a, b, c works to prove asymptotic stability (if $x_2 = x_3 = 0$ we can take any value for x_1.) The Jacobian at the origin is

$$\mathbf{J}(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Section 8.1

Exercise Solution 8.1.1. *Start with the continuity equation* $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ *and use the given fact that* $\frac{\partial \rho}{\partial t} = 0$ *to find* $\frac{\partial q}{\partial x} = 0$. *That is, q is independent of x. Conversely if* $\frac{\partial q}{\partial x} = 0$ *it is immediate that* $\frac{\partial \rho}{\partial t} = 0$, *so ρ does not depend on time.*

Exercise Solution 8.1.2. $u(x,t) = 3e^{-\pi^2 t^2} \sin(\pi x)$. *See left panel in Figure 8.54.*

Exercise Solution 8.1.3. $u(x,t) = 3e^{-\pi^2 t^2} \sin(\pi x) + 5e^{-36\pi^2 t} \sin(6\pi x)$. *See right panel in Figure 8.54.*

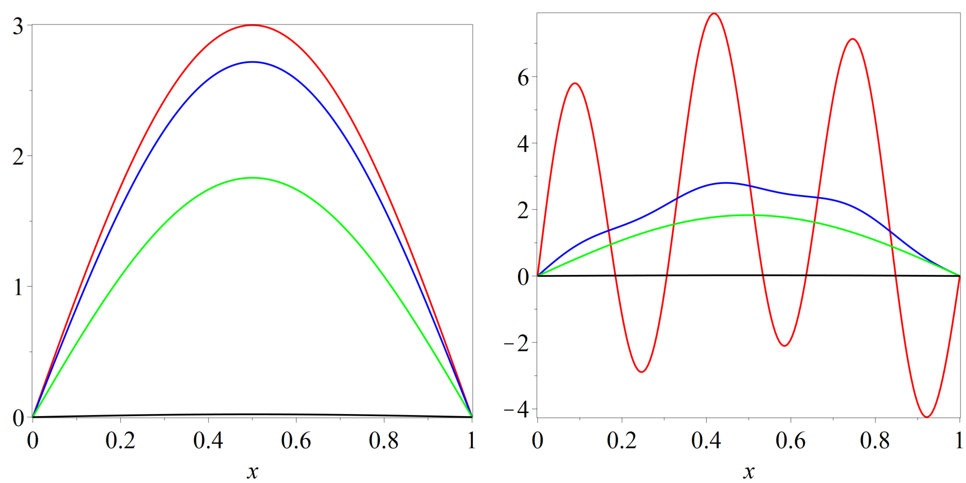

Figure 8.54: Figures for Exercises 8.1.2 (left) and 8.1.3 (right). In each case $t = 0$ is in red, $t = 0.01$ is blue, $t = 0.05$ is green, $t = 0.5$ is black. In each case the solution decays to 0 as t increases, at all points.

Exercise Solution 8.1.6. $u(x,t) = 3e^{-\pi^2 t} \cos(\pi x)$. *See top left panel in Figure 8.55.*

Exercise Solution 8.1.7. $u(x,t) = 4 + 3e^{-\pi^2 t} \cos(\pi x)$. *See top right panel in Figure 8.55.*

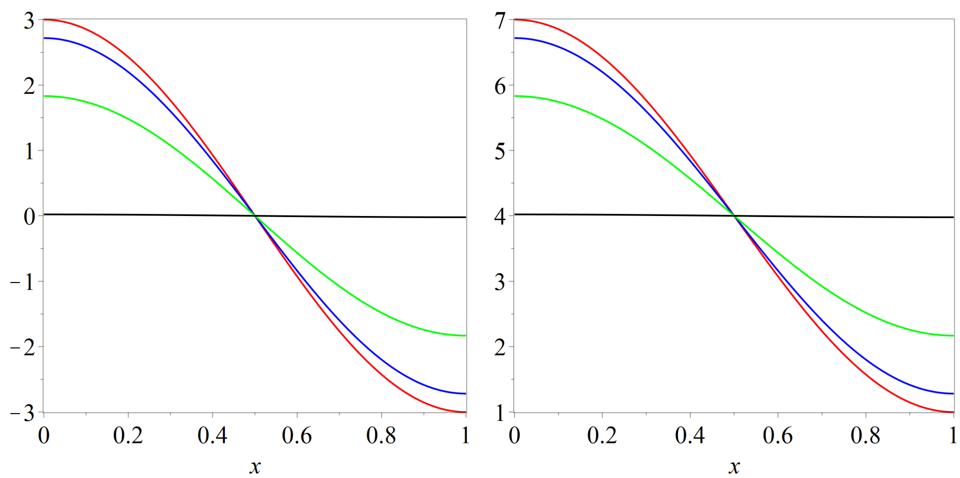

Figure 8.55: Figures for Exercises 8.1.6 (left) and 8.1.7 (right). In each case $t = 0$ is in red, $t = 0.01$ is blue, $t = 0.05$ is green, $t = 0.5$ is black. In each case the solution decays in time to a constant value (whatever the average value of $u(x, 0)$ is on the interval).

Section 8.2

Exercise Solution 8.2.1. *For $n \leq 1$ we obtain $s_n(x) = 0$ and for $n \geq 2$ find $s_n(x) = f(x) = 3\cos(2\pi x)$. Then $\|f - s_n\|_2 \approx 0.212$ for $n = 0, 1$ and $\|f - s_n\|_2 = 0$ for $n \geq 2$. This graph is omitted.*

Exercise Solution 8.2.3. *You should find that*

$$
\begin{aligned}
s_0(x) &= 1 \\
s_1(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} \\
s_3(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} - \frac{8\cos(\pi 3x/2)}{9\pi^2} \\
s_5(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} - \frac{8\cos(\pi 3x/2)}{9\pi^2} - \frac{8\cos(5\pi x/2)}{25\pi^2}.
\end{aligned}
$$

Also, $s_2 = s_1$ and $s_4 = s_3$. Then $\|f - s_0\|_2 \approx 0.816, \|f - s_1\|_2 \approx 0.098, \|f - s_5\|_2 \approx 0.022$. A plot is shown in Figure 8.56, left panel.

Exercise Solution 8.2.5. *The approximation s_{10} is*

$$s_{10}(x) \approx -0.053\cos(\pi x/3) + 0.186\cos(2\pi x/3) + \cdots - 0.026\cos(10\pi x/3).$$

The errors are $\|f - s_3\|_2 \approx 0.882, \|f - s_5\|_2 \approx 0.488, \|f - s_{10}\|_2 \approx 0.027$. A plot is shown in Figure 8.56, right panel.

Exercise Solution 8.2.7. *The coefficients here are $b_k = 4\sin(k\pi x)/(k\pi)$ when k is odd, $b_k = 0$ for k even. Then $\|f - s_1\|_2 = 0.435$, $\|f - s_3\|_2 = 0.315$, $\|f - s_{10}\|_2 = 0.201$. Plots of s_n for $n = 1, 3, 10$ are shown in Figure 8.57.*

Exercise Solution 8.2.8. *We find $s_1(x) = 0$, $s_n(x) = 3\sin(2\pi x)$ for $n = 2, 3$, and $s_n(x) = f(x) = 3\sin(2\pi x) - 4\sin(4\pi x)$ for $n \geq 4$. The errors are $\|f - s_1\|_2 = 3.536$, $\|f - s_2\|_2 = 2.828$, $\|f - s_{10}\|_2 = 0$. Graph here is omitted.*

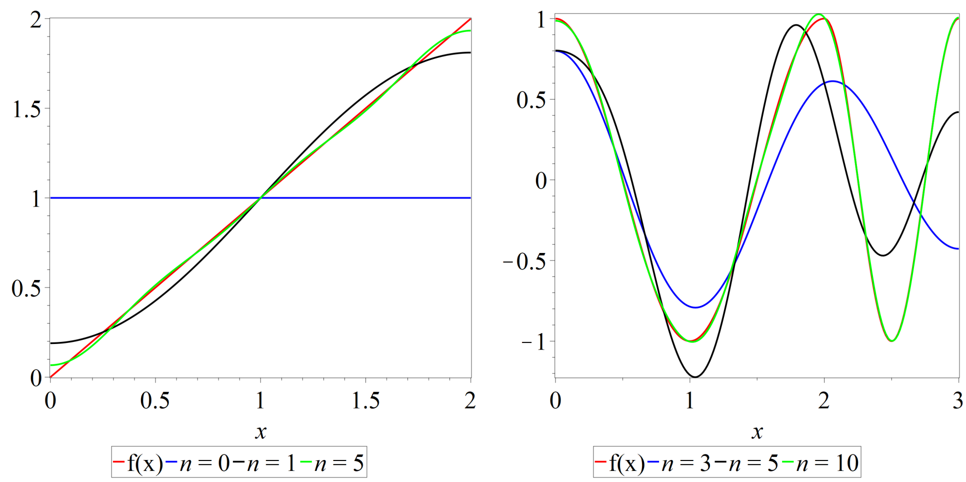

Figure 8.56: Graphs of $f(x)$ and $s_n(x)$ for various values of n for Exercises 8.2.3 (left) and 8.2.5 (right).

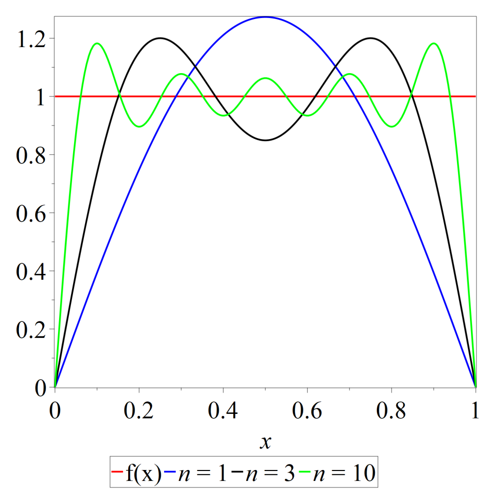

Figure 8.57: Graphs of $f(x)$ and $s_n(x)$ for various values of n for Exercise 8.2.7.

Section 8.3

Exercise Solution 8.3.1. *The approximate solution is*

$$u(x,t) \approx 0.918e^{-12.3t}\sin(1.57x) + 0e^{-49.3t}\sin(3.14x) + 0.133e^{-110t}\sin(4.71x).$$

Note $b_2 = 0$ here. Graph shown in the left panel of Figure 8.58.

Exercise Solution 8.3.2. *The approximate solution is*

$$u(x,t) \approx -0.360e^{-2.46t}\sin(1.57x) + e^{-9.86t}\sin(3.14x) - 0.388e^{-22.2t}\sin(4.71x).$$

Graph shown in the right panel of Figure 8.58.

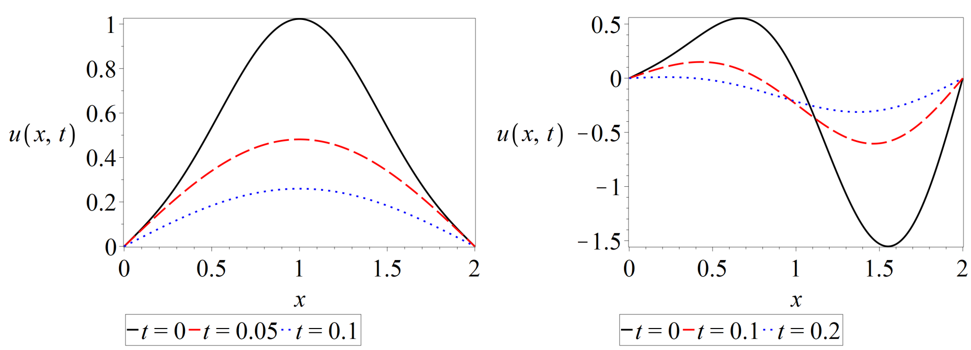

Figure 8.58: Solutions to Exercises 8.3.1 (left) and 8.3.2 (right).

Exercise Solution 8.3.5. *The approximate solution is*

$$u(x,t) \approx \frac{1}{30} - \frac{3}{\pi^4}e^{-4\pi^2 t}\cos(2\pi x) \approx -0.033 - 0.031e^{-39.4t}\cos(6.28x).$$

(The coefficient $a_2 = 0$ here). Graph shown in the left panel of Figure 8.59.

Exercise Solution 8.3.6. *The approximate solution is*

$$u(x,t) \approx 0.500 - 0.374e^{-2.46t}\cos(1.57x) + 0.162e^{-22.2t}\cos(4.71x) - 0.500e^{-39.4t}\cos(6.28x) + 0.188e^{-61.6t}\cos(7.85x).$$

Here $a_2 = 0$. Graph shown in the right panel of Figure 8.59.

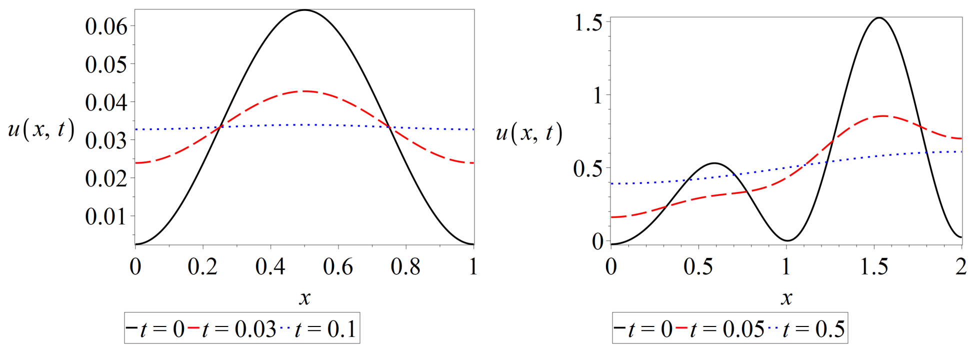

Figure 8.59: Solutions to Exercises 8.3.5 (left) and 8.3.6 (right).

Exercise Solution 8.3.9.

(a) The approximate solution is

$$u(x,t) \approx 1.01e^{-3.08t}\sin(0.785x) + 0.499e^{-27.7t}\sin(2.36x) - 0.207e^{-77t}\sin(3.92x) - 0.0172e^{-151t}\sin(5.50x).$$

Graph shown in Figure 8.60.

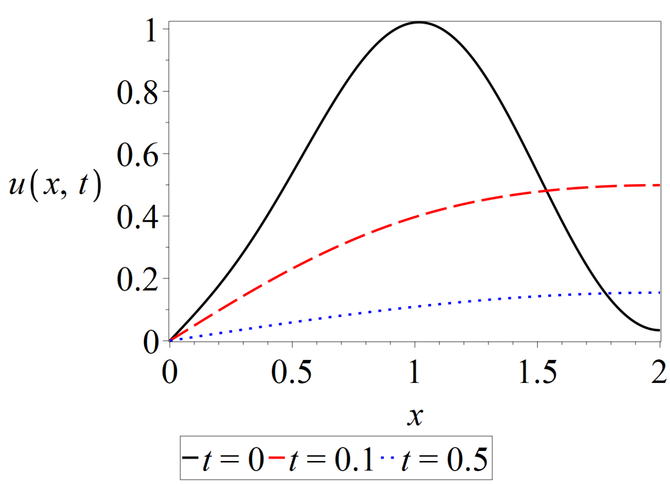

Figure 8.60: Solution to Exercise 8.3.9 part (a).

Exercise Solution 8.3.12. *The Fourier coefficients for $f(x)$ are all zero, of course. The Fourier cosine coefficients $a_0(t)$ to $a_3(t)$ for $r(x,t)$ with respect to x are*

$$a_0(t) = 2e^{-t}, \quad a_1(t) = -8e^{-t}/\pi^2, \quad a_2(t) = 0, \quad a_3(t = -8e^{-t}/(9\pi^2).$$

Solving for the $\phi_k(t)$ functions produces (rounded to three significant figures)

$$\phi 0(t) = 2 - 2e^{-t}, \quad \phi_1(t) = 0.552(e^{-2.47t} - e^{-t}, \quad \phi_2(t) = 0,$$
$$\phi_3(t) = 0.00425(e^{-22.2} - e^{-t}).$$

The approximation solution is

$$u(x,t) \approx \phi_0(t)/2 + \phi_1(t)\cos(\pi x/2) + \phi_2(t)\cos(\pi x) + \phi_3(t)\cos(3\pi x/2)$$

This is shown in the left panel of Figure 8.61.

Exercise Solution 8.3.14. *The Fourier coefficients for $f(x)$ are approximately $f_0 = 2.0, f_1 = -0.360, f_2 = -1.0, f_3 = 0.330, f_4 = 0.0, f_5 = 0.0208$. The Fourier cosine coefficients $a_0(t)$ to $a_5(t)$ for $r(x,t) = x - 2$ with respect to x are independent of time (since r is too) and given by $a_0(t) = 0, a_1(t0 = -1.62, a_2(t) = 0, a_3(t) = -0.180, a_4(t) = 0, a_5(t) - 0.0646$. More generally $a_k(t) = 0$ if k is even and $a_k(t) = -16/(k^2\pi^2)$ if k is odd.*

Solving for the $\phi_k(t)$ functions produces (rounded to three significant figures)

$$\phi 0(t) = 2, \quad \phi_1(t) = -0.876 + 0.516e^{-1.85t}, \quad \phi_2(t) = -e^{-7.40t},$$
$$\phi_3(t) = -0.018 + 0.342e^{-16.7t}, \quad \phi_4(t) = 0,$$
$$\phi_5(t) = -0.0014 + 0.0223e^{-46.3t}.$$

The approximate solution is

$$u(x,t) \approx 1 + \phi_1(t)\cos(\pi x/4) + \phi_2(t)\cos(\pi x/2) + \cdots + \phi_5(t)\cos(5\pi x/4).$$

This is shown in the right panel of Figure 8.61.

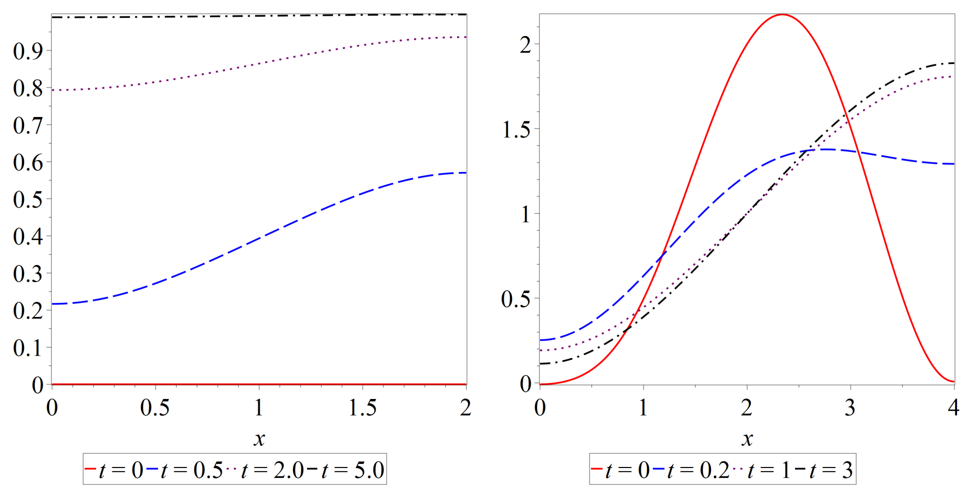

Figure 8.61: Solutions to Exercises 8.3.12 (left) and 8.3.14 (right).

Section 8.4

Exercise Solution 8.4.1. *The solution is* $\rho(x,t) = f(x-2t) = (x-2t)/((x-2t)^2+1)$. *See Figure 8.62.*

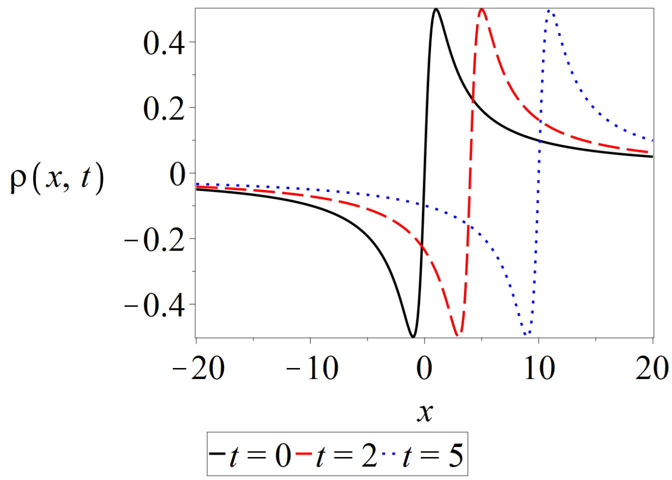

Figure 8.62: Solution to advection equation for Exercise 8.4.1.

Exercise Solution 8.4.4. *In this case the solution is* $u(x,t) = \cos(\pi t)\sin(\pi x)$ *and is exact (it is exact for any* $N \geq 2$*). Solution graphed in the left panel of Figure 8.63.*

Exercise Solution 8.4.5. *In this case the solution is* $u(x,t) = \cos(\pi t)\sin(\pi x) + 3\sin(2\pi t)\sin(2\pi x)/(2\pi)$ *and is exact (it is exact for any* $N \geq 4$*). Solution graphed in the right panel of Figure 8.63.*

Exercise Solution 8.4.8. *We find* $D = P_1 P_2$ *where* $P_1 = d/dt + I$ *and* $P_2 = d/dt + 8I$ *(or vice-versa). The solution or roots for* P_1 *and* P_2 *are* $c_1 e^{-t}$ *and* $c_2 e^{-8t}$ *for any constants* c_1, c_2.

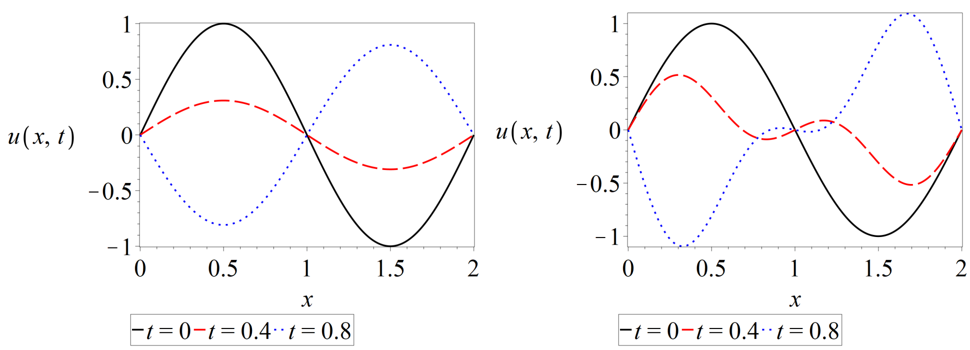

Figure 8.63: Solution to wave equation for Exercises 8.4.4 (left) and 8.4.5 (right).

Appendix A

Exercise Solution A.6.1.

(a) $\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = 4$, $\mathrm{Re}(w) = 1$, *and* $\mathrm{Im}(w) = -1$. *Also* $z + w = 4+3i$, $z-w = 2+5i$, $zw = 7+i$, *and* $z/w = -1/2+7i/2$. *Also* $|z| = 5$, $|w| = \sqrt{2}$, *and* $|zw| = |z||w| = 5\sqrt{2}$. *Also* $\overline{z} = 3 - 4i$, $\overline{w} = 1 + i$, *and* $\overline{zw} = 7-i$. *Finally,* $e^z = e^3 \cos(4)+ie^3 \sin(4)$, $e^w = e\cos(1)-ie\sin(1)$,

$$e^z e^w = e^4(\cos(1)\cos(4)+\sin(1)\sin(4))+ie^4(\sin(4)\cos(1)-\sin(1)\cos(4)),$$

and $e^{z+w} = e^4 \cos(3) + ie^4 \sin(3)$. *That* $e^z e^w = e^{z+w}$ *follows by applying the given trigonometric identity.*

(b) $\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = 0$, $\mathrm{Re}(w) = 0$, *and* $\mathrm{Im}(w) = 1$. *Also* $z+w = 3+i$, $z - w = 3 - i$, $zw = 3i$, *and* $z/w = -3i$. *Also* $|z| = 3$, $|w| = 1$, *and* $|zw| = |z||w| = 3$. *Also* $\overline{z} = 3$, $\overline{w} = -i$, *and* $\overline{zw} = -3i$. *Finally,* $e^z = e^3$, $e^w = e^i = \cos(1) + i\sin(1)$,

$$e^z e^w = e^3 \cos(1) + ie^3 \sin(1)$$

and $e^{z+w} = e^{3+i} = e^3 \cos(1) + ie^3 \sin(1)$.

(c) $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = \pi$, $\mathrm{Re}(w) = 1$, *and* $\mathrm{Im}(w) = \pi/2$. *Also* $z + w = 1 + 3i\pi/2$, $z - w = -1 + i\pi/2$, $zw = -\pi^2/2 + i\pi$, *and* $z/w = \frac{\pi^2}{2(1+\pi^2/4)} + i\frac{\pi}{1+\pi^2/4}$. *Also* $|z| = \pi$, $|w| = \sqrt{4+\pi^2}/2$, *and* $|zw| = |z||w| = \pi\sqrt{4+\pi^2}/2$. *Also* $\overline{z} = -i\pi$, $\overline{w} = 1-i\pi/2$, *and* $\overline{zw} = -\pi^2/2-i\pi$. *Finally,* $e^z = -1$, $e^w = ie$,

$$e^z e^w = -ie$$

and $e^{z+w} = e^{1+3i\pi/2} = -ie$.

Exercise Solution A.6.2. *Expand* $z^2 = (x + iy)^2 = x^2 + 2ixy - y^2$ *and set* $z^2 = i$ *to find* $x^2 - y^2 = 0$ *and* $2xy = 1$. *The solutions pairs are* (x, y) *equals* $(\sqrt{2}/2, \sqrt{2}/2)$ *and* $(-\sqrt{2}/2, -\sqrt{2}/2)$, *so that* $z = \sqrt{2}/2 + i\sqrt{2}/2$ *and* $z = -\sqrt{2}/2 - i\sqrt{2}/2$ *are the solutions.*

Exercise Solution A.6.3.

(a) *Roots* $z = 2$ *with multiplicity* 3, $z = i$ *with multiplicity* 1, $z = -3$ *with multiplicity* 2, *and* $z = -i$ *with multiplicity* 1. *The roots do not appear in conjugate pairs, so* $p(z)$ *does not have real coefficients.*

(b) *Roots* $z = -1 - i$ *with multiplicity* 2*,* $z = 0$ *with multiplicity* 7*, and* $z = i$ *with multiplicity* 4*. The roots do not appear in conjugate pairs, so* $p(z)$ *does not have real coefficients.*

(c) *Write* $z^2 + 1 = (z - i)(z + i)$ *so that* $p(z) = (z - i)^{14}(z + i)^{14}$*. The roots are then* $z = i$ *with multiplicity* 14 *and* $z = -i$ *with multiplicity* 14*. The roots are in conjugate pairs, so* $p(z)$ *has real coefficients (also clear if we just compute* $(z^2 + 1)^{14}$*).*

Exercise Solution A.6.4. *First, it's easy to see that* $z = 0$ *is a root, and we are given that* $z = i$ *is a root. Since* p *has real coefficients* $z = -i$ *must be a root. Thus* $p(z) = z(z - i)(z + i)q(z) = (z^3 + z)q(z)$ *for some quadratic polynomial. A polynomial division shows that* $q(z) = p(z)/(z^3 + z) = z^2 - 2z + 2$*. The two roots of* q *are* $z = 1 \pm i$*, and these are the two additional roots for* $p(z)$*.*

Exercise Solution A.6.5.

(a) *The zeros are* $z = 0$ *and* $z = 3$*. The poles are* $z = 1$ *and* $z = \pm 2i$*. The partial fraction decomposition is*

$$r(z) = \frac{-2/5}{z-1} + \frac{7/10 + 2i/5}{z - 2i} + \frac{7/10 - 2i/5}{z + 2i}.$$

(b) *The zeros are* $z = -1$ *and* -1 *(double root). The poles are* $z = 1$ *and* $z = -1 \pm i$*. The partial fraction decomposition is*

$$r(z) = \frac{4/5}{z-1} + \frac{1/10 + i/5}{z + 1 + i} + \frac{1/10 - i/5}{z + 1 - i}.$$

(c) *The only zero is* $z = 0$*. The poles are* $z = \pm i$ *and* $z = \pm 2i$*. The partial fraction decomposition is*

$$r(z) = \frac{1}{z - i} + \frac{1}{z + i} - \frac{1}{z - 2i} - \frac{1}{z + 2i}.$$

Appendix B

Exercise Solution B.6.1.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix}$$

Exercise Solution B.6.2.

$$\mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix}$$

Exercise Solution B.6.3.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$$

Exercise Solution B.6.4.

$$\mathbf{D} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -2 & 6 \\ 1 & 1 \end{bmatrix}$$

Exercise Solution B.6.5.

$$\mathbf{D} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

Exercise Solution B.6.6.

$$\mathbf{D} = \begin{bmatrix} 4+6i & 0 \\ 0 & 4-6i \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -1-2i & -1+2i \\ 3 & 3 \end{bmatrix}$$

Exercise Solution B.6.7.

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & -1 \\ 1 & 9 & 3 \end{bmatrix}$$

Exercise Solution B.6.8. *If we begin with* $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ *and conjugate both sides we obtain* $\overline{\mathbf{A}\mathbf{v}} = \overline{\lambda\mathbf{v}}$*. But from the familiar properties of conjugation we have* $\overline{\mathbf{A}\mathbf{v}} = \overline{\mathbf{A}}\overline{\mathbf{v}}$ *and* $\overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$*, so that*

$$\overline{\mathbf{A}}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

But since $\mathbf{A}$ *has real entries we have* $\overline{\mathbf{A}} = \mathbf{A}$ *and so*

$$\mathbf{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

This is precisely the statement that overline$\mathbf{v}$ *is an eigenvector for* $\mathbf{A}$ *with eigenvalue* $\overline{\lambda}$*.*

Thus if λ *is an eigenvalue for* $\mathbf{A}$ *so is* $\overline{\lambda}$*. This is an empty statement if* λ *is real, but it means that complex eigenvalues must come in conjugate pairs.*