Reading Exercises Key

For Version 1.12

Chapter 1

Reading Exercise Solution 1.1.1. *$a(t) = v'(t)$.*

Reading Exercise Solution 1.1.2. *Propulsive forces: the runner's "effort" via foot in contact with the ground; a tailwind. Resistive: internal forces/friction, air resistance/headwind.*

Reading Exercise Solution 1.1.3. *(a) P must have the dimension of length per time squared, i.e., acceleration; (b) we might interpret P as the maximum acceleration the runner is capable of, from a standing start.*

Reading Exercise Solution 1.1.4. *From $F = ma$ and $F = F_r + F_p$ we have $F_p + F_r = mv'(t)$. Then $F_p = mP$ and $F_k = -kmv(t)$ yield $mP - kmv(t) = mv'(t)$ or $v'(t) = P - kv(t)$.*

Reading Exercise Solution 1.1.5. *If the sprinter begins from a standing start, the graph of $v(t)$ should start at $v(0) = 0$ (if the runner starts at time $t = 0$) and rise to asymptotically approach a maximum velocity.*

Reading Exercise Solution 1.1.6. *(a), (b) (c) are all routine differentiations. For example, if $v(t) = 11/k - Ce^{-kt}$ then $v'(t) = Cke^{-kt}$ and $11 - kv(t) = 11 - k(11/k - Ce^{-kt}) = Cke^{-kt} = v'(t)$.*

Reading Exercise Solution 1.1.7. *We need to use the initial condition that $v(0.165) = 0$.*

Reading Exercise Solution 1.1.8. *The Hill-Keller ODE does not hold with $P = 11$ for $0 < t < 0.165$; in this time interval $P = 0$—Bolt was exerting no forward effort!*

Reading Exercise Solution 1.1.9. *In addition to the explicit modeling assumptions made, we ignore any but horizontal forces, variable runner effort, fatigue, perhaps the fact that maximum effort does not correspond to a constant P (e.g, right out of the blocks the propulsive force may differ than mid-race due to biomechanical differences in posture.)*

Reading Exercise Solution 1.2.1. *The rate of change is simply $u'(t)$, with units of micrograms per minute.*

Reading Exercise Solution 1.2.2. *The inflow rate is* rc_1*: each minute* r *microliters enter, carrying* c_1 *micrograms (*c_1 *micrograms per microliter times one microliter).*

Reading Exercise Solution 1.2.3. *The drug exits at* $-ru(t)/V$ *micrograms per minute; each minute* r *microliters exit, carrying* $u(t)/V$ *micrograms (*$u(t)/V$ *micrograms per microliter times one microliter).*

Reading Exercise Solution 1.2.4. *We find here that* $u'(t) = 0.04 - u(t)/6000$*.*

Reading Exercise Solution 1.2.5. *The solution here with* $u(0) = 0$ *is* $u(t) = 240(1 - e^{-t/6000})$*. After one week (*$t = 10080$ *minutes) we have* $u(10080) \approx 195.27$ *micrograms. After two weeks (*$t = 20160$ *minutes) we have* $u(20160) \approx 231.66$ *micrograms. These correspond to drug concentrations of* 0.976 *and* 1.158 *micrograms per microliter, respectively. As* $t \to \infty$ *the concentration of drug in the cochlear fluid approaches the inflow concentration of* 1.2 *micrograms per microliter.*

Reading Exercise Solution 1.3.1. *The population doubles to* $u(t) = 2u_0$ *when* $u_0e^{rt} = 2u_0$*, leading to* $t = \ln(2)/r$*; this is the population double time from* u_0 *to* $2u_0$*, then to* $4u_0$*, etc.*

Reading Exercise Solution 1.3.2. *Recall* $u'(t)/u(t) = r(1 - u(t)/K)$*.*

(a) If $u(t) \approx 0$ *then* $u'(t)/u(t) \approx r$*; the population growth rate is* r*.*

(b) If $u(t) = K$ *then* $u'(t)/u(t) = 0$ *and the population is not changing.*

(c) If $u(t) > K$ *then* $u'(t)/u(t) < 0$*, and also* $u'(t) < 0$*; the population is decreasing.*

Reading Exercise Solution 1.3.3. *Since* u *is in units of, e.g., "organisms" and* $u'(t)$ *is in unit of "organisms per time,"* r *must have unit reciprocal time.* K *has units of "organisms."*

Reading Exercise Solution 1.3.4. *See Figure 1.1. The solution grows more rapidly at first, but slows and limits to* K*. Larger values of* r *make the solution* $u(t)$ *approaches* K *more rapidly.*

Reading Exercise Solution 1.3.5. *Harvesting should decrease the rate of growth and limiting value of the population!*

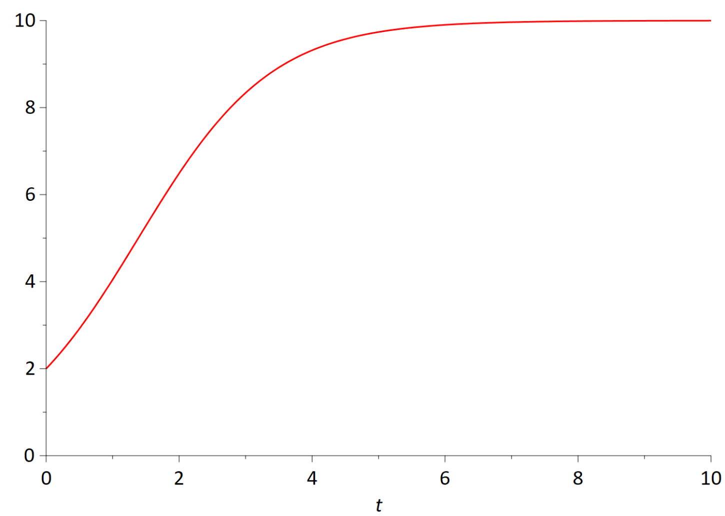

Figure 1.1: Logistic solution with $K = 10, r = 1, u_0 = 2$.

Reading Exercise Solution 1.3.6.

(a) *See Figure 1.2, left panel. The solution levels out at a smaller value than when* $h = 0$.

(b) *See Figure 1.2, right panel. This has an even larger impact.*

(c) *As* $t \to \infty$, $u(t) \to K(1 - h/r)$. *If* $h = r$, *it seems the population is doomed.*

Reading Exercise Solution 1.3.7. *See Figure 1.3, in which we use* $K = 10^5, r = 0.22, u_0 = 72148$ *and* $h = 0.2$; *better choices may be possible.*

Reading Exercise Solution 1.3.8. *The solution to the harvested logistic equation approaches* $K(1 - h/r)$ *if* $h < r$; *for the parameters chosen above the population will approach* $10^5(1 - 0.2/0.22) \approx 9100$. *But this is highly dependent on* r *and* h. *With* $h = 0.4$ *and* $r = 0.22$ *the population is doomed to extinction.*

Reading Exercise Solution 1.4.1. *A straightforward differentiation shows that* $u'(t) = 3t^2$, *and it is similarly easy to check that* $u(1) = 1 + 2 = 3$. *A general solution for* $u'(t) = 3t^2$ *is* $u(t) = t^3 + C$; *for the initial condition*

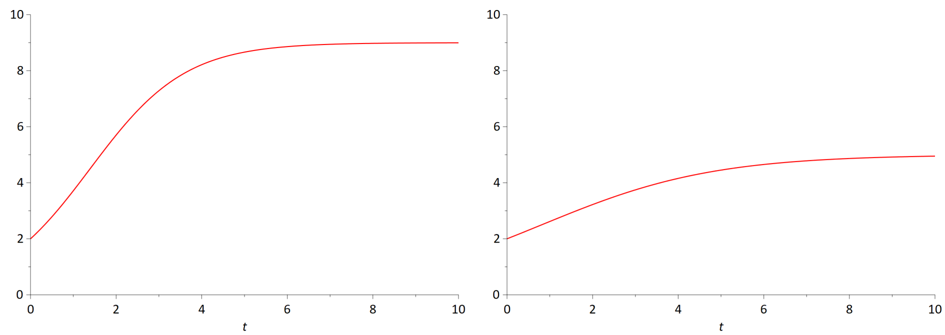

Figure 1.2: Harvested logistic solution with $K = 10, r = 1, u_0 = 2$ and $h = 0.1$ (left), $h = 0.5$ (right).

$u(t_0) = u_0$ we need $t_0^3 + C = u_0$, which is always solvable for $C = u_0 - t_0^3$. The solution is then $u(t) = t^3 + u_0 - t_0^3$. This works for any choice of t_0 and u_0.

Reading Exercise Solution 1.4.2. *Integrating shows that $u(t) = e^{2t}/2 + C$. Then $u(0) = 8$ yields $1/2 + C = 8$, so $C = 15/2$.*

Reading Exercise Solution 1.4.3. *With $u(t) = e^t + (3-e)t - 1$ it's easy to check that $u''(t) = e^t$, $u(1) = e + (3-e) - 1 = 2$, while $u'(1) = e + 3 - e = 3$. More generally with $u(t) = e^t + C_1 t + C_2$ the conditions $u(t_0) = u_0$ and $u'(t_0) = u'_0$ we require*

$$\begin{aligned} u_0 &= e^{t_0} + C_1 t_0 + C_2 \\ u'_0 &= e^{t_0} + C_1 \end{aligned}$$

with solution $C_1 = u'_0 - e^{t_0}$ and $C_2 = e^{t_0}(t_0 - 1) - t_0 u'_0 + u_0$. The solution can be expressed as

$$u(t) = e^t + (u'_0 - e^{t_0})t + e^{t_0}(t_0 - 1) - t_0 u'_0 + u_0.$$

For any choice of t_0, u_0, u'_0 we can solves for an appropriate C_1 and C_2.

Reading Exercise Solution 1.4.4. *Integrate $u''(t) = \sin(t)$ twice to find $u(t) = -\sin(t) + C_1 t + C_2$. Then $u(0) = 2$ forces $C_2 = 2$ and $u'(0) = 3$ forces $-1 + C_1 = 3$, so $C_1 = 4$. The solution is $u(t) = -\sin(t) + 4t + 2$.*

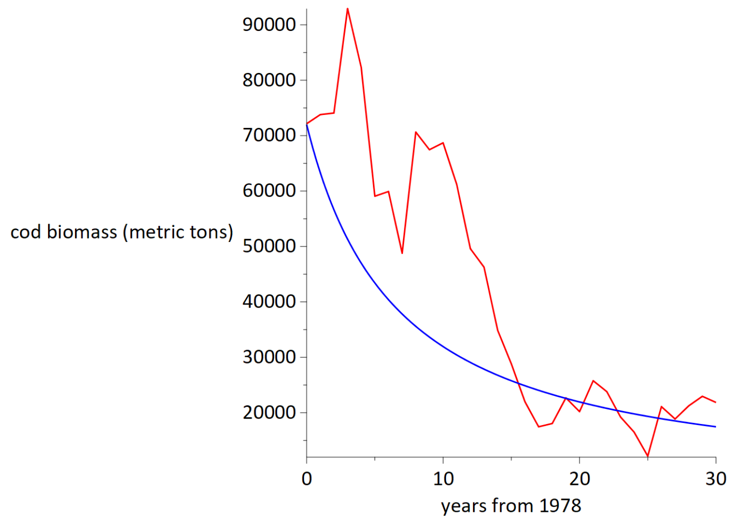

Figure 1.3: Data from Table 1.2 (red) and solution to harvested logistic solution with $K = 10^5, r = 0.22, u_0 = 72148$ and $h = 0.2$ (blue).

Reading Exercise Solution 1.5.1. *The quantity "liters per second" is volume per time, so* $[q] = L^3T^{-1}$. *The rate of change of surface area with respect to time has dimension* L^2T^{-1}. *The rate of change of the radius with respect to time is* LT^{-1}.

Reading Exercise Solution 1.5.2. *The dimension of* $v\,\Delta t$ *is computed as* $[v\,Deltat] = [v][\Delta t] = LT^{-1}T = L$. *A simple interpretation is that if an object is moving with velocity* v *(in one-dimension, anyway) then in a time interval* Δt *the object moves distance* $v\,\Delta t$, *with units a length or distance.*

Reading Exercise Solution 1.5.3. *From* $F = Gm_1m_2/r^2$ *we have* $G = Fr^2/(m_1m_2)$. *Since* $[F] = MLT^{-2}$, $[m_1] = [m_2] = M$, *and* $[r] = L$, *we have*

$$[G] = (MLT^{-2})(L^2)(M^{-2}) = M^{-1}L^3T^{-2}.$$

Reading Exercise Solution 1.5.4. *From the text,* $[E(t)] = ML^2T^{-2}$. *Then* $[E'(t)] = ML^2T^{-3}$ *and* $[E''(t)] = ML^2T^{-4}$.

Reading Exercise Solution 1.5.5. *The dimension of* $r(t)$ *is* $[r(t)] = L^3T^{-1}$ *(volume per time). Then* $\int_a^b r(t)\,dt$ *has dimension* $L^3T^{-1}T = L^3$, *and may be interpreted as the net amount of water that flowed into the tank from time* $t = a$ *to time* $t = b$.

Chapter 2

Reading Exercise Solution 2.1.1. *Differentiating* $e^{kt}v(t)+C_1$ *with respect to* t *yields* $e^{kt}(v'(t)+kv(t))$, *while differentiating* $Pe^{kt}/k+C_2$ *yields* Pe^{kt}, *so that we obtain exactly* $e^{kt}(v'(t)+kv(t)) = Pe^{kt}$. *Since* e^{kt} *is never zero, we have* $v'(t)+kv(t) = P$ *or* $v'(t) = P - kv(t)$.

Reading Exercise Solution 2.1.2. *Starting with general solution* $v(t) = P/k + Ce^{-kt}$, *the condition* $v(t_0) = 0$ *yields* $0 = P/k + Ce^{-kt_0}$ *with solution* $C = -Pe^{kt_0}/k$. *Then*

$$v(t) = \frac{P}{k} - \frac{P}{k}e^{kt_0}e^{-kt} = \frac{P}{k} - \frac{P}{k}e^{-k(t-t_0)}.$$

This works for any t_0.

Reading Exercise Solution 2.1.3. *With* $k = 1$ *the graph of* $v(t)$ *rises from* $v(0.165) = 0$ *asymptotically to* $v \approx 11$. *Taking* $k = 0.9$ *yields a limiting value of* $v \approx 12.2$. *The larger the value of* k, *the smaller the limiting value of* v, *and vice-versa.*

Reading Exercise Solution 2.1.4. *If we choose* $H(t) = \ln(t) + A$ *then* $e^{-H(t)} = e^{-A}/t$. *Multiplying both sides of the ODE* $u'(t) - u(t)/t = t^2$ *by this integrating factor yields* $e^{-A}(u'(t)/t - u(t)/t^2) = e^{-A}t$, *but the* e^{-A} *factor can be immediately cancelled. The rest of the computation proceeds as in the text.*

Reading Exercise Solution 2.1.5. *When* $h(t) = 0$ *we can take, for example,* $H(t) = 1$ *(or any constant) and then the ODE* $u'(t) = g(t)$ *becomes* $eu'(t) = eg(t)$ *after multiplication by the integrating factor* $e^{H(t)} = e$. *Of course we can cancel* e *and proceed as before, by integrating both sides of* $u'(t) = g(t)$.

Reading Exercise Solution 2.1.6. *In this case we multiply both sides of* $u'(t) - h(t)u(t) = 0$ *by* $e^{-H(t)}$ *to obtain* $e^{-H(t)}(u'(t) - h(t)u(t)) = 0$ *or* $d(u(t)e^{-H(t)})/dt = 0$. *Integrating produces* $u(t)e^{-H(t)} = C$ *for some constant* C, *and so* $u(t) = Ce^{H(t)}$.

Reading Exercise Solution 2.1.7.

(a) This is simply $u'(t)$.

(b) This is $k(u(t) - A)$ *(or* $-k(u(t) - A)$*).*

(c) *An appropriate ODE is* $u'(t) = k(u(t) - A)$ *(with* $k < 0$*) or* $u'(t) = -k(u(t) - A)$ *(with* $k > 0$*).*

Reading Exercise Solution 2.1.8. *The Newton Cooling ODE is linear, constant coefficient, and nonhomogeneous.*

Reading Exercise Solution 2.1.9. *The quantity* $u'(t)$ *has dimension* ΘT^{-1}*, which means that* $[k] = T^{-1}$*.*

Reading Exercise Solution 2.1.10. *If* $[V] = ML^2T^{-2}Q^{-1}$ *then from* $Rq'(t) + q(t)/C = V(t)$ *it follows that* $[q/C] = [V]$ *or* $[C] = [q]/[V] = M^{-1}L^{-2}T^2Q^2$*. Also from* $Rq'(t) + q(t)/C = V(t)$ *it follows that* $[Rq'] = [V]$ *so that* $[R] = [V]/[q'] = T[V]/[q] = ML^2T^{-1}Q^{-2}$*.*

Reading Exercise Solution 2.2.1. *From* $[F_r] = MLT^{-2}, [A] = L^2, [\rho] = ML^{-3}$*,* $[v] = LT^{-1}$ *and* $F_r = KA^a\rho^b v^c$ *we find that we need*

$$MLT^{-2} = M^b L^{2a-3b+c} T^{-c}$$

which yields $a = 1, b = 1$*, and* $c = 2$*. Thus* $F_r = KA\rho v^2$ *is the only dimensionally consistent choice, if these are the critically variables that determine* F_r*.*

Reading Exercise Solution 2.2.2. *This ODE is of the form* $v' = f(t, v) = g - kv^2/m = g(t)h(u)$ *where* $g(t) = 1$ *and* $h(u) = g - kv^2/m$*.*

Reading Exercise Solution 2.2.3. *Start by separating as* $du/(u^2+1) = dt$*. Integrate to find* $\arctan(u) = t + C$*, and so* $u = \tan(t + C)$*. Then* $u(0) = 0$ *forces* $C = 0$*. The solution is* $u(t) = \tan(t)$*.*

Reading Exercise Solution 2.2.4. *With the given parameters we find*

$$v(t) \approx 49.497(1 - e^{-0.396t})/(1 + e^{-0.396t}).$$

The solution graph is shown in Figure 2.4.

Reading Exercise Solution 2.2.5. *That* $\lim_{t\to\infty} v(t) = \sqrt{mg/k}$ *is easy to see since both exponentials limit to* 0*. This is the limiting velocity of the object. Given that* $[m] = M, [g] = LT^{-2}$*, and* $[k] = ML^{-1}$ *(easily deduced from* $F_r = kv^2$*) we find that* $[\sqrt{mg/k}] = (MLT^{-2}M^{-1}L)^{1/2} = LT^{-1}$ *has the dimension of velocity, so this makes perfect sense.*

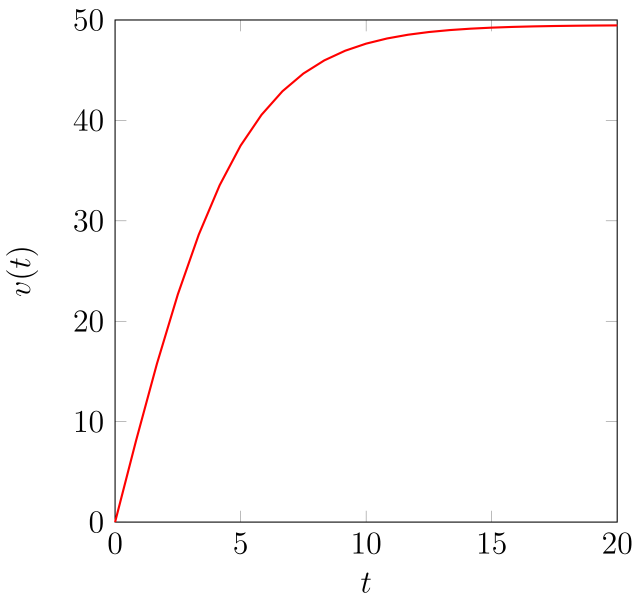

Figure 2.4: Graph of $v(t)$, solution to $v' = -kv^2/m + g$ with $v(0) = 0$, $m = 1, g = 9.8, k = 0.00$.

Reading Exercise Solution 2.3.1. *Use* $f(t,u) = -0.2(u-(10+5\sin(t/2)))$ *to compute* $f(10,5) \approx 0.041$ *and* $f(15,10) \approx 0.940$.

Reading Exercise Solution 2.3.2. *If* $u(t) = 10 + \frac{20}{29}\sin(t/2) - \frac{50}{29}\cos(t/2)$ *then* $u'(t) = \frac{10}{29}\cos(t/2) + \frac{25}{29}\sin(t/2)$. *Routine computation shows that* $u'(t) = -\frac{1}{5}u(t) + 2 + \sin(t/2)$, *which is the relevant Newton cooling ODE.*

Reading Exercise Solution 2.3.3. *The falling object ODE is of the form* $v' = f(t,v)$ *with* $f(t,v) = g - kv^2/m$, *and since* f *doesn't depend on* t, *this ODE is autonomous.*

Reading Exercise Solution 2.3.4. *A phase portrait would appear as in Figure 2.5.*

Reading Exercise Solution 2.3.5. *As* $t \to \infty$ *the solution with* $u(0) = -4$ *should asymptotically increase to* -1, *while* $u(0) = 0$ *should decrease to* -1 *and* $u(0) = 4$ *blows up. As* $t \to -\infty$ *solutions with* $u(0) = 0$ *and* $u(0) = 4$ *approach* $u = 3$, *while the* $u(0) = -4$ *solution blows up. See Figure 2.6.*

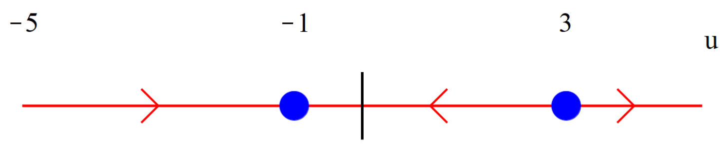

Figure 2.5: Phase portrait for $u' = u^2 - 2u - 3$.

Reading Exercise Solution 2.3.6. *The fixed point at $u = -1$ is asymptotically stable, that at $u = 3$ is unstable.*

Reading Exercise Solution 2.3.7. *The only difference is that the fixed point at $u = 3$ in Figure 2.5 would be an empty circle.*

Reading Exercise Solution 2.3.8. *Compute $f'(u) = -4$, so this fixed point is stable, but $f'(3) = 4$, so this fixed point is unstable.*

Reading Exercise Solution 2.3.9. *Solutions with $u(0) = K/2$ should asymptotically increase to K, those with $u(0) = 2K$ asymptotically decrease to K.*

Reading Exercise Solution 2.3.10. *A nonautonomous ODE $u' = f(t, u)$ typically has no fixed points. For example, if we try $u(t) = u^*$ in $u'(t) = u(t) + \sin(t)$ we obtain $0 = u^* + \sin(t)$, a contradiction since u^* was assumed constant.*

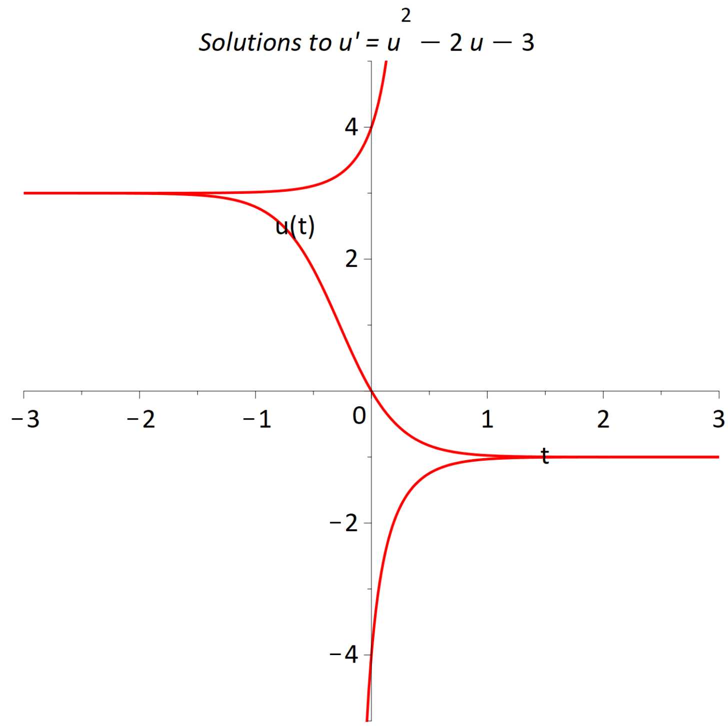

Figure 2.6: Solutions to $u' = u^2 - 2u - 3$.

Reading Exercise Solution 2.3.11. *Every solution to $u'(t) = 0$ is of the form $u(t) = c$ and so is a fixed point. Also, if a solution starts with $u(0) = u_0$ and $|u_0 - c| < \delta$ then $|u(t) - c| = |u_0 - c| < \delta$ for all t (and any δ), so $u(t) = c$ is stable.*

Reading Exercise Solution 2.3.12. *The phase portrait is as shown in Figure 2.7. It's clear all solutions approach $x(t) = 20$.*

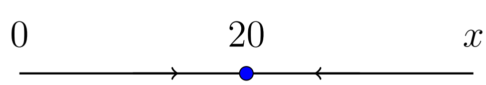

Figure 2.7: Phase portrait for $u'(t) = 0.2 - x(t)/100$.

Reading Exercise Solution 2.3.13. *When $h = r$ the harvested logistic equation becomes $u' = -u^2/K$. The only fixed point is $u = 0$; all solutions away from this fixed point satisfy $u' < 0$, so $u = 0$ is semistable.*

Reading Exercise Solution 2.4.1. *If* $p(x) = x^5 + x^3 + x + 5$ *then* p *is continuous (it's a polynomial). Also* $p(-2) = -32 - 8 - 2 + 5 = -37$ *and* $p(2) = 32 + 8 + 2 + 5 = 47$*. By the Intermediate Value Theorem* p *has a root in* $-2 < x < 2$*. Also,* $p'(x) = 5x^4 + 3x^2 + 1$ *which is positive for all* x*, so* p *is strictly increasing on* $-\infty < x < \infty$*, so any root is unique.*

Reading Exercise Solution 2.4.2. *For* $u(0) = -3/2$ *the interval is about* $(-1/2, \infty)$*. For* $u(0) = 1/2$ *the interval is* $(-\infty, \infty)$*. for* $u(0) = 2$ *the interval is about* $(-\infty, 0.5)$*.*

Reading Exercise Solution 2.4.3. *The ODE is of the form* $v' = f(v)$ *with* $f(v) = g - kv^r/m$ *(autonomous). On the region* $v > 0$ *the function* f *is continuous. Also,* $\frac{\partial f}{\partial v} = -rkv^{r-1}/m$ *is continuous for* $v > 0$*. We conclude that the ODE has a unique solution with* $v(t_0) = v_0$ *for* $v_0 > 0$*.*

Chapter 3

Reading Exercise Solution 3.1.1. *The ODE is* $u = f(t, u)$ *with* $f(t, u) = u(1 - u/(1 + 0.25\sin(2\pi t)))$*. Since* $1 + 0.25\sin(2\pi t) > 0$ *for all* t*, the function* f *is clearly continuous for all* t *and* u*. Also,* $\frac{\partial f}{\partial u} = 1 - 2u/(1 + 0.25\sin(2\pi t))$ *is also continuous for all* t *and* u*, so the ODE has a unique solution for any initial condition* $u(t_0) = u_0$*.*

Reading Exercise Solution 3.1.2. *Compute* $L(t) = 2 + 4(t - 1) = 4t - 2$*. The value of* $u(1 + h)$ *for* $h = 1, 0.1, 0.01, 0.001$ *is* $8, 2.42, 2.0402, 2.004002$*. The same values for* $L(1 + h)$ *are* $6, 2.4, 2.04, 2.004$*. The respective errors* $|u(1 + h) - L(1 + h)|$ *are* $2, 0.02, 0.0002, 0.000002$*. It appears the error is exactly* $2h^2$*, quadratic in* h*.*

Reading Exercise Solution 3.1.3. *Straightforward algebra shows that* $|u(1+h) - L(1+h)| = |2(1+h)^2 - (2+4h)| = |2h^2| = 2h^2$*, exactly as in the previous Reading Exercise.*

Reading Exercise Solution 3.1.4. *For the ODE given Euler's Method with step size* $h = 0.25$ *produces* $u_3 = u_2 + f(t_2, u_2) \approx 1.7070$ *and* $u_4 = u_3 + f(t_3, u_3) \approx 1.7119$*. The correct values for the solution are* $u(t_3) \approx 1.6025$ *and* $u(t_4) \approx 1.4086$*.*

Reading Exercise Solution 3.2.1. *With* $f(t, u) = u + t + 1$*,* $h = 0.5$*, and* $t_0 = 0, u_0 = 2$ *Euler's Method yields* $u_1 = u_0 + hf(t_0, u_0) = 2 + (0.5)(3) = 3.5$*. Then* $u_2 = u_1 + hf(t_1, u_1) = 3.5 + (0.5)(5) = 6.0$*.*

Reading Exercise Solution 3.2.2. *With* $f(t,u) = u + t + 1$, $h = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2$ *and* $u_2 = 7.5625$ *from the Example we find*

$$\begin{aligned} w &= u_2 + hf(t_2, u_2) = 12.34375 \\ m &= \frac{f(t_2, u_2) + f(t_3, w)}{2} = 12.203125 \\ u_3 &= u_2 + hm = 13.6640625. \end{aligned}$$

The second iteration produces

$$\begin{aligned} w &= u_3 + hf(t_3, u_3) = 21.74609 \\ m &= \frac{f(t_3, u_3) + f(t_3, w)}{2} = 20.45508 \\ u_4 &= u_3 + hm = 23.8916. \end{aligned}$$

The true value is $u(2) = 4e^2 - 4 \approx 25.556$.

Reading Exercise Solution 3.3.1. *For this ODE* $t_0 = 0, u_0 = 0$, *and* $f(t,u) = -2u + t^2$. *The analytical solution is* $u(t) = t^2/2 - t/2 + 1/4 - e^{-2t}/4$ *and* $u(1) = (1 - e^{-2})/4 \approx 0.216166$. *The RK4 method gives* $m_1 = f(0,0) = 0, m_2 = f(1/2, 0) = 1/4, m_3 = f(1/2, 1/8) = 0, m_4 = f(1,0) = 1$ *and* $m = 1/4$. *Then* $u_1 = u_0 + mh = 1/4$ *is the estimate for* $u(1)$.

Reading Exercise Solution 3.3.2. *The data for* h *versus* $u(t_k + h) - u_{k+1}$ *is shown in Table 3.1. The quantity* $(u(t_k+h) - u_{k+1})/h^2$ *stabilizes at around* 4.48, *so we would estimate* $C \approx 8.96$. *By no coincidence,* $u''(1.5) \approx 8.963$.

h	1	0.1	0.01	0.001
$u(t_k + h) - u_{k+1}$	6.438	4.635×10^{-2}	4.497×10^{-4}	4.484×10^{-6}

Table 3.1: Error $u(t_k + h) - u_{k+1}$ as a function of step size h for solution $u(t) = 2e^t - t - 1$ to ODE $u' = t + u$, $t_k = 1.5$.

Reading Exercise Solution 3.3.3. *A single Euler step of size* $h = 0.2$ *yields* $u_1 = u_0 + hf(t_0, u_0) \approx 0.598$. *Two steps of size* $h/2 = 0.1$ *yield* $u_{k+1/2} \approx 0.649$ *and* $\tilde{u}_1 \approx 0.58112$. *The local truncation error has magnitude about* $2|\tilde{u}_1 - u_1| \approx 0.03376$.

Reading Exercise Solution 3.4.1. *The condition* $s_3(k) = 0$ *leads to* $x(k, 3.78) = 30$ *or* $\frac{11}{k^2}(e^{-k(3.78-0.165)} - 1 + k(3.78 - 0.165)) = 30$ *with solution* $k = k^* \approx 0.9530$. *Then* $s_1(k^*) \approx 0.0537$ *and* $s_2(k^*) \approx 0.0009$.

Reading Exercise Solution 3.4.2. *The sum of squares is*

$$S(k) = (1.6e^{0.6k} - 2.1)^2 + (1.6e^{1.1k} - 2.45)^2 + (1.6e^{1.4k} - 2.82)^2.$$

A plot reveals a minimum around $k = 0.4$. *Setting* $S'(k) = 0$ *and solving for* k *yields* $k = k^* \approx 0.4205$ *(easily verified to be a minimum by graphing).*

Reading Exercise Solution 3.4.3. *In this case the sum of squares is*

$$S(k, A) = (Ae^{0.6k} - 2.1)^2 + (Ae^{1.1k} - 2.45)^2 + (Ae^{1.4k} - 2.82)^2.$$

Graphing $\ln(S(k, A))$ *shows a minimum somewhere around* $k = 0.4, A = 1.6$. *Solving* $\frac{\partial S}{\partial k} = 0, \frac{\partial S}{\partial A} = 0$ *yields* $k \approx 0.37$, $A \approx 1.66$.

Chapter 4

Reading Exercise Solution 4.1.1. *We have* $F_{spring} = -ku$ *and* $F_{damping} = -cu'$, *along with external force* $f(t)$ *on the mass. The total force is* $F = -ku - cu' + f(t)$, *so from* $F = ma$ *and* $a = u''$ *this yields* $mu'' = -ku - cu' + f(t)$ *or* $mu'' + cu' + ku = f(t)$.

Reading Exercise Solution 4.1.2.

(a) If $u(t) = A\cos(t) + B\sin(t)$ *then* $u''(t) = -A\cos(t) - B\sin(t)$ *and all terms cancel in* $u'' + u$, *for any* A *and* B.

(b) More generally if $u(t) = A\cos(\omega t) + B\sin(\omega t)$ *then* $u''(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$. *In* $mu'' + ku = 0$ *this becomes*

$$A(k - m\omega^2)\cos(\omega t) + B(k - m\omega^2)\sin(\omega t) = 0.$$

If $\omega = \sqrt{k/m}$ *the equation is satisfied for all* t.

Given that ωt *appears as an argument to sine or cosine, we must have* $[\omega] = T^{-1}$. *This also follows from* $[\omega] = [k]^{1/2}[m]^{-1/2} = T^{-1}$.

Reading Exercise Solution 4.1.3. *Start with* $u(t) = A\cos(t) + B\sin(t)$ *and require* $u(0) = 0.5$ *(initial position) and* $u'(0) = 0$ *(initial velocity). This leads to* $A = 0.5$ *and* $B = 0$, *so the solution is* $u(t) = 0.5\cos(t)$.

Reading Exercise Solution 4.1.4. *From* $u(t) = e^{-t}\cos(5t) + e^{-t}\sin(5t)/5$ *we have* $u'(t) = -26e^{-t}\sin(5t)/5$*, so the initial data is easy to check. Also,* $u''(t) = 26e^{-t}\sin(5t)/5 - 26e^{-t}\cos(5t)$*. Then*

$$\begin{aligned} u''(t) + 2u'(t) + 26u(t) &= 26e^{-t}\sin(5t)/5 - 26e^{-t}\cos(5t) \\ &\quad - 52e^{-t}\sin(5t)/5 + 26(e^{-t}\cos(5t) + e^{-t}\sin(5t)/5) = 0 \end{aligned}$$

after simplifying. The solution is a decaying sine/cosine wave, makes perfect sense.

Reading Exercise Solution 4.1.5. *With* $u(t) = u_{eq}$ *in the ODE we obtain* $15000u_{eq} = -450.8$*, which yields* $u_{eq} \approx= 0.03$ *meters or* 30 *mm. This is* $100\times(30/140) \approx 21.4$ *percent of the shock's range of travel, in the recommended range.*

Reading Exercise Solution 4.1.6. *When* $d(t) = 0$ *the ODE becomes* $my''(t) + cy'(t) + ky(t) = kL_0 - mg$ *and if* $y(t) = y_{eq}$ *then* $ky_{eq} = kL_0 - mg$*. Solve to find* $y_{eq} = L_0 - mg/k$*. That is,* $y(t)$ *is exactly the natural length of the spring compressed by an amount* mg/k *that stems from the weight of the table top mass* m*.*

Reading Exercise Solution 4.1.7. *The ODE becomes* $LQ''(t) + Q(t)/C = \cos(\omega t)$*. If* $Q(t) = A\cos(\omega t)$ *then* $Q''(t) = -A\omega^2\cos(\omega t)$ *and in the ODE this becomes* $(-L\omega^2 + 1/C)A\cos(\omega t) = \cos(\omega t)$*. This is a solution when* $A = 1/(1/C - L\omega^2) = C/(1 - LC\omega^2)$*. The amplitude* A *grows without limit as* $\omega \to 1/\sqrt{LC}$*.*

Reading Exercise Solution 4.2.1. *Translations for "ansatz" include "method of operation" or just "approach."*

Reading Exercise Solution 4.2.2. *The solution is of the form* $u(t) = c_1e^{-t} + c_2e^{-3t}$*;* $u(0) = 2$ *forces* $c_1 + c_2 = 2$*, while* $u'(t) = -c_1e^{-t} - 3c_2e^{-3t}$*, so* $u'(0) = 4$ *forces* $-c_1 - 3c_2 = 4$*. The solution is* $c_1 = 5, c_2 = -3$*, and then* $u(t) = 5e^{-t} - 3e^{-3t}$*. Solution graphed in Figure 4.8.*

Reading Exercise Solution 4.2.3. *The general solution* $u(t) = c_1e^{-t} + c_2e^{-3t}$ *with* $u(1) = -1$ *yields* $c_1e^{-1} + c_2e^{-3} = -1$*. Since* $u'(t) = -c_1e^{-t} - 3c_2e^{-3t}$ *the initial condition* $u'(1) = 6$ *leads to* $-c_1e^{-1} - 3c_2e^{-3} = 6$*. The solution for* c_1 *and* c_2 *is* $c_1 = 3e/2, c_2 = -5e^3/2$*. The solution* $u(t)$ *is*

$$u(t) = \frac{3e}{2}e^{-t} - \frac{5e^3}{2}e^{-3t} = \frac{3}{2}e^{-(t-1)} - \frac{5}{2}e^{-3(t-1)}.$$

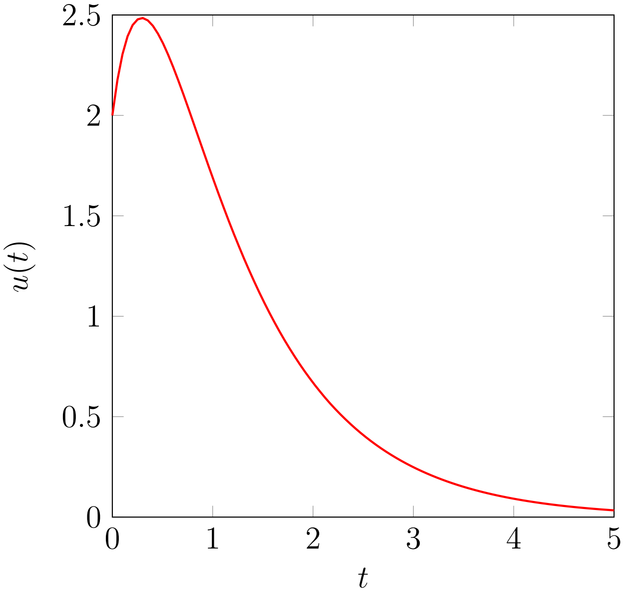

Figure 4.8: Graph of $u(t) = 5e^{-t} - 3e^{-3t}$.

Reading Exercise Solution 4.2.4. *We are given*

$$mu_1''(t)cu_1'(t) + ku_1(t) = 0$$
$$mu_2''(t)cu_2'(t) + ku_2(t) = 0.$$

Add and collect terms to find $m(u_1'' + u_2'') + c(u_1' + u_2') + k(u_1 + u_2) = 0$, *or* $m(u_1 + u_2)'' + c(u_1 + u_2)' + k(u_1 + u + 2) = 0$, *which is* $mu'' + cu' + ku = 0$.

Reading Exercise Solution 4.2.5. *If* $u_1(t) = \alpha u_2(t)$ *then* $u(t) = c_1u_1(t) + c_2u_2(t) = (c_1\alpha + c_2)u_2(t)$ *and* $u'(t) = (c_1\alpha + c_2)u_2'(t)$. *Then* $u(0) = u_0$ *forces* $(c_1\alpha + c_2)u_2(0) = u_0$ *and* $u'(0) = v_0$ *forces* $(c_1\alpha + c_2)u_2'(0) = v_0$. *If these equations are satisfied for some choice of* c_1 *and* c_2 *then multiply* $(c_1\alpha + c_2)u_2(0) = u_0$ *by* $u_2'(0)$ *and multiply* $(c_1\alpha + c_2)u_2'(0) = v_0$ *by* $u_2(0)$, *then subtract to find* $u_2'(t_0)u_0 = u_2(t_0)v_0$. *This last condition is necessary if there is a suitable choice for* c_1, c_2.

Reading Exercise Solution 4.2.6. *If* $u(t) = \cos(t)$ *then* $u''(t) = -\cos(t)$, *so* $u'' + u = 0$ *is clear. Also,* $u(0) = \cos(0) = 1$ *and* $u'(0) = -\sin(0) = 0$.

Reading Exercise Solution 4.2.7. *The characteristic equation* $3r^2 + 18r + 75 = 0$ *has roots* $r_1 = -3 + 4i$ *and* $r_2 = -3 - 4i$. *The complex-valued general* *solution is* $u(t) = c_1 e^{(-3+4i)t} + c_2 e^{(-3-4i)t}$*. Initial conditions dictate* $c_1 + c_2 = 0$ *and* $(-1+4i)c_1 + (-1-4i)c_2 = 4$*, with solution* $c_1 = -i/2, c_2 = i/2$*. The solution can be written as*

$$u(t) = -\frac{i}{2} e^{(-3+4i)t} + \frac{i}{2} e^{(-3-4i)t}.$$

Applying Euler's identity shows this is in fact $u(t) = e^{-3t}\sin(4t)$*.*

This result can be obtained with a real-valued general solution. The roots $-3 \pm 4i$ *indicate a general solution can be written as* $u(t) = d_1 e^{-3t}\cos(4t) + d_2 e^{-3t}\sin(4t)$*. Then* $u(0) = 0$ *implies* $d_1 = 0$*, so* $u(t) = d_2 e^{-3t}\sin(4t)$*. Then* $u'(t) = -3d_2 e^{-3t}\sin(4t) + 4d_2 e^{-3t}\cos(4t)$ *and* $u'(0) = 4d_2 = 4$ *means* $d_2 = 1$*. This yields the same solution as above.*

Reading Exercise Solution 4.2.8. *If* $u(t) = te^{-2t}$ *then* $u'(t) = -2te^{-2t} + e^{-2t}$ *and* $u''(t) = 4te^{-2t} - 4e^{-2t}$*. Substituting these into* $u''(t) + 4u'(t) + 4u(t) = 0$ *shows that it works. Also,* $u(0) = 0$ *and* $u'(0) = 1$*.*

Reading Exercise Solution 4.2.9. *If* $u(t) = c(t)e^{-\alpha t}$ *then* $u'(t) = -\alpha c(t)e^{-\alpha t} + c'(t)e^{-\alpha t}$ *and* $u''(t) = c(t)\alpha^2 e^{-\alpha t} - 2\alpha c'(t)e^{-\alpha t} + c''(t)e^{-\alpha t}$*. Substituting these into* $m(u''(t) + 2\alpha u'(t) + \alpha^2 u(t)) = 0$ *leads to* $me^{-\alpha t}c''(t) = 0$*. This means* $c''(t) = 0$ *and so* $c(t) = At + B$ *for some constants* A *and* B*.*

With $u(t) = c_1 e^{-\alpha t} + c_2 te^{-\alpha t}$ *the condition* $u(0) = u_0$ *leads to* $c_1 = u_0$*. Then* $u'(t) = -\alpha c_1 e^{-\alpha t} + c_2(-\alpha te^{-\alpha t} + e^{-\alpha t})$ *and* $u'(0) = v_0$ *leads to* $-\alpha c_1 + c_2 = v_0$*, so* $c_2 = v_0 + \alpha u_0$*.*

Reading Exercise Solution 4.3.1. *If* $u(t) = 4e^{-t} - 2e^{-3t} - e^{-2t}$ *then* $u'(t) = -4e^{-t} + 6e^{-3t} + 2e^{-2t}$ *and* $u''(t) = 4e^{-t} - 18e^{-3t} - 4e^{-2t}$*. Then* $u(0) = 1$*,* $u'(0) = 4$*, and straightforward algebra shows* $u''(t) + 4u'(t) + 3u(t) = e^{-2t}$*.*

Reading Exercise Solution 4.3.2. *In this case the general solution takes the form* $u(t) = c_1 e^{-t} + c_2 e^{-3t} - e^{-2t} - 5e^{-t}$*. The* $u(0) = 1$ *forces* $c_1 + c_2 - 6 = 1$ *or* $c_1 + c_2 = 7$*, and* $u'(0) = 4$ *forces* $-c_1 - 3c_2 + 7 = 4$ *or* $-c_1 - 3c_2 = -3$*. This yields* $c_1 = 9, c_2 = -2$ *and solution* $u(t) = 9e^{-t} - 2e^{-3t} - e^{-2t} - 5e^{-t} = 4e^{-t} - 2e^{-3t} - e^{-2t}$*, exactly as before.*

Reading Exercise Solution 4.3.3. *To find the minimum set*

$$u'(t) \approx 8.817e^{-14.56t} - 14.237e^{-22.40t} = 0.$$

The solution is $t \approx 0.611$ *seconds, maximum displacement* $u \approx -0.117$ *meters, or* 117 *mm. The shock will not bottom out under these circumstances.*

Reading Exercise Solution 4.4.1. *Since the roots of the characteristic equation $r^2 + 2r + 10 = 0$ are $r = -1 \pm 3i$, it's clear the transient portion provides a general solution to the homogeneous equation $u''(t) + 2u'(t) + 10u(t) = 0$. It is also straightforward to check that the periodic portion is a particular solution to the nonhomogeneous equation. With the given values for A and B verification that $\sqrt{A^2 + B^2} = 1/\sqrt{\omega^4 - 16\omega^2 + 100}$ is also straightforward. A plot of this amplitude is shown in Figure 4.9.*

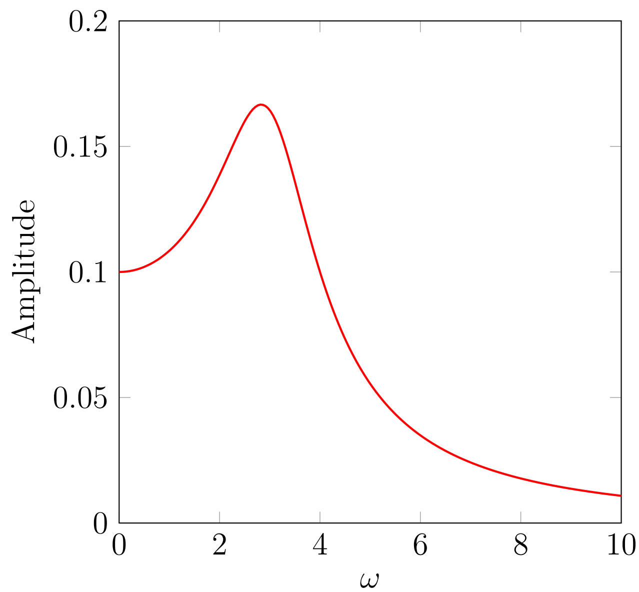

Figure 4.9: Amplitude of periodic response as a function of driving frequency ω.

Reading Exercise Solution 4.4.2. *We have $\psi(\omega) = \frac{1}{\sqrt{(\omega^2-10)^2+\omega^2}}$. The maximum occurs when $\psi'(\omega) = 0$. If we write $\psi(\omega) = (g(\omega))^{-1/2}$ with $g(\omega) = (\omega^2 - 10)^2 + \omega^2$ then $\psi'(\omega) = -\frac{1}{2}(g(\omega))^{-3/2}g'(\omega) = -\frac{g'(\omega)}{(g(\omega))^{3/2}}$. Thus $\psi'(\omega) = 0$ exactly when $g'(\omega) = 0$ and this yields*

$$g'(\omega) = 4(\omega^2 - 10)\omega + 2\omega = 0.$$

The only nonnegative solutions are $\omega = 0$ and $\omega = \sqrt{38}/2 \approx 3.08$.

Reading Exercise Solution 4.4.3. *From* $G(\omega) = \frac{1}{\sqrt{(m\omega^2-k)^2+c^2\omega^2}}$ *we have* $G(0) = 1/\sqrt{k^2} = 1/k$ *since* $k > 0$. *Also*

$$\begin{aligned}\lim_{\omega\to\infty} m\omega^2 G(\omega) &= \lim_{\omega\to\infty} \frac{m\omega^2}{\sqrt{(m\omega^2-k)^2+c^2\omega^2}}\\ &= \lim_{\omega\to\infty} \frac{m}{\sqrt{(m-k/\omega^2)^2+c^2/\omega^2}}\\ &= \frac{m}{\sqrt{m^2}}\\ &= 1.\end{aligned}$$

Reading Exercise Solution 4.4.4. *Resonance as defined requires* $c^2 < 2mk$, *so if* $2mk < c^2 < 4mk$ *the system is underdamped, but resonance does not occur. However, if* $c^2 < 2mk$ *the resonance frequency of the system is* $\omega_{res} = \frac{\sqrt{4mk-2c^2}}{2m}$ *while the natural frequency is* $\omega_{nat} = \frac{\sqrt{4mk-c^2}}{2m}$. *Since* $2mk - c^2 < 4mk - c^2$, *the resonant frequency is strictly less than the natural frequency.*

Reading Exercise Solution 4.4.5. *The* $\sin(3t)$ *term has period* $2\pi/3$ *and the* $\sin(4t)$ *term has period* $2\pi/4$. *The period of a linear combination will be the least common multiple of these two periods, or* 2π *times the least common multiple of* $1/3$ *and* $1/4$, *which is* 1. *The period is* 2π.

Reading Exercise Solution 4.4.6. *The function* $u(t)$ *satisfies* $u(0) = u'(0) = 0$.

Reading Exercise Solution 4.5.1. *If* $|x| \le 0.1\sqrt{k_1/k_2}$ *then* $|x|^2 \le 0.01k_1/k_2$. *Multiply by* $|x|$ *and find* $|x|^3 \le 0.01k_1|x|/k_2$ *or* $k_2|x^3| \le 0.01k_1|x|$.

Reading Exercise Solution 4.5.2. *We seek constants* α, β, γ *such that* $[u_c] = [m^\alpha k^\beta u_0^\gamma]$, *which leads to* $M^0L^1T^0 = (M^\alpha)(MT^{-2})^\beta(L^\gamma)$ *or* $M^0L^1T^0 = M^{\alpha+\beta}L^\gamma T^{-2\beta}$. *This yields equations* $\alpha+\beta = 0, \gamma = 1, -2\beta = 0$ *with solution* $\alpha = 0, \beta = 0, \gamma = 1$. *The only characteristic time scale that can be formed in this manner is* $u_c = u_0$, *the initial displacement.*

Reading Exercise Solution 4.5.3. *Starting with* $t_c = \frac{c}{k}\left(\frac{mk}{c^2}\right)^\alpha$ *and substituting in* $\alpha = 1/2$ *yields* $t_c = \frac{c}{k}\frac{\sqrt{mk}}{c} = \frac{m}{k}$, *comparable to the period of the system.*

Reading Exercise Solution 4.5.4. *Starting with* $t_c = \frac{c}{k}\left(\frac{mk}{c^2}\right)^\alpha$ *and substituting in* $\alpha = 1$ *yields* $t_c = \frac{c}{k}\frac{mk}{c^2} = \frac{m}{k}$. *This is the time scale that appears in the exponent in the solution* $u(t) = c_1 e^{-\frac{ct}{2m}}\cos(\omega t) + c_2 e^{-\frac{ct}{2m}}\sin(\omega t)$ *and characterizes the rate at which the solution decays; the solution is diminished to about one percent of its initial value after* $t \approx 5t_c$ *time constants.*

Reading Exercise Solution 4.5.5. *In this case* $\bar{u}(\tau) = u(t)/u_c = (3\tau)^2/7 = 9\tau^2/7$. *Then* $du/dt = 2t$, *while* $d\bar{}/d\tau = (7/3)(18/7\tau) = 6\tau = 2t$.

Reading Exercise Solution 4.5.6. *Consider* $u_c = r^\alpha K^\beta h^\gamma$ *where* $[r] = T^{-1}, [K] = N, [h] = NT^{-1}$. *This leads to* $N = T^{-\alpha-\gamma}N^{\beta+\gamma}$ *so that* $-\alpha-\gamma = 0$ *and* $\beta + \gamma = 1$. *Treating* γ *as a free variable leads to* $\alpha = -\gamma$ *and* $\beta = 1 - \gamma$, *so that* $u_c = r^{-\gamma}K^{1-\gamma}h^\gamma$ *or*

$$u_c = K\left(\frac{h}{rK}\right)^\gamma$$

for some γ.

Reading Exercise Solution 4.5.7. *In this case we need* $h \le (0.05)(10^6)(0.03) = 1500$ *individuals per year. If* K *or* r *increases, the condition* $h/(rK) \le 0.05$ *or* $h \le (0.05)rK$ *will yield larger values of* h. *This makes sense—if the species reproduces more rapidly or has a larger carrying capacity then more harvesting should be possible with no greater impact on the population.*

Chapter 5

Reading Exercise Solution 5.1.1. *A phase line portrait for* $u'(t) = -ku(t) + r_0$ *is shown in Figure 5.10. There is a single equilibrium solution at* $u = r_0/k$ *and this equilibrium is asymptotically stable.*

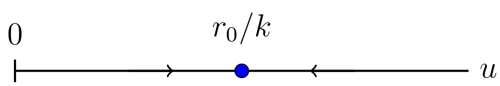

Figure 5.10: Phase portrait for $u'(t) = -ku(t) + r_0$.

Reading Exercise Solution 5.1.2. *The solution is*

$$u(t) = \begin{cases} 8.67 + 1.33e^{-kt}, & 0 \le t \le 12 \\ 14.45 - 5.61e^{-k(t-12)}, & t > 12. \end{cases}$$

A plot is shown in Figure 5.11.

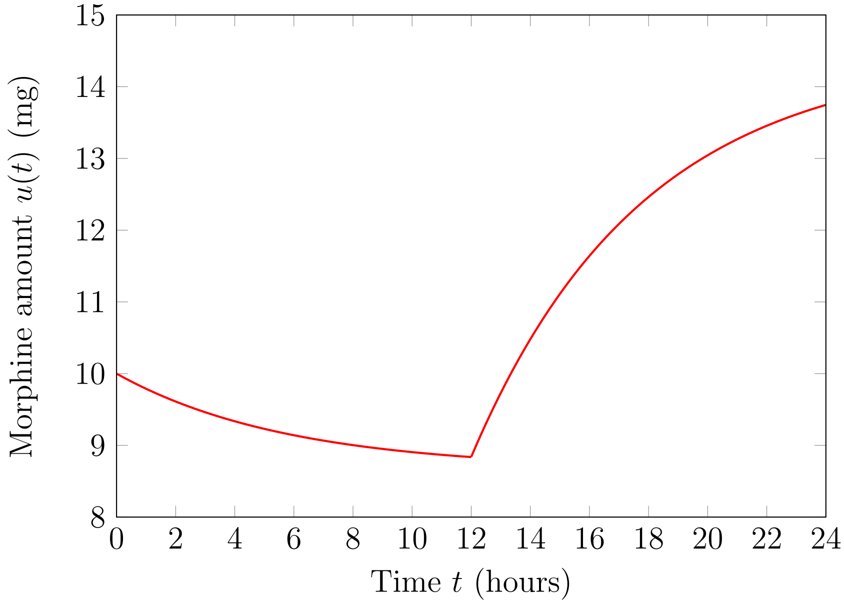

Figure 5.11: Solution to $u' = -ku + r(t)$ with $r(t) = 1.5$ for $0 < t < 12$, $r(t) = 2.5$ for $t > 12$, initial data $u(0) = 10$.

Reading Exercise Solution 5.1.3. *If the rate in for* $0 < t < 12$ *is* 1.5 *mg per hour then* $u' = -ku + 1.5$ *still holds, and with* $u(0) = 10$ *the solution* $u_1(t)$ *is still* $u_1(t) = 8.67 + 1.33e^{-kt}$ *on this interval. At* $t = 12$ *we should see a* 5 *mg instantaneous jump in the amount of morphine in the patient's system, so "just after"* $t = 12$ *there should be* $u_1(12) + 5 \approx 13.84$ *mg in the patient. For* $t > 12$ *the rate in is still* 1.5 *mg per hour, so the amount* $u_2(t)$ *in the patient's system should still obey* $u_2' = -ku_2 + 1.5$*, but with the condition* $u_2(12) = 13.84$*. The solution to this initial value problem is* $u_2(t) = 8.67 + 5.17e^{-k(t-12)}$*. A plot of the amount* $u(t)$ *of morphine in the patient's system is shown in Figure 5.12.*

Reading Exercise Solution 5.2.1. *An easy differentiation.*

Reading Exercise Solution 5.2.2. *For a fixed choice of* $a, M > 0$ *define the function* $\phi(t) = \frac{e^{t^2}}{Me^{at}}$*. Then* $\ln(\phi(t)) = t^2 - at - \ln(M)$*. It is clear that for any choice of* a *and* M *we have*

$$\lim_{t\to\infty} \ln(\phi(t)) = \infty$$

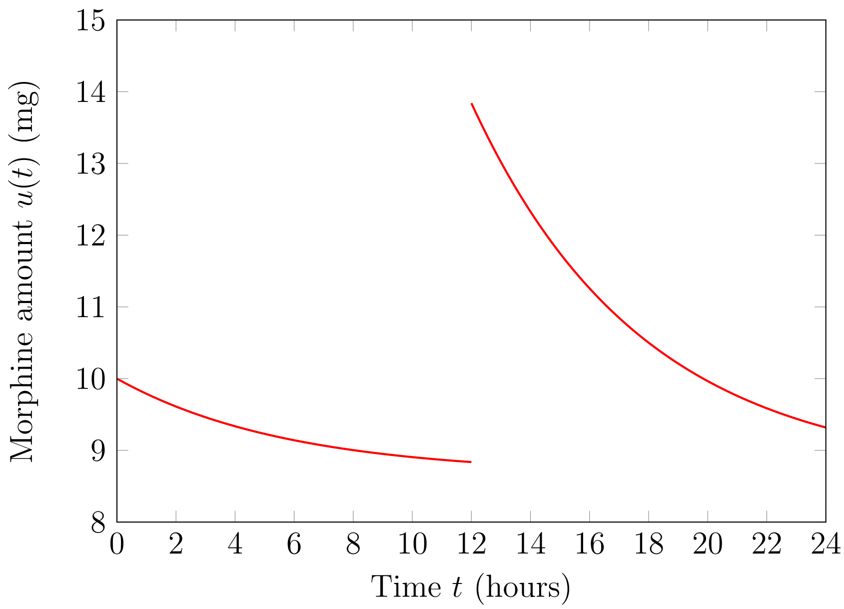

Figure 5.12: Solution to $u' = -ku + r(t)$ with $r(t) = 1.5$ for $t > 0$ but with instantaneous 5 mg dose at time $t = 12$, initial data $u(0) = 10$.

from which we can conclude that

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{at}} = \infty.$$

In particular, the quantity on the left will exceed 1 *when* t *is sufficiently large, and so* $|f(t)| \leq Me^{at}$ *cannot holds for any fixed* M *and* a *for all* $t \geq 0$.

Reading Exercise Solution 5.2.3. *Routine integrations.*

Reading Exercise Solution 5.2.4. *The transform of* t^2 *is* $2/s^3$, *the transform of* $-3\sin(3t)$ *is* $(-3)(3)/(s^2+9) = -9/(s^2+9)$, *and the transform of* 5 *is* $5/s$. *So* $F(s) = 2/s^3 - 9/(s^2+9) + 5/s$.

Reading Exercise Solution 5.2.5. *Laplace transform* $u'(t)$ *to obtain* $sU(s) - u(0)$ *or* $sU(s) - 3$ *after filling in the initial condition. From linearity the Laplace transform of* $-2u(t)$ *is* $-2U(s)$, *so the ODE becomes* $sU(s) - 3 = -2U(s)$. *Solve for* $U(s) = 3/(s+2)$. *A reverse lookup in the transform table shows* $u(t) = 3e^{-2t}$.

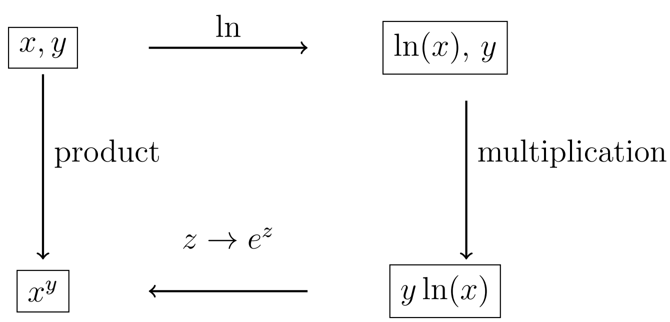

Figure 5.13: Commutative diagram for computing x^y.

Reading Exercise Solution 5.2.6. *It might look like Figure 5.13.*

Reading Exercise Solution 5.2.7. *Compute* $f''(t) = 4e^{-2t}(t-1) = 4te^{-2t} - 4e^{-2t}$*. The Laplace transform of* $4te^{-2t}$ *is (from line 4 in the transform table and linearity) equal to* $4/(s+2)^2$*. The transform of* $-4e^{-2t}$ *is* $-4/(s+2)$*. Then* $F(s) = 4/(s+2)^2 - 4/(s+2) = -4(s+1)/(s+2)^2$*.*

Reading Exercise Solution 5.2.8. *Laplace transforming* $u''(t)$ *produces* $s^2U(s) - su(0) - u'(0) = s^2U(s) - s + 3$*, while the transform of* $3u'(t)$ *is* $3(sU(s) - u(0)) = 3(sU(s) - 1)$*. The transform of the left side of the ODE is then* $s^2U(s) - s + 3 - 3sU(s) - 9 + 2U(s)$*, and of course the transform of the right side is* 0*. All in all* $s^2U(s) - s + 3 - 3sU(s) - 3 + 2U(s) = 0$*. Collect all* $U(s)$ *terms on the right and everything else on the right to find* $(s^2 - 3s + 2)U(s) = s$*. Then* $U(s) = s/(s^2 - 3s + 2)$*. A partial fraction decomposition shows*

$$U(s) = \frac{2}{s+2} - \frac{1}{s+1}.$$

From the Laplace transform table we find $u(t) = 2e^{-2t} - e^{-t}$*.*

Reading Exercise Solution 5.2.9. *The transform of* $f(t) = t$ *is* $F(s) = 1/s^2$*, so the transform of* te^{-t} *is* $F(s+1) = 1/(s+1)^2$ *(using* $a = -1$ *in the Theorem).*

Reading Exercise Solution 5.2.10. *If* t *has dimension* T *or time and we want to exponentiate* $-st$ *then* st *should be dimensionless. Thus* s *should have dimension* T^{-1}*, just as a frequency does.*

Reading Exercise Solution 5.2.11. *Given that the denominator factors as* $(s+2)(s+5)$ *we should expect terms involving* e^{-2t} *and* e^{-5t} *in the inverse transform. And in fact a partial fraction expansion shows*

$$F(s) = \frac{3s+3}{s^2+7s+10} = \frac{4}{s+5} - \frac{1}{s+2}$$

so that $f(t) = 4e^{-5t} - e^{-2t}$.

Reading Exercise Solution 5.2.12. *Given that* $F(s) = (6s+2)/(s^2+4)$ *has poles at* $s = 2i$ *and* $s = -2i$ *expect the inverse transform to contain* e^{2it} *and* e^{-2it}. *But these corresponds to* $\sin(2t)$ *and* $\cos(2t)$. *Writing*

$$F(s) = \frac{6s+2}{s^2+4} = 6\frac{s}{s^2+4} + \frac{2}{s^2+4}$$

an inverse table lookup shows that $f(t) = 6\cos(2t) + \sin(2t)$.

Reading Exercise Solution 5.2.13. *If* $f(t) = 2 + 3e^{-t}$ *then* $F(s) = 2/s + 3/(s+1) = \frac{5s+2}{s^2+s}$. *Then*

$$\begin{aligned}
\lim_{t\to 0^+} f(t) &= 5 \\
\lim_{s\to\infty} sF(s) &= \lim_{s\to\infty} \frac{5s^2+2s}{s^2+s} = 5 \\
\lim_{t\to\infty} f(t) &= 2 \\
\lim_{s\to 0^+} sF(s) &= \lim_{s\to 0^+} \frac{5s^2+2s}{s^2+s} = \lim_{s\to 0^+} \frac{5s+2}{s+1} = 2.
\end{aligned}$$

Reading Exercise Solution 5.3.1. *If* $t < 2$ *then all Heaviside functions equal* 0 *and* $q(t) = 0$. *If* $2 < t < 5$ *then* $H(t-2) - H(t-5) = 1$, *but* $H(t-5) - H(t-7) = 0$ *and* $H(t-7) = 0$, *so* $q(t) = t^2$. *For* $5 < t < 7$ *we have* $H(t-2) - H(t-5) = 0$ *(since both Heaviside functions are "on") and* $H(t-5) - H(t-7) = 1$ *while* $H(t-7) = 0$, *so* $q(t) = e^t$. *For* $t > 7$ *only* $H(t-7) = 1$ *and* $q(t) = \cos(t)$.

Reading Exercise Solution 5.3.2. *If* $c < 0$ *then* $H(t-c) = 1$ *for all* $t \geq 0$ *and so* $\mathcal{L}(H(t-c)) = 1$ *in this case. As far as the Laplace transform is concerned,* $H(t-c)$ *is the constant function* 1 *if* $c < 0$.

Reading Exercise Solution 5.3.3. *Write* $\phi(t) = H(t-3)f(t-3)$ *where* $f(t) = e^t$. *Then* $F(s) = 1/(s-1)$ *and so* $\mathcal{L}(\phi) = e^{-3s}/(s-1)$.

Reading Exercise Solution 5.3.4. *Define* $f(t) = (t+7)^2$ *so that* $f(t-7) = t^2$ *and* $\phi(t) = H(t-7)f(t-7)$*. Since* $f(t) = t^2 + 14t + 49$ *we compute* $F(s) = 2/s^3 + 14/s^2 + 49/s$ *and so* $\mathcal{L}(\phi) = e^{-7s}F(s)$*.*

Reading Exercise Solution 5.3.5. *Take* $g(t) = t^2$ *and* $c = 7$*, so that* $f(t) = g(t+7) = (t+7)^2 = t^2+14t+49$*. Then* $F(s) = 2/s^3+14/s^2+49/s$ *and so* $\mathcal{L}(H(t-7)t^2) = \mathcal{L}(H(t-7)g(t)) = e^{-7s}F(s) = e^{-7s}(2/s^3+14/s^2+49/s)$*.*

Reading Exercise Solution 5.3.6. *Laplace transform both sides of* $u'(t) = -u(t)+H(t-1)$ *and substitute in* $u(0) = 0$ *to obtain* $sU(s) = -U(s)+e^{-s}/s$*. Solve for* $U(s) = e^{-s}/(s(s+1))$*. A partial fraction decomposition shows that* $1/(s(s+1)) = 1/s-1/(s+1)$ *so that* $U(s) = e^{-s}/s-e^{-s}/(s+1)$*. The inverse transform of* e^{-s}/s *is* $H(t-1)$ *and the inverse transform of* $e^{-s}/(s+1)$ *is* $H(t-1)e^{-(t-1)}$*, so* $u(t) = H(t-1)(1-e^{-(t-1)})$*.*

Reading Exercise Solution 5.3.7. *Given the mass is at rest at equilibrium at* $t = 0$ *and no forces act on it until* $t = 3$*, it should clearly stay at rest. Only for* $t > 3$ *should it move, and given that the force is constant the mass should approach a position* $F/k = 10/10 = 1$ *as* $t \to \infty$*.*

Reading Exercise Solution 5.4.1. *Yes, you have been paying attention. No, the Dirac delta function is not a function.*

Reading Exercise Solution 5.4.2. *If* $a < b < t_0 - \epsilon$ *then the integrand is identically zero on this interval, so*

$$\int_a^b \phi_\epsilon(t)\,dt = \int_a^b 0\,dt = 0.$$

A similar analysis holds if $t_0 + \epsilon < a < b$*.*

Reading Exercise Solution 5.4.3. *If* $a < b < 0$ *then for* $\epsilon > 0$ *sufficiently close to* 0 *it follows that* $b < -\epsilon$ *and so* $\phi_\epsilon(t) = 0$ *for* $a < t < b$*. Then* $\displaystyle\int_a^b \phi_\epsilon(t)\,dt = 0$ *for all such* ϵ *and so*

$$\int_a^b \delta(t)\,dt = \lim_{\epsilon\to 0^+} \int_a^b \phi_\epsilon(t)\,dt = \lim_{\epsilon\to 0^+} 0 = 0.$$

Similar reasoning holds for $0 < a < b$*, since then* $\phi_\epsilon(t) = 0$ *for* $a < t < b$ *once* $\epsilon < a$*.*

Reading Exercise Solution 5.4.4. *With $g(t) = t^2 + t$ we have antiderivative $G(t) = t^3/3 + t^2/2$, so (at least if $\epsilon < 2$) we have*

$$\begin{aligned}\int_0^5 \frac{1}{2\epsilon}(H(t-2+\epsilon) - H(t-2-\epsilon))g(t)\,dt &= \frac{1}{2\epsilon}\int_{2-\epsilon}^{2+\epsilon} g(t)\,dt \\ &= \frac{1}{2\epsilon}\int_{2-\epsilon}^{2+\epsilon} (t^2+t)\,dt \\ &= \frac{1}{2\epsilon}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_{t=2-\epsilon}^{t=2+\epsilon} \\ &= 6 + \frac{\epsilon^2}{3}\end{aligned}$$

after simplifying. As $\epsilon \to 0^+$ this limits to $g(2) = 6$.

Reading Exercise Solution 5.4.5. *This is essentially the argument of Reading Exercise 132.*

Reading Exercise Solution 5.4.6. *With $g(t) = t^2 + t$*

1. *We find (using the fact that $\frac{1}{2\epsilon}(H(t-2+\epsilon) - H(t-2-\epsilon)) = 1$ for $2 \le t \le 2+\epsilon$, zero elsewhere)*

   $$\begin{aligned}\int_2^5 \frac{1}{2\epsilon}(H(t-2+\epsilon) - H(t-2-\epsilon))g(t)\,dt &= \frac{1}{2\epsilon}\int_2^{2+\epsilon} g(t)\,dt \\ &= \frac{\epsilon^2}{6} + \frac{5\epsilon}{4} + 3,\end{aligned}$$

   at least once $\epsilon < 3$. As $\epsilon \to 0^+$ this limits to $3 = g(2)/2$. It's as if the Dirac mass is half in the interval, half out.

2. *This integral is exactly $g(t_0)$, by the sifting property of the Dirac function.*

3. *From part (b) the limit is*

   $$\lim_{t_0 \to 2^+} \int_2^5 \delta(t-t_0)g(t)\,dt = \lim_{t_0 \to 2^+} g(t_0) = g(2)$$

   since g is continuous. This treats the Dirac mass as if it is entirely inside the interval.

Reading Exercise Solution 5.4.7. *From the formula for $u(t)$ we can compute that $u'(t) = 10e^{-(t-1)}\cos(3(t-1)) - 10e^{-(t-1)}\sin(3(t-1))/3$. As $t \to 1^+$ we find u' limits to 10, and to the momentum of the particle just after the blow lands is 1 kg times 10 meters per second or 10 kg-meters per second. This is the same as the total impulse of the hammer blow, 10 newton-second (and note that kg-meters per second is the same as newton-seconds.)*

Reading Exercise Solution 5.5.1. *Laplace transforming $mg''(t) + cg'(t) + kg(t) = \delta(t)$ and using the initial data yields $(ms^2 + cs + k)G(s) = 1$, so $G(s) = 1/(ms^2 + cs + k)$ as advertised.*

Reading Exercise Solution 5.5.2. *Write $a = 2304 = 2 \cdot 10^3 + 3 \cdot 10^2 + 0 \cdot 10^1 + 4 \cdot 10^0$ and $b = 137 = 1 \cdot 10^2 + 3 \cdot 10^1 + 7 \cdot 10^0$ and foil to obtain product*

$$\begin{aligned} a \cdot b &= (2 \cdot 1)10^5 + (2 \cdot 3 + 3 \cdot 1)10^4 + (2 \cdot 7 + 3 \cdot 3 + 0 \cdot 1)10^3 \\ &\quad + (3 \cdot 7 + 0 \cdot 3 + 4 \cdot 1)10^2 + (7 \cdot 0 + 4 \cdot 3)10^1 + (4 \cdot 7)10^0 \\ &= 2 \cdot 10^5 + 9 \cdot 10^4 + 23 \cdot 10^3 + 25 \cdot 10^2 + 12 \cdot 10^1 + 28 \cdot 10^0 \end{aligned}$$

or $(2|3|0|4) * (3|1|7) = (2|9|23|25|12|28)$. But performing the carries leads to $a \cdot b = 315,648$.

Reading Exercise Solution 5.5.3. *The convolution of e^{-t} and e^{-2t} is computed as*

$$\begin{aligned} (e^{-t}) * (e^{-2t}) &= \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau \\ &= \int_0^t e^{-2t+\tau}\, d\tau \\ &= e^{-2t} \int_0^t e^{\tau}\, d\tau \\ &= e^{-2t}(e^t - 1) \\ &= e^{-t} - e^{-2t}. \end{aligned}$$

Reading Exercise Solution 5.5.4. *They do!*

Reading Exercise Solution 5.5.5. *The transform of $H(t - t_0)f(t - t_0)$ is, from the Second Shifting Theorem, equal to $e^{-st_0}F(s)$. The transform of δ_{t_0} is e^{-st_0}, so it is clear that $\mathcal{L}(\delta_{t_0} * f) = \mathcal{L}(\delta_{t_0})\mathcal{L}(f)$ is consistent with the Convolution Theorem.*

Reading Exercise Solution 5.6.1. *In addition to those listed, the Newton cooling ODE, the RC/RLC circuits, or compartment/salt tank models might be subject to control, in the right setting.*

Reading Exercise Solution 5.6.2. *The ODE here is* $y'(t) = -0.05y(t) - 0.5(1 + 19e^{-t})$. *Any number of techniques show that the solution is* $y(t) = 10(1 - e^{-t})$.

Reading Exercise Solution 5.6.3. *With initial condition* $y(0) = y_0$ *the solution is* $y(t) = 10(1-e^{-t})+y_0e^{-0.05t}$, *which approaches the desired setpoint as* $t \to \infty$.

Reading Exercise Solution 5.6.4. *With initial condition* $y(0) = y_0$ *the solution is* $y(t) = 5\sin(2\pi t/24) + y_0e^{-0.05t}$, *which approaches the desired setpoint* $r(t) = 5\sin(2\pi t/24)$ *as* $t \to \infty$, *for any* y_0.

Reading Exercise Solution 5.6.5. *The solution to* $y'(t) = -ky(t)+0.4u(t)$ *with* $y(0) = 0$ *and* $u(t) = 1 + 19e^{-t}$ *is* $y(t) = 8 - 8e^{-t}$. *The incubator temperature does not track the desired setpoint.*

Reading Exercise Solution 5.6.6. *The governing controlled ODE is* $y'(t) = -ky(t) + Ku(t)$. *If we substitute* $u(t) = K_p(r(t) - y(t))$ *into this ODE we obtain* $y'(t) = -ky(t) + KK_p(r(t) - y(t))$ *or*

$$y'(t) = -(k + KK_p)y(t) + KK_pr(t).$$

With $k = 0.05, K = 0.5$, $K_p = 1$, *and setpoint* $r(t) = 10 - 10e^{-t}$ *the ODE is*

$$y'(t) = -0.55y(t) + 5(1 - e^{-t}).$$

The solution with initial data $y(0) = 0$ *is* $y(t) \approx 9.09 + 11.1e^{-t} - 20.2e^{-0.55t}$. *This differs from the setpoint* $r(t)$, *even in the limit that* $t \to \infty$. *Larger values for* K_p *improve the situation. In fact solving the ODE with* K_p *left undefined and taking the limit as* $t \to \infty$ *shows that* $y(t) \to 100K_p/(10K_p+1)$.

Reading Exercise Solution 5.6.7. *The solution is approximately*

$$y(t) \approx 1.764e^{-0.55t} - 1.764\cos(2\pi t/24) + 3.71\sin(2\pi t/24).$$

A plot of $y(t)$ *and* $r(t)$ *is shown in Figure 5.14.*

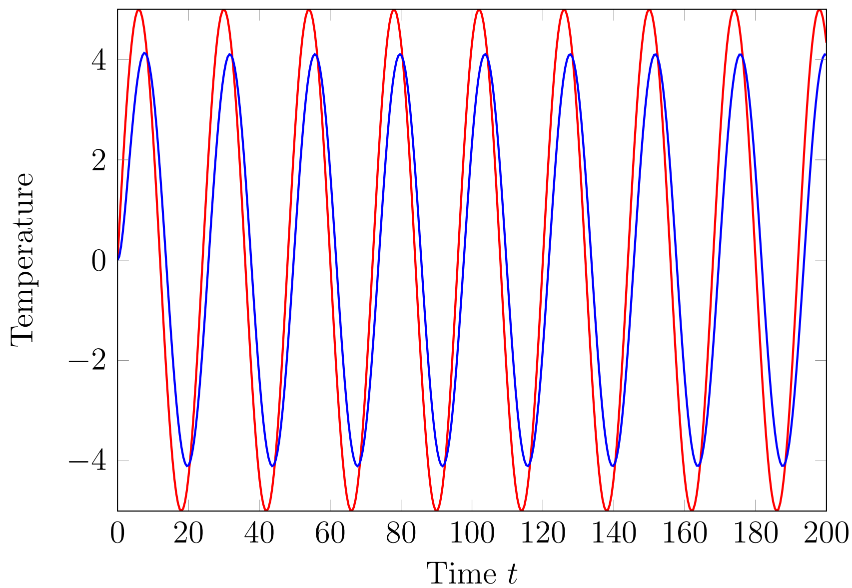

Figure 5.14: Setpoint $r(t)$ (red) and system process variable $y(t)$ (temperature, blue).

Reading Exercise Solution 5.6.8. *We need* $kr_0/(KK_p + k) < 0.1$, *or* $0.5/(0.5K_p + 0.05) < 0.1$, *which leads to* $K_p > 9.9$.

Reading Exercise Solution 5.6.9. *In this case* $G(s) = 1/(2s - 0.9)$ *and* $Y(s) = \frac{10}{s(2s-0.9)(s+1)}$. *Then* $y(t) \approx 11.11 - 3.49e^{-t} - 7.66e^{0.45t}$ *contains a growing exponential and* $y(t)$ *is unbounded. This disastrous control strategy is to turn up the heat when the incubator is too hot, turn up the cooling when the incubator is too cold.*

Reading Exercise Solution 5.6.10. *The solution here is* $y(t) \approx 10 - 2.08e^{-t} - 4.73e^{-0.15t}\sin(0.69t) - 7.92e^{-0.15t}\cos(0.69t)$. *The solution still stabilizes at the appropriate value, but oscillates while doing so. The poles of* $G(s)$ *lie at approximately* $-0.15 \pm 0.69i$, *which is precisely where* $Y(s)$ *has poles, and this explains the behavior of* $y(t)$.

Reading Exercise Solution 5.6.11. $G(s)$ *is continuous up to* $s = 0$ *so that* $\lim_{s\to 0^+} G(s) = G(0) = 1$.

Chapter 6

Reading Exercise Solution 6.1.1. *For this system* $f_1(x_1, x_2, t) = tx_1 - 3x_2 + t^2 = a_{1,1}(t)x_1 + a_{1,2}x_2 + b_1(t)$ *with* $a_{1,1}(t) = t, a_{1,2}(t) = -3$*, and* $b_1(t) = t^2$*. Also* $f_2(x_1, x_2, t) = -\sin(t)x_1 + 3tx_2 - t = a_{2,1}(t)x_1 + a_{2,2}x_2 + b_2(t)$ *with* $a_{2,1}(t) = -\sin(t), a_{2,2}(t) = 3t$*, and* $b_2(t) = -t$*. This is of the form for a linear system.*

Reading Exercise Solution 6.1.2. *Take* $x_1 = \theta$ *and* $x_2 = \dot{\theta}$*. Then* $\dot{x}_1 = x_2$ *is clear. The equation* $\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = 0$ *can be solved for* $\ddot{\theta} = -\frac{g}{L}\sin(\theta)$*, or in terms of* x_1 *and* x_2*,* $\dot{x}_2 = -\frac{g}{L}\sin(x_1)$*. The coupled ODE's*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1)\end{aligned}$$

is equivalent to the scalar ODE $\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = 0$*.*

Reading Exercise Solution 6.1.3.

(a) If the rest length of mass 1 is L_1 *then based on the figure it's pretty clear the amount it is stretched is* $x_1 - L_1$*. And given that the distance between mass 1 and mass 2 is* $x_2 - x_1$*, the amount spring 2 is stretched is the deviation from this, so* $x_2 - x_1 - L_2$*.*

(b) The force on mass m_1 *due to the first spring follows from Hooke's law and is* $-k_1(x_1 - L_1)$*. The force exerted by spring 2 also follows from Hooke's law and is* $k_2(x_2 - x_1 - L_2)$*, since that spring in on the right (e.g., if the spring is lengthened so that* $x_2 - x_1 - L_2 > 0$ *the force is to the right. The velocity of mass 1 is* $\dot{x}_1$*, so viscous damping specifies force* $-c_1\dot{x}_1$ *for some* $c_1 \geq 0$*.*

(c) The force exerted by spring 2 on mass m_2 *follows from Hooke's law and is* $-k_2(x_2 - x_1 - L_2)$*, since that spring in on the left (e.g., if the spring is lengthened so that* $x_2 - x_1 - L_2 > 0$ *the force is to the left, the negative direction). The velocity of mass 2 is* $\dot{x}_2$*, so viscous damping specifies force* $-c_2\dot{x}_2$ *for some* $c_2 \geq 0$*.*

Reading Exercise Solution 6.1.4. *From Newton's second law,* $F = m_1\ddot{x}_1$ *where* F *is the sum of all forces on* m_1*. From the last Reading Exercise this net force is* $-k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_1) - c_1\dot{x}_1$ *(spring 1, spring 2, and friction). So*

$$m_1\ddot{x}_1 = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_1) - c_1\dot{x}_1$$

From Newton's second law, $F = m_2\ddot{x}_2$ *where* F *is the sum of all forces on* m_2*. From the last Reading Exercise this net force is* $-k_2(x_2 - x_1 - L_2) - c_2\dot{x}_2$ *(spring 2 and friction). So*

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1 - L_2) - c_2\dot{x}_2.$$

Reading Exercise Solution 6.1.5. *Start with*

$$\dot{w}_2 = -\frac{k_1 + k_2}{m_1}w_1 - \frac{c_1}{m_1}w_2 + \frac{k_2}{m_1}w_3$$

Now use $w_2 = \dot{u}_1$ *to find that* $\dot{w}_2 = \ddot{u}_1$ *on the left. Fill in* $w_1 = u_1, w_2 = \dot{u}_1,$ *and* $w_3 = u_2$ *and obtain exactly the equation* $\ddot{u}_1 = -\frac{k_1+k_2}{m_1}u_1 + \frac{k_2}{m_1}u_2 - \frac{c_1}{m_1}\dot{u}_1$.

Similarly, start with

$$\dot{w}_4 = \frac{k_2}{m_2}w_1 - \frac{k_2}{m_2}w_3 - \frac{c_2}{m_2}w_4.$$

Use $w_4 = \dot{u}_2$ *to find* $\dot{w}_4 = \ddot{u}_2$ *on the left. Fill in* $w_1 = u_1, w_3 = u_2,$ *and* $w_4 = \dot{u}_2$ *to obtain*

$$\ddot{u}_2 = \frac{k_2}{m_2}u_1 - \frac{k_2}{m_2}u_2 - \frac{c_2}{m_2}\dot{u}_2.$$

Reading Exercise Solution 6.2.1. *The matrix* $\mathbf{A}$ *is*

$$\mathbf{A} = \begin{bmatrix} -3.601 & 4.64 \\ 3.19 & -4.64 \end{bmatrix}$$

and has eigenvalues $\lambda_1 = -0.238$*,* $\lambda_2 = -8.002$ *with corresponding eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} 0.810 \\ 0.587 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -0.725 \\ 0.688 \end{bmatrix}.$$

Then

$$\mathbf{w}_1(t) = e^{-0.238t}\begin{bmatrix} 0.810 \\ 0.587 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{w}_2(t) = e^{-8.002t}\begin{bmatrix} -0.725 \\ 0.688 \end{bmatrix}.$$

The eigenvalues should have negative real part since the amount of LSD in each compartment should clearly decay to zero if there is no input after time $t = 0$.

Reading Exercise Solution 6.2.2.

(a) *A general solution would be* $\mathbf{u}(t) = c_1\mathbf{w}_1(t) + c_2\mathbf{w}_2(t)$ *or*

$$\mathbf{u}(t) = c_1 e^{-0.238t}\begin{bmatrix}0.810\\0.587\end{bmatrix} + c_2 e^{-8.002t}\begin{bmatrix}-0.725\\0.688\end{bmatrix}.$$

(b) *At* $t = 0$ *we need* $0.81c_1 - 0.725c_2 = 140$ *and* $0.587c_1 + 0.688c_2 = 0$, *with solution* $c_1 \approx 98.02$ *and* $c_2 \approx -83.57$. *The solution with the required initial data is*

$$\mathbf{u}(t) = 98.02e^{-0.238t}\begin{bmatrix}0.810\\0.587\end{bmatrix} - 83.57e^{-8.002t}\begin{bmatrix}-0.725\\0.688\end{bmatrix}.$$

A plot of both components, $u_1(t)$ *and* $u_2(t)$, *is shown in Figure 6.15.*

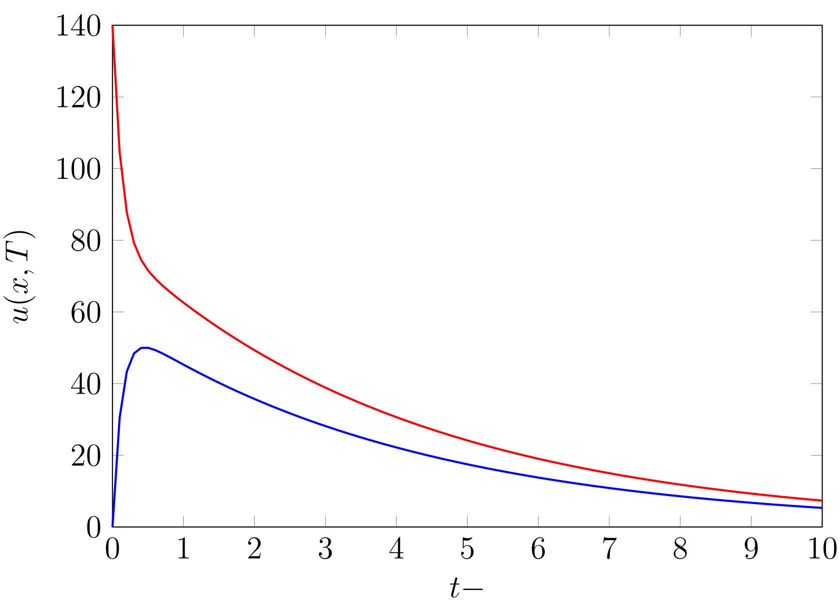

Figure 6.15: Plot of $u_1(t)$ (red) and $u_2(t)$ (blue).

Reading Exercise Solution 6.2.3. *In this case we find*

$$\mathbf{x}(t) = d_1 e^{-t}\begin{bmatrix}-\cos(2t) + 2\sin(2t)\\5\cos(2t)\end{bmatrix} + d_2 e^{-t}\begin{bmatrix}\sin(2t) + 2\cos(2t)\\-5\sin(2t)\end{bmatrix}$$

after picking off the real and imaginary parts of $e^{(-1-2i)t}\langle -1 + 2i, 5\rangle$.

Reading Exercise Solution 6.2.4. *Substituting $t = 0$ into the general solution $\mathbf{w}(t)$ yields $\mathbf{w}(0) = \langle(-c_2 + c_4)/\alpha, c_1/2 - c_3, -\alpha c_2 - c_4/\alpha, c_1 + c_3\rangle$, and setting this equal to $\langle 1, 0, 0, 0\rangle$ gives the asserted equations. The c_1, c_3 equations are decoupled from c_2, c_4 and homogeneous, so $c_1 = c_3 = 0$, while $c_2 = -\sqrt{2}/3$ and $c_4 = 2\sqrt{2}/3$. Then*

$$\mathbf{w}(t) = -\frac{\sqrt{2}}{3}\begin{bmatrix} -\cos(t/\alpha)/\alpha \\ \sin(t/\alpha)/2 \\ -\alpha\cos(t/\alpha) \\ \sin(t/\alpha) \end{bmatrix} + \frac{2\sqrt{2}}{3}\begin{bmatrix} \cos(\alpha t)/\alpha \\ -\sin(\alpha t) \\ -\cos(\alpha t)/\alpha \\ \sin(\alpha) \end{bmatrix}.$$

Reading Exercise Solution 6.3.1. *Transforming the ODEs and substituting in the initial data yields*

$$\begin{aligned} sX_1(s) - 2 &= 3X_1(s) - X_2(s) + 3/(s-1) \\ sX_2(s) - 1 &= -X_1(s) + 3X_2(s). \end{aligned}$$

Solve to find $X_1(s) = \frac{-s+5}{2(s^2-3s+2)}$ and $X_2(s) = \frac{s+1}{2(s^2-3s+2)}$. Inverse transforming yields $x_1(t) = 3e^{2t}/2 - 2e^t$ and $x_2(t) = 3e^{2t}/2 - e^t$.

Reading Exercise Solution 6.3.2.

(a) *We find*

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -6 & -7 \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t) = e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

(b) *Write $\mathbf{f}(t) = e^t\mathbf{w}$ with $\mathbf{w} = \langle 5, 10\rangle$ and substitute $\mathbf{x}_p(t) = e^t\mathbf{v}$ into the ODE to find $\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{w}$ after cancelling the e^t terms. Then $(\mathbf{A} - \mathbf{I})\mathbf{v} = -\mathbf{w}$ so*

$$\mathbf{v} = -(\mathbf{A} - \mathbf{I})^{-1}\mathbf{w} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

Then $\mathbf{x}_p(t) = e^t\mathbf{v}$.

(c) *A general homogeneous solution is*

$$\mathbf{x}_h(t) = c_1 e^{-4t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(d) *From* $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t)$ *the initial data* $x_1(0) = 1$, $x_2(0) = 3$ *yields equations* $-c_1 - c_2 + 7 = 1$, $2c_1 + c_2 - 4 = 3$ *with solution* $c_1 = 1$, $c_2 = 5$.

Reading Exercise Solution 6.4.1. *For* $m = 1$ *the sum yields*

$$\begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}.$$

For $m = 2$ *the sum yields*

$$\begin{bmatrix} 25/2 & -5 \\ 15 & -5 \end{bmatrix}.$$

For $m = 5$ *the sum yields*

$$\begin{bmatrix} 251/12 & -91/10 \\ 273/10 & -164/15 \end{bmatrix} \approx \begin{bmatrix} 20.92 & -9.1 \\ 27.3 & -10.93 \end{bmatrix}.$$

For $m = 10$ *the sum yields approximately*

$$\begin{bmatrix} 21.40 & -9.34 \\ 28.02 & -11.29 \end{bmatrix}.$$

Reading Exercise Solution 6.4.2. *We know that* $\mathbf{B} + (-\mathbf{B}) = \mathbf{0}$, *so*

$$e^{\mathbf{B}+(-\mathbf{B})} = e^{\mathbf{0}} = \mathbf{I}.$$

But since $\mathbf{B}$ *and* $-\mathbf{B}$ *commute under multiplication it follows that* $e^{\mathbf{B}+(-\mathbf{B})} = e^{\mathbf{B}}e^{-\mathbf{B}}$. *Combine these last two facts to find that*

$$e^{\mathbf{B}}e^{-\mathbf{B}} = \mathbf{I}.$$

Reading Exercise Solution 6.4.3. 1. *We find that*

$$\frac{d(e^{t\mathbf{A}})}{dt} = \begin{bmatrix} -4e^{-t} & 2e^{-t} \\ -6e^{-t} & 3e^{-t} \end{bmatrix}.$$

if we differentiate component-by-component.

2. *A routine matrix multiplication shows that*

$$\mathbf{A}e^{t\mathbf{A}} = e^{t\mathbf{A}}\mathbf{A} = \begin{bmatrix} -4e^{-t} & 2e^{-t} \\ -6e^{-t} & 3e^{-t} \end{bmatrix}$$

which is the same as $\frac{d(e^{t\mathbf{A}})}{dt}$ *from part (a).*

Reading Exercise Solution 6.4.4. *The matrix here is*

$$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 3$ *and* $\lambda_2 = 1$. *We conclude that*

$$e^{t\mathbf{D}} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix}.$$

The solution with $\mathbf{x}(0) = \langle 2, 5 \rangle$ *is*

$$\mathbf{x}(t) = e^{t\mathbf{D}}\mathbf{x}(0) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2e^{3t} \\ 5e^{t} \end{bmatrix}.$$

Reading Exercise Solution 6.4.5. *The eigenvalues of* $\mathbf{A}$ *are* $\lambda_1 = 0$ *and* $\lambda_2 = -1$ *with corresponding eigenvectors* $\mathbf{v}_1 = \langle 1, 2 \rangle$ *and* $\mathbf{v}_2 = \langle 2, 3 \rangle$ *(the ordering of the eigenvalues and scaling of the eigenvectors may vary). Then*

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also,

$$e^{t\mathbf{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} -3 + 4e^{-t} & 2 - 2e^{-t} \\ -6 + 6e^{-t} & 4 - 3e^{-t} \end{bmatrix}.$$

Reading Exercise Solution 6.4.6. *There is a double eigenvalue* $\lambda = -2$ *but only one eigenvector,* $\mathbf{v} = \langle -1, 2 \rangle$, *so we cannot form* $\mathbf{P}$. *This matrix has a defective eigenvalue and is not diagonalizable.*

Chapter 7

Reading Exercise Solution 7.1.1. *The expression* $r_1 u_1 \left(\frac{K_1 - u_1 - a u_2}{K_1} \right)$ *decreases as* u_2 *increases (since* $K_1 - u_1 - a u_2$ *is a decreasing function of* u_2 *when* $a > 0$*). This makes sense—a larger population for* u_2 *decreases the growth rate of the* u_1 *species. Similar remarks apply to the expression* $r_2 u_2 \left(\frac{K_2 - u_2 - b u_1}{K_2} \right)$, *with the roles of* u_1 *and* u_2 *reversed.*

Reading Exercise Solution 7.1.2. *Each solution pair* $(u_1(t), u_2(t))$ *has* $\dot{u}_1 = \dot{u}_2 = 0$ *(since both components are constant) and are easily seen to be solutions to the right hand side of the ODEs. The solution* $(u_1, u_2) = (0, 0)$ *is mutual extinction,* $(u_1, u_2) = (K_1, 0)$ *is species* 1 *at carrying capacity, species* 2 *extinct,* $(u_1, u_2) = (0, K_2)$ *is species* 2 *at carrying capacity, species* 1 *extinct. The choice*

$$u_1 = \frac{K_1 - K_2 a}{1 - ab}, \quad u_2 = \frac{K_2 - K_1 b}{1 - ab}$$

can also be verified to satisfy the ODEs and corresponds to mutual coexistence, provide both functions are positive.

Reading Exercise Solution 7.1.3. *Many critiques are possible, for example, once someone is sick and bedridden their contact with susceptible people may drop to near zero (if isolated), as opposed to when they are sick and not bedridden. The model makes no obvious attempt to model the length of the illness, and of course assumes a closed community.*

Reading Exercise Solution 7.1.4. *We find that* $\dot{S} + \dot{I} + \dot{R} = 0$ *at all times. This makes sense: the total population remains constant.*

Reading Exercise Solution 7.1.5. *The substitution* $\sin(\theta) = \theta$ *in the nonlinear pendulum ODE leads to*

$$\ddot{\theta} + c\dot{\theta} + g\theta/L = 0.$$

If we let $x_1 = \theta$ *and* $x_2 = \dot{\theta}$ *then* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -gx_1/l - cx_2$*, which can be formulated as* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -g/L & -c \end{bmatrix}.$$

The characteristic equation for $\mathbf{A}$ *is* $\lambda(\lambda + c) + g/L = 0$ *or* $\lambda^2 + c\lambda + g/L = 0$*, with roots*

$$-\frac{c}{2} \pm \frac{\sqrt{c^2 - 4g/L}}{2}.$$

These are the eigenvalues of $\mathbf{A}$*. When* $0 < c \leq 2\sqrt{g/L}$ *we have* $c^2 - 4g/L < 0$ *and the eigenvalues are complex and conjugate with negative real part* $-c/2$*. If* $c > 2\sqrt{g/L}$ *then* $c^2 > 4g/L$*, so* $c^2 - 4g/L > 0$ *and the eigenvalues are real. Moreover, it is always the case that* $c^2 - 4g/L < c^2$ *so that* $\sqrt{c^2 - 4g/L} < c$*.*

As a result $-c + \sqrt{c^2 - 4g/L} < 0$ *(and* $-c - \sqrt{c^2 - 4g/L} < 0$ *too). Thus both eigenvalues are real and negative.*

Thus when $c > 0$ *the solution will decay in time; the pendulum comes to rest.*

Reading Exercise Solution 7.1.6. *In this case the direction vector is* $\langle 0, 1\rangle$*. That is,* $\dot{x}_1 = 0$ *and* $\dot{x}_2 = 1$*. This indicates that (since* $x_1 = u$*)* $\dot{u} = 0$*, so the mass is not moving. Also (since* $x_2 = \dot{u}$*) that* $\ddot{u} = 1$*, that is, the mass is accelerating in the direction of increasing* u*.*

Reading Exercise Solution 7.1.7. *The solution is* $u(t) = \frac{2\sqrt{3}}{3}e^{-t/2}\sin(t\sqrt{3}/2) + 2e^{-t/2}\cos(t\sqrt{3}/2)$*. Equivalently,* $x_1(t) = \frac{2\sqrt{3}}{3}e^{-t/2}\sin(t\sqrt{3}/2) + 2e^{-t/2}\cos(t\sqrt{3}/2)$ *and* $x_2(t) = -\frac{4\sqrt{3}}{3}e^{-t/2}\sin(t\sqrt{3}/2)$*. The function* $x_1(t)$ *oscillates and decays to zero, as does* $x_2 = \dot{x}_1$*, consistent with the solution trajectory show in the text.*

Reading Exercise Solution 7.1.8. *Solving* $p_1(1 - p_1/2) - 0.1p_1p_2 = 0$ *and* $2p_2(1 - p_2/3) - 0.3p_1p_2 = 0$ *yields solutions* $(x, y) = (0, 0), (2, 0), (0, 3)$*, and* $(20/13, 30/13) \approx (1.54, 2.31)$*. The point* $(0, 0)$ *is mutual extinction,* $(2, 0)$ *is the extinction of species 2,* $(0, 3)$ *is the extinction of species 1, and* $(1.54, 2.31)$ *is mutual coexistence.*

Reading Exercise Solution 7.1.9. *The solutions are now* $(x, y) = (0, 0), (2, 0), (0, 3)$*.*

Reading Exercise Solution 7.2.1.

(a) A general solution in this case is $\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{v} + c_2e^{\lambda t}(\mathbf{v}_1 + t\mathbf{v})$*. If* $\lambda < 0$ *then clearly* $\lim_{t\to\infty} e^{\lambda t} = 0$*. It's also easy to check that* $\lim_{t\to\infty} te^{\lambda t} = 0$ *(e.g, use L'Hopital's rule on* $t/e^{-\lambda t}$*).*

(b) Also clear, since then $\lim_{t\to\infty} e^{\lambda t} = \infty$ *and* $\lim_{t\to\infty} te^{\lambda t} = \infty$*. If either* c_1 *or* c_2 *is nonzero, the solution will grow without bound.*

Reading Exercise Solution 7.2.2. *The inequality* $x_2 > \frac{k_b+k_e}{k_a}x_1$ *is easily manipulated to* $-(k_b + k_e)x_1 + k_ax_2 > 0$ *(multiply by* k_a*, subtract* $(k_b + k_e)x_1$ *from both sides. Thus* $\dot{x}_1 = -(k_b + k_e)x_1 + k_ax_2 > 0$*, so solutions move in the direction of increasing* x_1 *(to the right). Similar computations show that* $x_2 < \frac{k_b+k_e}{k_a}x_1$ *is equivalent to* $-(k_b + k_e)x_1 + k_ax_2 < 0$*, so in this region* $\dot{x}_1 = -(k_b+k_e)x_1+k_ax_2 < 0$ *and solutions move in the direction of decreasing* x_1 *(to the left).*

Reading Exercise Solution 7.3.1. *If* $u_1(t) = 0$ *for all* t *and* $u_2(t)$ *satisfies the logistic equation* $\dot{u}_2 = 2u_2(1 - u_2/3)$ *then the pair* (u_1, u_2) *satisfies the coupled system* $\dot{u}_1 = u_1(1 - u_1/2) - 0.1u_1u_2$ *(both sides of this ODE are identically zero here) and* $\dot{u}_2 = 2u_2(1 - u_2/3) - 0.3u_1u_2$ *(satisfied since* $u_1 = 0$*). We conclude that this is the solution to the coupled system with data* $u_1(t_0) = 0$*,* $u_2(t_0) = u_0$*. In this setting the* u_1 *species is extinct and the* u_2 *species exists alone, and will approach its carrying capacity.*

Reading Exercise Solution 7.3.2. *At this point we are in the region where* $\dot{u}_1 < 0$*, so the population of species 1 is declining.*

Reading Exercise Solution 7.3.3. *From the left panel, at such a point* $\dot{u}_1 < 0$ *so the population of species 1 is declining. From the right panel we see also that* $\dot{u}_2 < 0$*, so the population of species 2 is also declining.*

Reading Exercise Solution 7.3.4. *If* $u_2(t) = 0$ *for all* t *and* $u_1(t)$ *satisfies the logistic equation* $\dot{u}_1 = u_1(1 - u_1/2)$ *then the pair* (u_1, u_2) *satisfies the coupled system* $\dot{u}_1 = u_1(1 - u_1/2) - 0.1u_1u_2$ *(since* u_2 *is identically zero here) and* $\dot{u}_2 = 2u_2(1 - u_2/3) - 0.3u_1u_2$ *(satisfied since* $u_2 = 0$*, so both sides of the ODE are identically zero). We conclude that this is the solution to the coupled system with data* $u_2(t_0) > 0$*,* $u_2(t_0) = 0$*. In this setting the* u_2 *species is extinct and the* u_1 *species exists alone, and will approach its carrying capacity.*

Reading Exercise Solution 7.3.5. *First, the right sides of the competing species equations are continuous with continuous partial derivatives, so the solution through any point in the plane is unique. If* $u_1(t_0) = 0$ *with* $u_2(t_0) > 0$ *or* $u_1(t_0) > 0$ *with* $u_2(t_0) = 0$ *the previous Reading Exercises show that the solution will stay on the corresponding axis (*$u_1 = 0$ *or* $u_2 = 0$*). If we start at* $u_1(t_0) = u_2(t_0) = 0$ *then both* $u_1(t)$ *and* $u_2(t)$ *are identically zero. In each case, the solution remains in the first quadrant.*

If $u_1(t_0) > 0$ *and* $u_2(t_0) > 0$ *then the solution must also remain in the first quadrant, so if at some time* $t = t^*$ *we have* $u_1(t^*) = 0$ *then we have two distinct solutions curves (one with* $u_1(t) > 0$ *for* $t < t^*$*, the other of the form* $u_1(t) = 0$ *for* $t < t^*$*) both passing though the same point,* $(0, u_2(t^*))$*, in violation of the existence-uniqueness theorem.*

Reading Exercise Solution 7.3.6. *The point* $(0, 0)$ *is mutual extinction. The point* $(2, 0)$ *is the first species at its carrying capacity and the second species extinct, while the point* $(0, 3)$ *is the second species at its carrying* *capacity and the first species extinct. The point* $(1.54, 2.31)$ *is mutual coexistence.*

Reading Exercise Solution 7.3.7. *Solutions with each set of initial conditions are shown in Figure 7.16.*

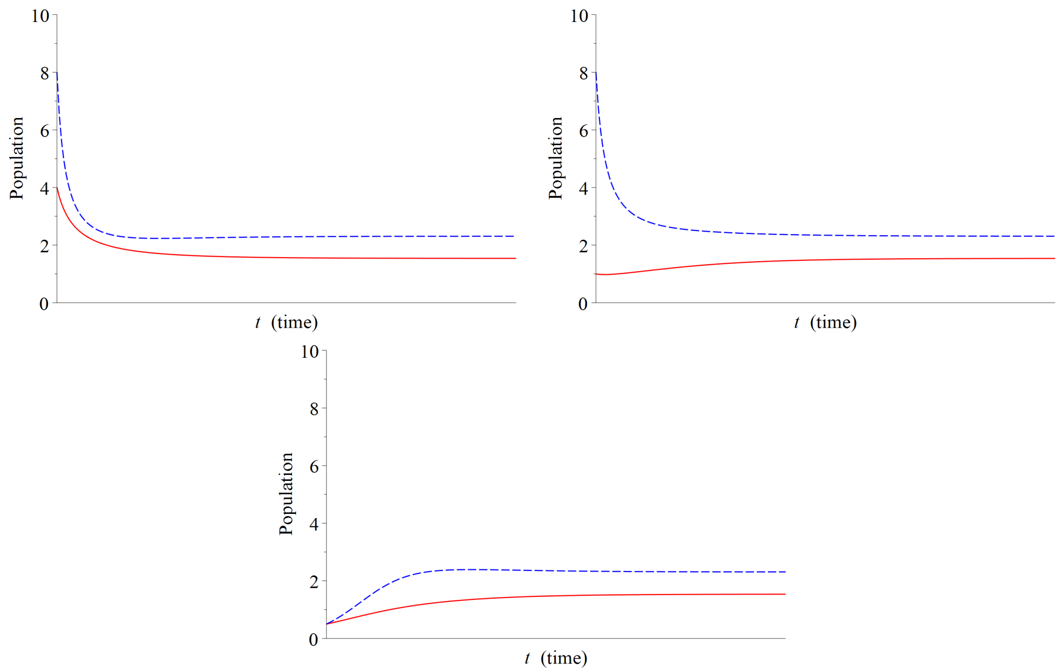

Figure 7.16: Solutions for $u_1(t)$ (solid red curves) and $u_2(t)$ (dashed blue curves). Top left panel: $u_1(0) = 4, u_2(0) = 8$. Top right panel: $u_1(0) = 1, u_2(0) = 8$. Bottom panel: $u_1(0) = 0.5, u_2(0) = 0.5$.

Reading Exercise Solution 7.3.8. *The linearizations of* f_1 *and* f_2 *for this system at the point* $(1.54, 2.31)$ *are*

$$L_1(u_1, u_2) \approx -0.769(u_1 - 1.54) - 0.154(u_2 - 2.31)$$
$$L_2(u_1, u_2) \approx -0.692(u_1 - 1.54) - 1.54(u_2 - 2.31),$$

respectively. The corresponding matrix is

$$\mathbf{A} \approx \begin{bmatrix} -0.769 & -0.154 \\ -0.692 & -1.54 \end{bmatrix}$$

and $\mathbf{A}$ *has approximate eigenvalues* -0.65 *and* -1.66*. This means that* $(1.54, 2.31)$ *is a stable node for the fixed point* $(1.54, 2.31)$ *in the linearized system* $\dot{u}_1 = L_1(u_1, u_2)$*,* $\dot{u}_2 = L_2(u_1, u_2)$*. From the Hartman-Grobman Theorem, the fixed point* $(1.54, 2.31)$ *behaves like a stable node for the nonlinear system.*

Reading Exercise Solution 7.4.1. *The ODE for* $u_1(t)$ *is* $\dot{u}_1 = r_1 u_1(K_1 - u_1 - a u_2)/K_1$*. With* $u_1 = K_1 v_1$*,* $\dot{u}_1 = K_1 \dot{v}_1$*, and* $\dot{u}_2 = K_2 \dot{v}_2$ *this ODE becomes*

$$K_1 \dot{v}_1 = r_1 K_1 v_1 (K_1 - K_1 v_1 - a K_2 v_2)/K_1.$$

Divide by K_1 *to obtain* $\dot{v}_1 = r_1 v_1 (K_1 - K_1 v_1 - a K_2 v_2)/K_1 = r_1 v_1 (1 - v_1 - a K_2 v_2 / K_1)$*. With* $\bar{a} = K_2 a / K_1$ *this becomes* $\dot{v}_1 = r_1 v_1 (1 - v_1 - \bar{a} v_2)$*. The same computation with the roles and parameters the first and second species reversed yields* $\dot{v}_2 = r_2 v_2 (1 - v_2 - \bar{b} v_1)$ *with* $\bar{b} = K_1 b / K_2$*.*

Reading Exercise Solution 7.4.2. *If we set* $r_1 v_1 (1 - v_1 - \bar{a} v_2) = 0$ *we can divide both sides by* r_1 *and find* $v_1(1 - v_1 - \bar{a} v_2) = 0$*. The implies that either* $v_1 = 0$ *or* $1 - v_1 - \bar{a} v_2 = 0$*. The second ODE similarly leads to* $v_2 = 0$ *or* $1 - v_2 - \bar{b} v_1 = 0$*. Clearly* $(v_1, v_2) = (0, 0)$ *is a solution. Taking* $v_1 = 0$ *and* $v_2 = 1$ *yields a solution, as does* $v_1 = 1, v_2 = 0$*. Finally, if we impose* $1 - v_1 - \bar{a} v_2 = 0$ *and* $1 - v_2 - \bar{b} v_1 = 0$ *and solve simultaneously we find* $(v_1, v_2) = ((\bar{a} - 1)/(\bar{a}\bar{b} - 1), (\bar{b} - 1)/(\bar{a}\bar{b} - 1))$*.*

Reading Exercise Solution 7.4.3. *We find* $\bar{a} = K_2 a / K_1 = 0.3$ *and* $\bar{b} = K_1 b / K_2 = 0.3$*. Then* $1/\bar{a} = 10/3$ *and* $1/\bar{b} = 10/3$*, both greater than* 1*.*

Reading Exercise Solution 7.4.4. *This is straightforward; the figure is correct.*

Reading Exercise Solution 7.4.5. *The value for* $1/\bar{a}$ *has two possibilities, greater than or less than one, as does* $1/\bar{b}$*. This gives* $2 \times 2 = 4$ *possibilities, as listed.*

Reading Exercise Solution 7.4.6. *When* $1 < 1/\bar{a}$ *and* $1 < 1/\bar{b}$ *it looks like all solutions converge to the fixed point* $(v_1, v_2) = ((\bar{a} - 1)/(\bar{a}\bar{b} - 1), (\bar{b} - 1)/(\bar{a}\bar{b} - 1))$ *(corresponding to fixed point* $u_1 = (K_1 - K_2 a)/(1 - ab)$ *and* $u_2 = (K_2 - K1b)/(1 - ab)$ *in the original system. When* $1 > 1/\bar{a}$ *and* $1 > 1/\bar{b}$ *it appears that solutions converge to either* $v_1 = 1, v_2 = 0$ *or* $v_1 = 0, v_2 = 1$*, depending on the initial conditions. These corresponds to* $u_1 = K_1, u_2 = 0$ *or* $u_1 = 0, u_2 = K_2$*.*

Reading Exercise Solution 7.4.7. *When $\bar{a} < 1$ and $\bar{b} < 1$ the eigenvalues of the Jacobian at $(1, 0)$ are $-r_1 < 0$ and $r_2(1 - \bar{b}) > 0$, so this is a saddle. When $\bar{a} < 1$ and $\bar{b} < 1$ the eigenvalues of the Jacobian at $(0, 1)$ are $-r_2 < 0$ and $r_1(1 - \bar{a}) > 0$, so this is also a saddle. In brief, when competition is mutually low, the fixed points ¡m¿(1,0)¡/m¿ and ¡m¿(0,1)¡/m¿ in which only one species survives are unstable.*

When $\bar{a} > 1$ and $\bar{b} > 1$ the eigenvalues of the Jacobian at $(1, 0)$ are $-r_1 < 0$ and $r_2(1-\bar{b}) < 0$, so this is a stable node. Similarly when $\bar{a} > 1$ and $\bar{b} > 1$ the eigenvalues of the Jacobian at $(0, 1)$ are $-r_2 < 0$ and $r_1(1-\bar{a}) < 0$ and this is also a stable node.

Reading Exercise Solution 7.4.8. *That $T^2 - 4D = (r_1 v_1^* - r_2 v_2^*)^2 + 4\bar{a}\bar{b} r_1 r_2 v_1^* v_2^*$ is a straightforward algebra computation. The quantity on the right is clearly positive as it is a sum of a square and a positive term. Then we have $D > 0$ and $T^2/4 - D$. From the analysis in the appendix, both eigenvalues of $\mathbf{J}(v_1^*, v_2^*)$ are in fact real, so this is a stable node.*

Reading Exercise Solution 7.5.1. *With $\mathbf{f}(1.0, \mathbf{x}^2) \approx \langle 0.483, -0.783\rangle$ we find*

$$\mathbf{x}^3 = \langle 0.960, 0.720\rangle > +0.5\langle 0.483, -0.783\rangle \approx \langle 1.202, 0.328\rangle$$

and with $\mathbf{f}(1.5, \mathbf{x}^3) \approx \langle 0.0043, -0.846\rangle$

$$\mathbf{x}^4 = \langle 1.202, 0.328\rangle + 0.5\langle 0.0043, -0.846\rangle \approx \langle 1.204, -0.0949\rangle.$$

Reading Exercise Solution 7.5.2.

(a) The system is linear, of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}.$$

The solution with the given initial data is $\mathbf{x}(t) = \langle 3e^{-t} - 2e^{-2t}, 6e^{-t} - 6e^{-2t}\rangle$ and $\mathbf{x}(2) \approx \langle 0.36937, 0.70212\rangle$.

(b) The estimate is $\langle 0.34167, 0.66028\rangle$, error $\|\mathbf{x}(2) - \mathbf{x}^n\| \approx 0.05017$.

(c) The estimate is $\langle 0.36968, 0.70158\rangle$, error 6.177×10^{-4}.

(d) The estimate is $\langle 0.36937, 0.70211\rangle$, error 5.68×10^{-6}.

(e) *For Euler the error is* 0.004584*, for the improved Euler method the error is* 4.47×10^{-6}*, and for RK4 the error is* 4.77×10^{-10}*. The Euler method error dropped by about a factor of* 10*, improved Euler by about a factor of* 100*, and RK4 by about a factor of* 10^4*, all as expected.*

Reading Exercise Solution 7.5.3. *In this case* $\lambda = 1$*, so we have* $x^{k+1} = (1-h)x^k$ *with* $x^0 = 1$*. Then it's easy to see that in general* $x^k = (1-h)^k$*, and with* $h = 1.5$ *this becomes* $x^k = (-0.5)^k$ *or* $x^k = (-1/2)^k$*. With* $h = 2.2$ *we have* $x^k = (-1.2)^k$*.*

Reading Exercise Solution 7.5.4. *Start with the inequality* $0 < 1 - h\lambda < 1$*. Multiply through by* -1 *to obtain* $-1 < h\lambda - 1 < 0$*, add* 1 *throughout, and divide by* λ *(assumed positive) to find* $0 < h < 1/\lambda$*. For the system* $x'(t) = -x(t)$ *we have* $\lambda = 1$*, so the bound on the step size becomes* $0 < h < 1$ *to maintain positive iterates. This is in accord with the figure.*

Reading Exercise Solution 7.5.5. *The bound in this case is* $0 < h < 2/\lambda$ *where* λ *should be chosen as* $\max(\lambda_1, \lambda_2)$ *with* $\lambda_1 = 1, \lambda_2 = 100$*, so* $\lambda = 100$ *and the stability bound on* h *is* $0 < h < 0.02$*. In the figure we see that* $h = 0.001$ *yields a good approximation, but* $h = 0.02005$ *(which violates the bound) causes the iterates for the* x_1 *component to grow.*

Reading Exercise Solution 7.5.6. *The true solution is* $x_1(t) = 1.05e^{-t} - 0.05e^{-100t}$*,* $x_2(t) = 1.05e^{-t} + 0.04e^{-100t}$*. A bit of experimentation shows that* $h < 0.2$ *is necessary in the improved Euler method for the iterates to decay (and approximate the true solution). For the RK4 method* $h < 0.27$ *is necessary.*

Reading Exercise Solution 7.6.1. *Given* $E(u,v) = mgL(1 - \cos(u)) + \frac{1}{2}mL^2v^2$ *compute* $\nabla E = \langle mgL\sin(u), mL^2v \rangle$ *and so the critical point of* E *occur when* $v = 0$ *and* $u = k\pi$ *for some integer* k*. A second derivative test shows that the points* $u = 2k\pi, v = 0$ *are local minima.*

Reading Exercise Solution 7.6.2. *These trajectories correspond to the pendulum swinging "over the top", around and around.*

Reading Exercise Solution 7.6.3. *Constant functions would be a first integral for any system of ODEs (if allowed) and carry no information at all about the system's behavior.*

Reading Exercise Solution 7.6.4.

(a) *We find $\dot{x}_1 = x_2$ and $\dot{x}_2 = -kx_1/m$, or $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}) = \langle x_2, -kx_1/m\rangle$.*

(b) *Compute $\nabla E = \langle ku_1, mu_2\rangle$ and then $\nabla E(\mathbf{u}) \cdot \mathbf{f}(\mathbf{u}) = 0$ for any $\mathbf{u}$.*

(c) *The solution trajectories are level curves $E(x_1, x_2) = c$ for some constant c, or $\frac{k}{2}x_1^2 + \frac{m}{2}x_2^2 = c$. This is the equation of an ellipse in the x_1x_2 phase plane. It indicates that (given $x_1 = x, x_2 = \dot{x}$) the solutions are periodic without decay.*

Reading Exercise Solution 7.6.5.

(a) *See the left panel in Figure 7.17.*

(b) *See the right panel in Figure 7.17.*

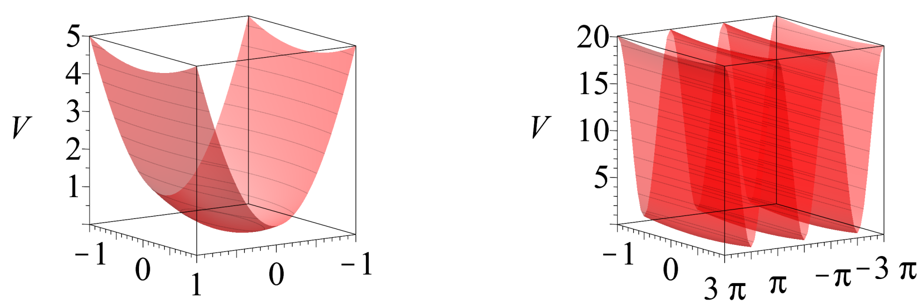

Figure 7.17: left panel: Function $V(\mathbf{u}) = mgL(1 - \cos(u_1)) + \frac{1}{2}mL^2u_2^2$ with $m = L = 1$, $g = 9.8$, on region $-1 \leq u_1, u_2 \leq 1$. Right panel: same, on region $-3\pi < u_1 < 3\pi$, $-1 < u_2 < 1$.

Reading Exercise Solution 7.6.6. *The right side of the $\dot{x}_1$ equation is the model we've seen before for logistic growth of a prey species $rx_1(K - x_1)/K$ and the effect of predation by the second species, $-rax_1x_2/K$ (jointly proportional to x_1 and x_2.) The right side of the $\dot{x}_2$ equation captures the beneficial effect that x_2 predation on x_1 has on the x_2 population (bx_1x_2), the negative effect of the x_3 predation on x_2 ($-cx_2x_3$), and the exponential decay of the x_2 species ($-dx_2$). Finally, the $\dot{x}_3$ equation captures the sustenance this population gains from eating the x_2 species (ex_2x_3), and the exponential decay of the x_3 species in the absence of food ($-fx_3$).*

Reading Exercise Solution 7.6.7. *The point* $(0,0,0)$ *is obviously mutual extinction. The point* $(K,0,0)$ *corresponds to the extinction of the second and third species, with the first at carrying capacity. The fixed point* $(0, f/e, -d/c)$ *is not physical since the third population is negative. The fourth point* $(d/b, (Kb-d)/(ab), 0)$ *corresponds to extinction of the third species and the coexistence of the first two species, if* $Kb - d > 0$*. Finally* $(K - af/e, f/e, (Kbe - fac - de)/(ec))$ *corresponds to the mutual coexistence of all three species, if* $K - af/e > 0$ *and* $Kbe - fac - de > 0$*.*

Reading Exercise Solution 7.6.8. *The Jacobian at* $\mathbf{x}^* = \langle 1, K-1, 0\rangle$ *is*

$$\mathbf{J}(\mathbf{x}^*) = \begin{bmatrix} -r/K & -r/K & 0 \\ K-1 & 0 & 1-K \\ 0 & 0 & K-2 \end{bmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^3 + (2 - K + r/K)\lambda^2 + (r/K)\lambda - (K-1)(K-2)r/K.$$

A bit of tedious computation (aided by software, preferably) shows that

$$\begin{aligned} D_1 &= \frac{-K^2 + 2K + r}{K} \\ D_2 &= \frac{r(K(K-2)^2 + r}{K^2} \\ D_3 &= -\frac{r^2(K-1)(K-2)(K(K-2)^2 + r}{K^3}. \end{aligned}$$

If $r > 0$ *and* $1 < K < 2$ *then (using the hint that* $-K^2 + 2K + r = r + 1 - (K-1)^2$*) we see that the denominator of* D_1 *is positive, since* $r + 1 > 1$ *and* $0 < (K-1)^2 < 1$*. Hence* D_1 *is positive. The quantity* D_2 *is always positive, and if* $1 < K < 2$ *we see that* $(K-1)(K-2) < 0$*, which makes it clear that* $D_3 > 0$*. The roots of* $p(\lambda)$ *(the eigenvalues of the Jacobian at* $\mathbf{x}^*$*) all have negative real part, and so this fixed point is asymptotically stable. Perhaps this can be interpreted that the carrying capacity for the base species in this food chain must be high enough to support the second species, but if* $K < 2$ *there is not enough of the second species to support the third.*

Reading Exercise Solution 8.1.1. *Many possibilities.*

Reading Exercise Solution 8.1.2. *The dimensions:*

- *For a pipe with water we have* $[\rho] = ML^{-1}$ *and* $[q] = MT^{-1}$.
- *For a wire with electrons we have* $[\rho] = QL^{-1}$ *and* $[q] = QT^{-1}$.
- *For a roads with cars we have* $[\rho] = L^{-1}$ *and* $[q] = T^{-1}$ *if cars are a dimensionless quantity, a simple count. Of we assign them the dimension "N"* $[\rho] = NL^{-1}$ *and* $[q] = NT^{-1}$.
- *For a river with pollutant we have* $[\rho] = ML^{-1}$ *and* $[q] = MT^{-1}$.
- *For a metal bar with thermal energy (energy has dimension* ML^2T^{-2} *we have* $[\rho] = MLT^{-2}$ *and* $[q] = ML^2T^{-3}$.

Reading Exercise Solution 8.1.3. *With 19 people in the building at time* $t = t_0$*, if* 14 *enter and* 7 *exit over the next hour, there should be* $19+14-7 = 26$ *people in the building. But this assumes that people are conserved. If there are* 27 *people at the end of the hour, it means there was a net creation of one person during that hour—maybe it's a hospital and someone was born. Or if there are* 25 *people, someone died. Again, maybe it's a hospital.*

Reading Exercise Solution 8.1.4. *This is a routine differentiation. For the graphs see Figure 8.18.*

Reading Exercise Solution 8.1.5. *That* $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ *is a routine computation. It's also easy to see that* $u(x,0) = \sin(\pi x)$*, as well as* $u(0,t) = 0$ *and* $u(1,t) = 0$ *for* $t > 0$*. The solution for* $t > 0$ *is just a scaled down version (vertically) of* $\sin(\pi x)$*, and seems perfectly reasonable—the heat energy flows out of the ends of the bar.*

Reading Exercise Solution 8.1.6. *Similar to the previous Reading Exercise 8.1.5. The condition* $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ *is a routine computation. It's also easy to see that since* $\frac{\partial u}{\partial x}(x,t) = -\pi e^{-\pi^2 t}\sin(\pi x)$ *we have* $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$*, and clearly* $u(x,0) = 1 + \cos(\pi x)$*. The solution seems perfectly reasonable—the heat energy can't flow out the ends of the bar, so the solution (temperature) settles to the constant value* 1*.*

Reading Exercise Solution 8.1.7. *For* $u(x,t) = e^{-\alpha\lambda^2 t}\sin(\lambda x)$ *compute*

$$\frac{\partial u}{\partial t} = -\alpha\lambda^2 e^{-\alpha\lambda^2 t}\sin(\lambda x)$$
$$\frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\alpha\lambda^2 t}\sin(\lambda x).$$

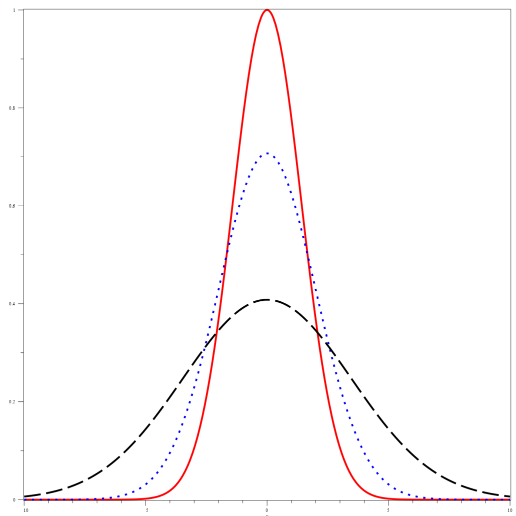

Figure 8.18: Solution $u(x,t) = \frac{e^{-\frac{x^2}{4(t+1)}}}{\sqrt{t+1}}$ to heat equation with $\alpha = 1$ at times $t = 0$ (solid/red), $t = 1$ (dotted/blue) and $t = 5$ (dashed/black).

Then it's easy to see that $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$. *Virtually the same computation works for* $v(x,t)$.

Reading Exercise Solution 8.1.8. *If* $\gamma = 0$ *then the expression* T *becomes just* $T(t) = C$, *a constant (the actual ODE for* T *is just* $T'(t) = 0$*). The ODE for* $X(x)$ *becomes* $X''(x) = 0$ *with general solution* $X(x) = c_1 x + c_2$. *Then* $u(x,t) = T(t)X(x) = C_1 x + C_2$ *(with* $C_1 = Cc_1, C_2 = Cc_2$*) is the solution to the heat equation.*

Reading Exercise Solution 8.1.9. *Start with*

$$u(x,t) = C_1 e^{-\alpha\lambda^2 t}\sin(\lambda x) + C_2 e^{-\alpha\lambda^2 t}\cos(\lambda x).$$

Then

$$\frac{\partial u}{\partial x}(x,t) = C_1 \lambda e^{-\alpha\lambda^2 t}\cos(\lambda x) - C_2 \lambda e^{-\alpha\lambda^2 t}\sin(\lambda x).$$

The condition $\frac{\partial u}{\partial x}(0,t) = C_1 \lambda e^{-\alpha\lambda^2 t} = 0$ *forces* $C_1 = 0$ *or* $\lambda = 0$. *If* $\lambda = 0$ *then* $u(x,t) = C_2$, *a constant. If* $\lambda \neq 0$ *then we must have* $C_1 = 0$, *so*

$$u(x,t) = C_2 e^{-\alpha\lambda^2 t}\cos(\lambda x)$$

and

$$\frac{\partial u}{\partial x}(x,t) = -C_2 \lambda e^{-\alpha\lambda^2 t}\sin(\lambda x).$$

The condition that $\frac{\partial u}{\partial x}(L,t) = 0$ *means that* $-C_2\sin(\lambda L) = 0$. *Taking* $C_2 = 0$ *leads to* $u(x,t) = 0$, *which is of no value. We thus need* $\sin(\lambda L) = 0$, *which (as in the Dirichlet data case) leads to* $\lambda = \pi j/L$ *for some integer* j.

Reading Exercise Solution 8.2.1. *From the identity* $\cos(x)\cos(y) = \frac{\cos(x+y)+\cos(x-y)}{2}$ *we find*

$$\begin{aligned}
\int_0^L \cos(j\pi x/L)\cos(k\pi x/L)\,dx &= \int_0^L \left(\frac{\cos(\pi(j+k)x/L) + \cos(\pi(j-k)x/L)}{2}\right)dx \\
&= \left(\frac{L\sin(\pi(j+k)x/L)}{2\pi(j+k)} + \frac{L\sin(\pi(j-k)x/L)}{2\pi(j-k)}\right)\Big|_{x=0}^{x=L} \\
&= \frac{L\sin(\pi(j+k))}{2\pi(j+k)} + \frac{L\sin(\pi(j-k))}{2\pi(j-k)} \\
&+ \frac{L\sin(0)}{2\pi(j+k)} + \frac{L\sin(0)}{2\pi(j-k)} \\
&= 0
\end{aligned}$$

since j *and* k *are distinct integers (so* $j-k \neq 0$, *and note that* $j+k > 0$*).*

If $j = k$ *then we can use the identity with* $x = y$, *in which case this identity becomes* $\cos^2(y) = 1/2 + \cos(2y)/2$. *Then*

$$\begin{aligned}
\int_0^L \cos^2(j\pi x/L)\,dx &= \int_0^L (1/2 + \cos(2j\pi x/L)/2)\,dx \\
&= \left(\frac{x}{2} + \frac{L\sin(2j\pi x/L)}{2j\pi}\right)\Big|_{x=0}^{x=L} \\
&= \frac{L}{2}
\end{aligned}$$

since $\sin(2j\pi) = 0$ *(and* $j > 0$, *so no division by zero).*

Reading Exercise Solution 8.2.2. *Here* $s_0(x) = s_1(x) = 0$, *while* $s_2(x) = s_3(x) = s_4(x) = f(x) = \cos(2\pi x)$. *Once* $n = 2$ *the approximation is exact and increasing* n *cannot improve the approximation.*

Reading Exercise Solution 8.2.3. *Compute*

$$\begin{aligned}
a_0 &= 2\int_0^1 x\,dx = 1 \\
a_k &= 2\int_0^1 x\cos(k\pi x)\,dx = 2\left(\frac{x\sin(k\pi x)}{k\pi}\Big|_{x=0}^1 - \int_0^1 \frac{\sin(k\pi x)}{k\pi}\,dx\right) \\
&= -2\frac{\cos(k\pi x)}{k^2\pi^2}\Big|_{x=0}^1 \\
&= 2\frac{1-(-1)^k}{k^2\pi^2}
\end{aligned}$$

(integrate by parts for a_k). It's easy to see that $a_k = 0$ if k is even and $a_k = 4/(k^2\pi^2)$ if k is odd. Then the Fourier cosine series for f yields

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1,\ odd}^{\infty} \frac{\cos(k\pi x)}{k^2}.$$

Using $x = 0$ on the right yields value $1/2$, since $\cos(k\pi/2) = 0$ when k is odd. With $x = 0$ we obtain (using $f(0) = 0$ and pointwise convergence)

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1,\ odd}^{\infty} \frac{1}{k^2}.$$

Subtract $1/2$ from both sides and multiply by $\pi^2/4$ to find

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Reading Exercise Solution 8.3.1. *Now the solution becomes*

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4(1-(-1)^k)}{k^3\pi^3} e^{-2k^2\pi^2 t}\sin(k\pi x) = \sum_{k=1,odd}^{\infty} \frac{8}{k^3\pi^3} e^{-2k^2\pi^2 t}\sin(k\pi x).$$

In effect, time moves twice as fast when $\alpha = 2$. A plot is shown in Figure 8.20.

Reading Exercise Solution 8.3.2. *From*

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-\alpha k\pi^2 t/L^2}\cos(k\pi x/L)$$

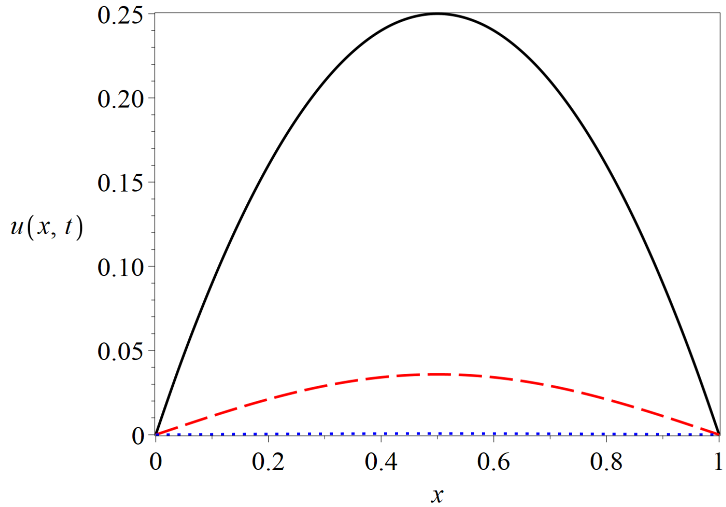

Figure 8.19: Solution $u(x,t)$ to heat equation with diffusivity $\alpha = 2$ on interval $0 \leq x \leq 1$, boundary conditions $u(0,t) = u(1,t) = 0$ and initial data $u(x,0) = x(1-x)$. Solution u at time $t = 0$ shown as solid black graph, $t = 0.1$ as dashed red graph, $t = 0.3$ as dotted blue graph.

it's clear that the solution decays to $a_0/2$*, since all terms in the summation decay rapidly to zero. But from the formula for the* a_k *we have*

$$\frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx.$$

Reading Exercise Solution 8.3.3. *Use* $\sin(y)\sin(z) = (\cos(y-z) - \cos(y+z))/2$ *with* $y = \pi(j+1/2)x/L$ *and* $z = \pi(k+1/2)x/L$ *to write*

$$\sin\left(\frac{\pi(j+1/2)x}{L}\right)\sin\left(\frac{\pi(k+1/2)x}{L}\right) = \frac{1}{2}\cos\left(\frac{\pi(j-k)x}{L}\right) - \frac{1}{2}\cos\left(\frac{\pi(j+k)x}{L}\right).$$

Then

$$\begin{aligned}&\int_0^L \sin\left(\frac{\pi(j+1/2)x}{L}\right)\sin\left(\frac{\pi(k+1/2)x}{L}\right)\,dx\\ &= \frac{1}{2}\int_0^L \cos\left(\frac{\pi(j-k)x}{L}\right)\,dx - \frac{1}{2}\int_0^L \cos\left(\frac{\pi(j+k)x}{L}\right)\,dx\\ &= \left.\frac{\sin(\pi(j-k)x/L)}{2\pi(j-k)/L}\right|_{x=0}^{x=L} - \left.\frac{\sin(\pi(j+k)x/L)}{2\pi(j+k)/L}\right|_{x=0}^{x=L}.\end{aligned}$$

All function evaluations in this last line are of the form $\sin(m\pi)$ *for some integer* m*, and so all are zero. Note that* $j-k$ *and* $j+k$ *are never zero too.*

Also, from $\sin^2(y) = 1/2 - \cos(2y)/2$ *with* $y = \pi(j+1/2)x/L$ *we have*

$$\sin^2\left(\frac{\pi(j+1/2)x}{L}\right) = \frac{1}{2} + \frac{\cos((2j+1)\pi x)}{2L}.$$

Then

$$\begin{aligned}\int_0^L \sin^2\left(\frac{\pi(j+1/2)x}{L}\right)\,dx &= \int_0^L \left(\frac{1}{2} + \frac{\cos((2j+1)\pi x)}{2L}\right)dx \\ &= \left(\frac{x}{2} + \frac{\sin((2j+1)\pi x)}{4\pi(j+1/2)L}\right)\Big|_{x=0}^{x=L} \\ &= \frac{L}{2}\end{aligned}$$

since all the $\sin$ *evaluations are at a multiple of* π*.*

Reading Exercise Solution 8.3.4. *The coefficients turn out to be* $c_j = 8(-1)^j/(\pi^2(2j+1)^2)$ *so*

$$s_n(x) = \sum_{j=0}^{n} \frac{8(-1)^j \sin((j+1/2)\pi x/2)}{\pi^2(2j+1)^2}.$$

The graph of $s_n(x)$ *for* $n = 0, 5, 10$ *is shown in Figure* **??**.

Reading Exercise Solution 8.3.5. *If* $x = 1/2$ *then* $\cos(k\pi x) = \cos(k\pi/2) = 0$ *when* k *is odd and* $\cos(k\pi x) = \cos(k\pi/2) = (-1)^{k/2}$ *when* k *is even. Then*

$$s_n(1/2) = 1 + 2\sum_{k=2,\ even}^{n} (-1)^{k/2}(-1)^{k/2} = 1 + 2\sum_{k=2,\ even}^{n} 1 = n+1$$

and $s_n(1/2) = n$ *when* n *is odd. For example,* $s_6(1/2) = 1+2(1+1+1) = 7$*, and* $s_5(1/2) = 1+2(1+1) = 5$*. The quantity* $s_n(1/2)$ *is the height of the delta function that develops as* n *increases, so this makes perfect sense.*

Reading Exercise Solution 8.3.6. *If* ρ *has dimensions of mass per length* (ML^{-1}) *then from the analysis of Section 1.5,* $\partial\rho/\partial t$ *has dimension* $ML^{-1}T^{-1}$*. Similarly if* q *has the dimension mass per time* (MT^{-1}) *then* $\partial q/\partial x$ *has dimension* $ML^{-1}T^{-1}$*. And it is given that* r *has the dimension of* $ML^{-1}T^{-1}$*.*

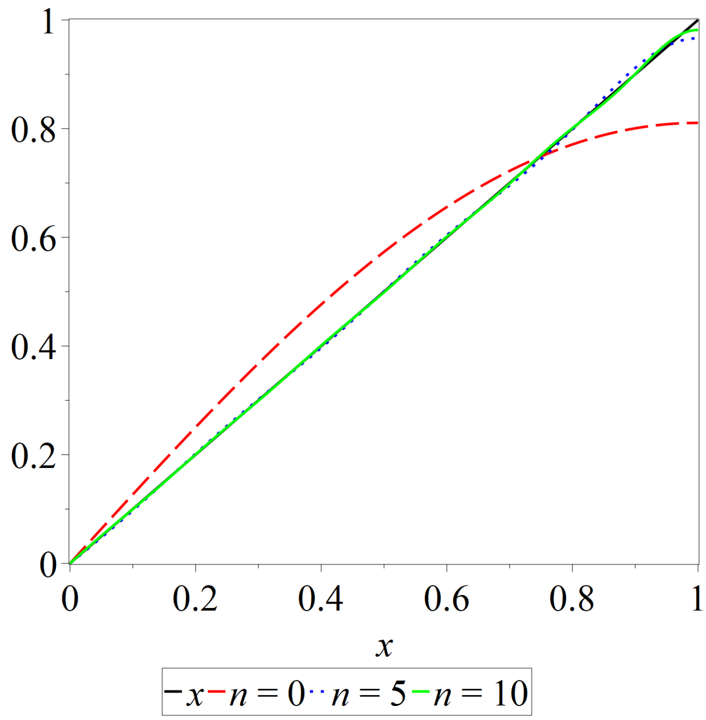

Figure 8.20: Function $f(x) = x$ and approximations $s_0(x)$, $s_5(x)$, and $s_{10}(x)$ on $0 \leq x \leq 2$.

Reading Exercise Solution 8.3.7. *The total amount of pollutant mass in the bar at time $t = T$ should be the total amount present at time $t = 0$ plus the total amount introduced by the source term $r(x, t)$ because none can enter or exit at the boundaries. Since f is the initial density, the integral*

$$\int_0^L f(x)\,dx$$

simply tallies the total amount of stuff present at time $t = 0$. The integral

$$\int_0^L r(x,t)\,dx$$

(integrating only in x) is the rate at which stuff is being created in the pipe or bar at time t, so the double integral

$$\int_0^T \int_0^L r(x,t)\,dx\,dt$$

tallies up the total amount of stuff created from $t = 0$ *to* $t = T$. *In this example we have*

$$\int_0^L f(x)\,dx = \int_0^2 x^2(2-x)^2\,dx = 16/15$$

and

$$\int_0^T \int_0^L r(x,t)\,dx\,dt = \int_0^5 \int_0^2 r(x,t)\,dx\,dt = 2(1-e^{-5}) \approx 1.987.$$

By time $t = 5$ *the stuff has diffused to a mostly constant concentration, which in a conduit of length 2 yields a concentration of* $(16/15 + 1.987)/2 \approx 1.527$. *That's what the graph shows.*

Reading Exercise Solution 8.4.1.

(a) *In the figure, consider the fluid at position* $x = x_0 - c\Delta t$ *at time* $t = t_0$. *Over the next* Δt *seconds this fluid will move to the right a distance of* $c\Delta t$ *(rate multiplied by time) and end up at position* $x = x_0$, *ready to cross this* x *coordinate. All the fluid that was in the region* $x_0 - c\Delta t < x < x_0$ *did cross* $x = x_0$ *in this* Δt *time interval. Thus total amount of fluid that flowed past* $x = x_0$ *is the amount that was in the interval* $x_0 - c\Delta \leq x_0$ *at time* t_0, *and this is (since* ρ *is the density of the pollutant in the fluid)*

$$\int_{x_0 - c\Delta t}^{x_0} \rho(x,t_0)\,dx.$$

(b) *The integral in part (a) when divided by* Δt *yields the average rate at which fluid flowed past* $x = x_0$ *in the time interval* $t_0 \leq t \leq t_0 + \Delta t$, *so the instantaneous rate is*

$$q(x_0,t_0) = \lim_{\Delta t \to 0} \left(\int_{x_0 - c\Delta t}^{x_0} \rho(x,t_0)\,dx \right).$$

(c) *Take the hint and let* P *be an antiderivative for* $\rho(x,t)$ *with respect to* x. *Then from part (b) we have*

$$q(x_0,t_0) = c \lim_{\Delta t \to 0} \left(\frac{P(x_0,t_0) - P(x_0 - c\Delta t, t_0)}{c\Delta t} \right).$$

The limit on the right is $\frac{\partial P}{\partial x}(x_0,t_0)$, *from the very definition of the derivative. But* $\frac{\partial P}{\partial x}(x_0,t_0) = \rho(x_0,t_0)$, *so we have shown that* $q = c\rho$.

(d) *The flux* q *has dimension* MT^{-1} *(mass per time),* ρ *has dimension* ML^{-1} *(mass per length) and* c *has the dimension velocity,* LT^{-1}*. Note that the dimension of* $c\rho$ *is thus* $(ML^{-1})(LT^{-1}) = MT^{-1}$*, the same as* q*.*

Reading Exercise Solution 8.4.2. *Start with* $\rho(x,t) = f(x - ct)$*. By the chain rule we have*

$$\frac{\partial \rho}{\partial t} = -cf'(x - ct) \quad \textit{and} \quad \frac{\partial \rho}{\partial x} = f'(x - ct).$$

Then $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = -cf'(x-ct) + cf'(x-ct) = 0$ *as advertised. Also,* $\rho(x,0) = f(x - c0) = f(x)$*.*

Reading Exercise Solution 8.4.3. *From the solution formula for the advection equation we find that* $\rho(6,3) = f(6-3\cdot3) = f(-3) = e^{-9}$*. This point lies on the characteristic* $x - 3t = -3$ *since* $(6) - (3)(3) = -3$*. This line intersects the* x *axis at* $x = -3$ *(when* $t = 0$*) and here* $\rho(-3,0) = f(-3) = e^{-9}$*. The function* ρ *equals* e^{-9} *all along this characteristic.*

Reading Exercise Solution 8.4.4. *If* $c = 0$ *the characteristic curves would be of the form* $x = x_0$*, vertical lines. Given that* ρ *is constant on such a curve, this says that the value of* ρ *at any time* $t > 0$ *and at a point* $x = x_0$ *is equal to* $f(x_0)$*, whatever the initial concentration was at time* $t = 0$*. Nothing is moving.*

If $c < 0$ *then the characteristic curves are lines sloping to the left instead of the right.*

Reading Exercise Solution 8.4.5. *In each case compute*

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)(x) &= \frac{\partial x}{\partial t} + 3\frac{\partial x}{\partial x} = 3 \\
\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)(xe^t) &= \frac{\partial (xe^t)}{\partial t} + 3\frac{\partial (xe^t)}{\partial x} = xe^t + e^t \\
\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)((x-3t)^2) &= \frac{\partial (x-3t)^2}{\partial t} + 3\frac{\partial (x-3t)^2}{\partial x} \\
&= 2(x-3t)(-3) + (3)2(x-3t) = 0.
\end{aligned}$$

Reading Exercise Solution 8.4.6. *It can be helpful to put an unspecified function into the computation, something for the differential operators to* *act upon. In this case we'll use $u(t)$. After doing this just "FOIL" the composition as*

$$\begin{aligned}\left(\frac{d}{dt}+2I\right)\left(\frac{d}{dt}+I\right)(u(t)) &= \left(\frac{d}{dt}+2I\right)\left(\frac{du}{dt}+u\right)\\ &= \frac{d}{dt}\left(\frac{du}{dt}+u\right)+2I\left(\frac{du}{dt}+u\right)\\ &= \frac{d^2u}{dt^2}+\frac{du}{dt}+2\frac{du}{dt}+2u\\ &= \frac{d^2u}{dt^2}+3\frac{du}{dt}+2u.\end{aligned}$$

Taking u out of the computation after the fact yields $\left(\frac{d}{dt}+2I\right)\left(\frac{d}{dt}+I\right)=\frac{d^2}{dt^2}+3\frac{d}{dt}+2I.$

The same computation works if we reverse the order of the differential operators.

Since $\left(\frac{d}{dt}+I\right)(e^{-t})=-e^{-t}+e^{-t}=0$, *it follows that*

$$\begin{aligned}\left(\frac{d}{dt}+2I\right)\left(\frac{d}{dt}+I\right)(e^{-t}) &= \left(\frac{d}{dt}+2I\right)(0)\\ &= 0.\end{aligned}$$

But this says that $(\frac{d^2}{dt^2}+3\frac{d}{dt}+2I)(e^{-t})=0$, *that is,* $u(t)=e^{-t}$ *satisfies the ODE* $d^2u/dt^2+3du/dt+2u=0$. *The same conclusion holds for* $u(t)=e^{-2t}$ *if we apply the differential operators in the order* $\left(\frac{d}{dt}+I\right)\left(\frac{d}{dt}+2I\right)$.

Reading Exercise Solution 8.4.7. *We are given* $\rho(x,t)=f(x-ct)+g(x+ct)$. *Compute*

$$\begin{aligned}\frac{\partial^2\rho}{\partial t^2} &= c^2f''(x-ct)+c^2g''(x+ct)\\ \frac{\partial^2\rho}{\partial x^2} &= f''(x-ct)+g''(x+ct).\end{aligned}$$

It's easy to see that for any f and g (twice-differentiable, anyway) we have $\frac{\partial^2\rho}{\partial t^2}-c^2\frac{\partial^2\rho}{\partial x^2}=0$.

Reading Exercise Solution 8.4.8. *The dimension of the tension is* $MLT-2$ *(here* T *is time, not tension!) and the dimension of* λ *is* ML^{-1}*. Then the tension divided by* λ *has dimension* L^2T^{-2}*. Since* c *is the square root,* c *has the dimension* LT^{-1}*.*

Reading Exercise Solution 8.4.9. *The fundamental frequency of the guitar string stems from the first terms in the wave equation solution and these involve* $\cos(ck\pi t/L)$ *and* $\sin(ck\pi t/L)$ *with* $k = 1$*. With* $c = \sqrt{T/\lambda}$ *the corresponding radial frequency is* $\omega = \frac{\pi\sqrt{T/\lambda}}{L}$*, which is a frequency of* $f = \frac{\sqrt{T/\lambda}}{2L}$ *Hz (just* $\omega/(2\pi)$*). We want* $f = 110$*, so substitute* $L = 0.66$ *and* $\lambda = 3.5 \times 10^{-3}$ *into* $f = 110$ *and solve for* T *to find* $T = 73.79$ *newtons.*

Reading Exercise Solution 8.4.10. *If* c *is close to zero then the backward light cone becomes narrower, and encompasses a smaller portion of the* x *axis. In the extreme case that* $c = 0$ *the backward light cone would become a vertical line. Conversely, if* c *is large the light cone spreads out, and as* $c \to \infty$ *the cone would become a half-space that encompasses the entire* x *axis. This means that anything that happens at time* $t = 0$ *could affect future events at any* x *coordinate and any time* $t > 0$*, since information propagates arbitrarily rapidly.*

Appendix A

Reading Exercise Solution A.1.1. $\mathrm{Re}(z) = -3$, $\mathrm{Im}(z) = 5$, $\overline{z} = -3 - 5i$, *and* $|z| = \sqrt{34}$.

Reading Exercise Solution A.2.1. $z+w = 3+i$, $z-w = 1-3i$, $zw = 4+i$, *and* $-i$.

Reading Exercise Solution A.3.1. $e^{i\pi} = -1$, $e^{i\pi/2} = i$, *and* $e^{i\pi/4} = \sqrt{2}/2 + i\sqrt{2}/2$.

Reading Exercise Solution A.4.1. *With* $u = z^2$ *the quartic (fourth degree) polynomial becomes* $u^2 + 5u + 6 = 0$*, with solutions* $u = -2$ *and* $u = -3$*. Then* $z^2 = u$ *gives* $z = i\sqrt{2}, -i\sqrt{2}, i\sqrt{3}$*, and* $-i\sqrt{3}$ *as solutions to the quartic equation.*

Reading Exercise Solution A.4.2. *If* $z = 1 - i$ *and* $w = 3 + i$ *then* $\overline{z} = 1 + i$, $\overline{w}3 - i$*, and then* $\overline{zw} = (\overline{z})(\overline{w}) = 4 + 2i$, $\overline{z/w} = \overline{z}/\overline{w} = 1/5 + 2i/5$*, and* $\overline{e^z} = e^{\overline{z}} = e^{1+i}$*.*

Appendix B

Reading Exercise Solution B.1.1.

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} -18 \\ -6 \end{bmatrix}.$$

Reading Exercise Solution B.1.2. *Here*

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & -1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}.$$

Then

$$\mathbf{A} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \mathbf{b}.$$

Reading Exercise Solution B.1.3.

(a) *Here*

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Then

$$\mathbf{A} \begin{bmatrix} 3+2t \\ t \end{bmatrix} = \mathbf{b}$$

for any choice of t.

(b) *Here*

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

But subtracting twice $x_1 - 2x_2 = 3$ from $2x_1 - 4x_2 = 5$ leads to $0 = -1$, so the equations are inconsistent (we don't even need the matrix formulation)

Reading Exercise Solution B.2.1. *Compute*

$$\mathbf{B}\mathbf{A} = \begin{bmatrix} 3 & -3 \\ 13 & 22 \end{bmatrix}.$$

Reading Exercise Solution B.2.2. *We find*

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{B})\mathbf{C} = \begin{bmatrix} -18 & 12 \\ 39 & 9 \end{bmatrix}.$$

Reading Exercise Solution B.2.3. *They are*

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Both $\mathbf{I}_2\mathbf{C}$ *and* $\mathbf{C}\mathbf{I}_2$ *equal* $\mathbf{C}$*, of course.*

Reading Exercise Solution B.2.4. *The inverse is*

$$\mathbf{A}^{-1} = \begin{bmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{bmatrix}.$$

It is straightforward to check that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$*, and also that* $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Reading Exercise Solution B.2.5. *If we write out*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we find that we need $a + 2b = 1$ *and* $2a + 4b = 0$*, which are inconsistent. (We also require* $c + 2d = 0$ *and* $2c + 4d = 1$*, also inconsistent.)*

Reading Exercise Solution B.2.6. *The determinants are, from left to right,* $-11, 19, -32$*, and* 1*, respectively. The determinant of the* 4×4 *matrix is* $(1)(-11) - (2)(19) + (4)(-32) - (3)(1) = -180$.

Reading Exercise Solution B.2.7. *The computation is*

$$\begin{aligned} \left| \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right| &= a \left| \begin{bmatrix} e & f \\ h & i \end{bmatrix} \right| - (b) \left| \begin{bmatrix} d & f \\ g & i \end{bmatrix} \right| + (c) \left| \begin{bmatrix} d & e \\ g & h \end{bmatrix} \right| \\ &= (a)(ei - fh) - (b)(di - fg) + (c)(dh - eg) \\ &= aei - afh + bfg - bdi + cdh - ceg. \end{aligned}$$

Reading Exercise Solution B.2.8. *The determinant of* $\mathbf{A}$ *will be* $A_{1,1}$ *times the determinant of the* $(n-1)\times(n-1)$ *diagonal matrix with diagonal entries* $A_{i,i}$ *for* $2 \le i \le n$*. This* $(n-1)\times(n-1)$ *matrix has determinant* $A_{2,2}$ *times the determinant of the* $(n-2)\times(n-2)$ *diagonal matrix with diagonal entries* $A_{i,i}$*,* $3 \le i \le n$*. The pattern is the simple; an induction shows that the determinant of* $\mathbf{A}$ *is* $A_{1,1}A_{2,2}\cdots A_{n,n}$*.*

Reading Exercise Solution B.3.1. *Compute*

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

so that $\mathbf{v}_1 = \langle 1, 2\rangle$ *is an eigenvector for* $\mathbf{A}$ *with eigenvalue* $\lambda_1 = 1$*. Also*

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

so $\mathbf{v}_2 = \langle 1, 3\rangle$ *is an eigenvector for* $\mathbf{A}$ *with eigenvalue* 3*.*

Reading Exercise Solution B.3.2. *We have*

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Then

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

If $\mathbf{v}_1 = \langle x, y\rangle$ *then* $(\mathbf{A} - 4\mathbf{I})\mathbf{v}_1 = \mathbf{0}$ *leads to equations* $-3x + 3y = 0$ *and* $3x - 3y = 0$*. Any choice* $x = y$ *will work, e.g.,* $x = y = 1$*. An eigenvector for the eigenvalue* $\lambda = 4$ *is then* $\mathbf{v}_1 = \langle 1, 1\rangle$*. Any nonzero multiple will also work.*

Reading Exercise Solution B.3.3. *Form*

$$\mathbf{A} - (1-2i)\mathbf{I} = \begin{bmatrix} -4+2i & 2 \\ -10 & 4-2i \end{bmatrix}.$$

If $\mathbf{v} = \langle x, y\rangle$ *then* $(\mathbf{A} - (1-2i)\mathbf{I})\mathbf{v} = \mathbf{0}$ *leads to equations* $(-4+2i)x + 2y = 0$ *and* $-10x + (4-2i)y = 0$*. These equations are dependent—multiplying the first by* $2+i$ *yields the second. Thus either equation can be used to determine an eigenvector. We can take* $x = 1$ *in the first equation and solve for* $y = 2-i$*. An eigenvector with eigenvalue* $1-2i$ *is thus* $\mathbf{v} = \langle 1, 2-i\rangle$*.*

Reading Exercise Solution B.3.4. *Compute*

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

with determinant $(\lambda - 2)^2$. *Of course* $\mathbf{Av} = 2\mathbf{Iv} = 2\mathbf{v}$ *for any* $\mathbf{v}$, *so all vectors are eigenvectors for* $\mathbf{A}$, *with eigenvalue* 2.

Reading Exercise Solution B.3.5. *Compute*

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

with determinant $(\lambda - 2)^2$, *which is the characteristic polynomial. If we seek an eigenvector* $\mathbf{v}$, *first form*

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

With $\mathbf{v} = \langle x, y\rangle$ *the equation* $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$ *leads to* $y = 0$ *and the only condition on* $\mathbf{v}$. *So anything of the form* $\mathbf{v} = \langle x, 0\rangle$ *with* $x \neq 0$ *is an eigenvector for* $\mathbf{A}$. *All such vectors are scalar multiples of* $\langle 1, 0\rangle$.

Reading Exercise Solution B.3.6. *Such a vector* $\mathbf{v}$ *would satisfy* $\mathbf{Av} = 0\mathbf{v} = \mathbf{0}$. *But according to the cited theorem, this means* $\mathbf{A}$ *cannot be invertible.*

Conversely, according to the theorem, if $\mathbf{A}$ *is not invertible then there is a nonzero vector* $\mathbf{v}$ *such that* $\mathbf{Av} = \mathbf{0}$. *Such a* $\mathbf{v}$ *is an eigenvector for* $\mathbf{A}$ *with eigenvalue* 0 *(by definition).*

Reading Exercise Solution B.3.7. *When*

$$\mathbf{A} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

we can compute

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} a-\lambda & b \\ 0 & d-\lambda \end{bmatrix}$$

which has determinant $(\lambda - a)(\lambda - d)$. *This makes it easy to see that the eigenvalues (the roots of* $(\lambda - a)(\lambda - d) = 0$*) are precisely* $\lambda = a$ *and* $\lambda = d$.

When

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

we can compute

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} a - \lambda & 0 \\ c & d - \lambda \end{bmatrix}$$

which also has determinant $(\lambda - a)(\lambda - d)$*, and so eigenvalues* $\lambda = a$ *and* $\lambda = d$*.*

Reading Exercise Solution B.4.1. *If* $D = 0$ *then the formula for the eigenvalues in terms of* T *and* D *yields*

$$\lambda_1 = (T + |T|)/2$$
$$\lambda_2 = (T - |T|)/2$$

using $\sqrt{T^2} = |T|$*. If* $T > 0$ *then* $|T| = T$*, so* $\lambda_1 = T$ *and* $\lambda_2 = 0$*. If* $T < 0$ *then* $|T| = -T$*, so* $\lambda_1 = 0$ *and* $\lambda_2 = T$*. If* $T = 0$ *then both eigenvalues are* 0*.*

Reading Exercise Solution B.4.2. *If* $D = T^2/4$ *then* $T^2 - 4D = 0$ *and the formula for the eigenvalues in terms of* T *and* D *yields* $\lambda_1 = \lambda_2 = T/2$*.*

Appendix C

Reading Exercise Solution C.1.1. *The ODE is*

$$0.001q''(t) + 4q'(t) + 200000q(t) = 5$$

with $q(0) = q'(0) = 0$*. The solution is*

$$q(t) = \frac{1}{40000} - \frac{1}{40000}e^{-2000t}\cos(14000t) - \frac{1}{280000}e^{-2000t}\sin(14000t).$$

The current is

$$I(t) = \frac{5}{14}e^{-2000t}\sin(14000t).$$

A plot of $I(t)$ *is shown in Figure 3.21. This system appears to be underdamped, easily confirmed by looking at the roots of the characteristic equation.*

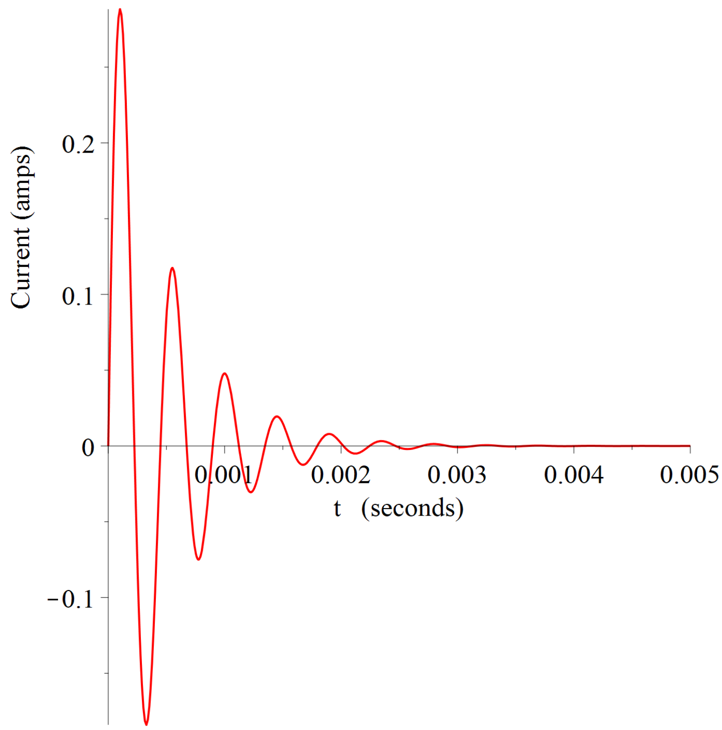

Figure 3.21: Current through RLC circuit governed by $0.001q''(t) + 4q'(t) + 200000q(t) = 5$.

Reading Exercise Solution C.1.2. *The solution is*

$$\begin{aligned} q(t) &= (-7.57 \times 10^{-7})e^{-2000t}\sin(14000t) - (5.09 \times 10^{-6})e^{-2000t}\cos(14000t) \\ &\quad + (2.08 \times 10^{-7})\sin(2000t) + (5.09 \times 10^{-6})\cos(2000t). \end{aligned}$$

Then

$$\begin{aligned} I(t) &= (0.0728)e^{-2000t}\sin(14000t) - (0.0004)e^{-2000t}\cos(14000t) \\ &\quad - (0.01)\sin(2000t) + (0.0004)\cos(2000t). \end{aligned}$$

A plot of $I(t)$ is shown in Figure 3.22. The transient has died out by about $t = 0.004$ seconds.

Reading Exercise Solution C.1.3. *The solution is shown in Figure 3.23.*

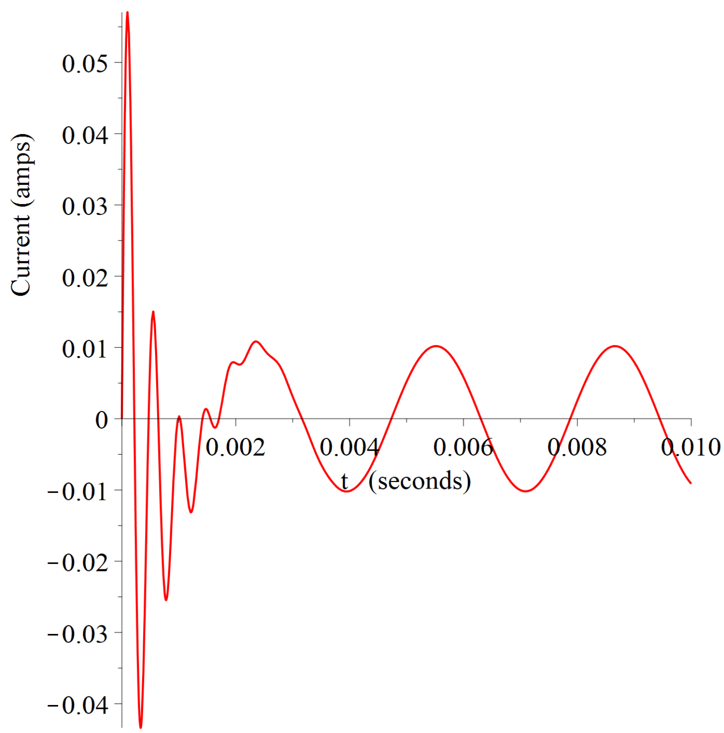

Figure 3.22: Response of RLC circuit governed by $0.001q''(t) + 4q'(t) + 200000q(t) = \cos(2000t)$.

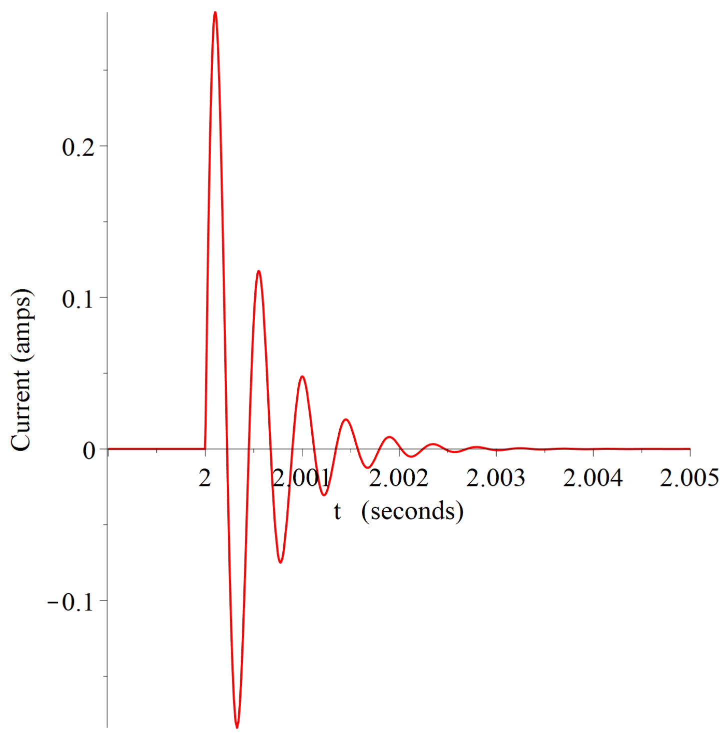

Figure 3.23: Response of RLC circuit governed by $0.001q''(t) + 4q'(t) + 200000q(t) = 5H(t-2)$.