## Instructor Notes

This worksheet is meant to act as a capstone to the differential calculus portion of a typical undergraduate Calculus I class, and as such asks the students to recall and apply a variety of differential calculus concepts including: differential equations, numerical differentiation, optimization, and limits. It is also meant to provide enough background information and guidance to act as a standalone asynchronous activity, although going through it in groups or with instructor guidance could certainly be helpful. It is expected that the students will submit a report containing all of their problem solutions, but the submission format is left up to the instructor.

One of the goals of this project is to give students experience applying calculus concepts in a way that they likely haven't had much in-class experience with. In addition to the theoretical work and hand computations that they're probably used to, this activity asks them to work with discrete data (rather than continuous functions) and to use computational technology (in the form of spreadsheet software). The data tables included with this module are packaged as spreadsheets, though text versions are also included. If you have any particularly ambitious students you might also want to show them how to perform some of these computations using a programming language like Octave[1], Python[2], or R[3].

Students should be provided with data table files as well as the handout document, which is divided into 5 parts. Part 1 is a brief introduction to the exponential growth model and its defining differential equation (the Law of Natural Growth), as well as an introduction to performing basic computations in spreadsheets. The worksheet was written with Microsoft Excel in mind, but free alternatives like Google Sheets[4] and OpenOffice Spreadsheets[5] would work just as well (the functions will be identical, but you might have to tell the students how to insert plots). Even if you plan to assign this as a worksheet to students to complete without instructor guidance, it is recommended to have everyone complete this part together in order to ensure that everyone understands how to use the computational technology that all following parts will depend on.

Parts 2 and 3 explore the process of fitting the exponential growth model to a set of noisy bacterial growth data. Part 2 focuses mostly on the transformation of the exponential growth model into a linear model using semi-log plots, while Part 3 asks the students to derive and then apply the least squares criterion to fit a linear model to their semi-log plot. The intercept is fixed at the initial population in order to leave only a single parameter, in which case the problem can be solved using only univariate optimization techniques from Calculus I.

Parts 4 and 5 explore the process of fitting the logistic growth model to an extended data set. Part 4 explores the theoretical properties of the logistic growth model and its behavior (which involves evaluating limits), while Part 5 takes the students through the process of fitting their model (which involves numerical differentiation). This part of the worksheet follows a page break, in case you would like to use Parts 1–3 alone for a shorter activity.

[1] https://www.gnu.org/software/octave/index

[2] https://www.python.org/

[3] https://www.r-project.org/about.html

[4] https://www.google.com/sheets/about/

[5] https://www.openoffice.org/

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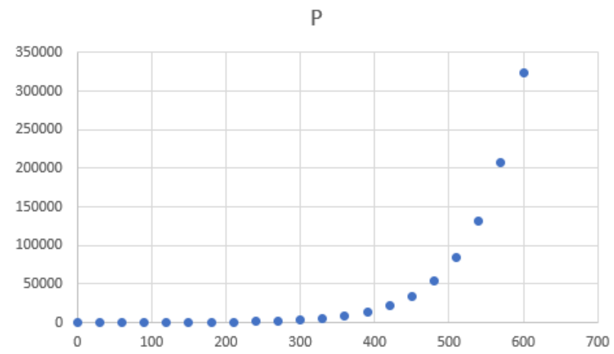
The following pages include solutions to the worksheet's activities. See the student handout for the full text of the problem statements.

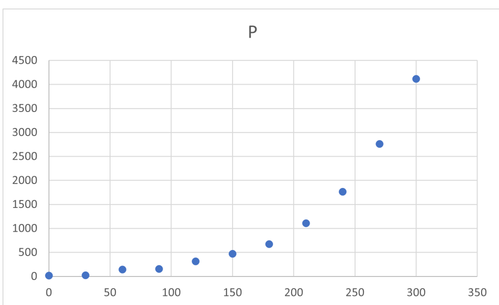
## Part 1: Exponential Growth and Computational Technology

This part is not meant to be graded. The students should apply the formula `=40*EXP(0.015*A2)` to the second column of the data table and then generate the following plot.



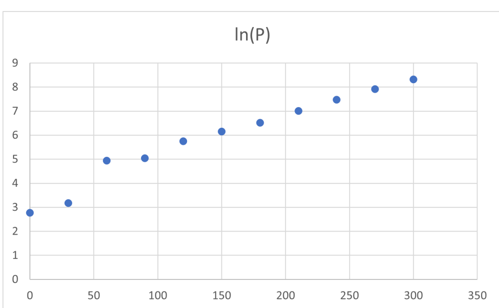

## Part 2: Preliminary Analysis and Semi-Log Plots

**2a.** The students should get the following plot.




**2b.** 0.02 or anything near it would be a reasonable guess.

**2c.** The students should get the following plot.



**2d.** The derivation is:

$$\begin{aligned}
\ln(P) &= \ln(16e^{rt}) \\
&= \ln(16) + \ln(e^{rt}) && \text{since } \ln(AB) = \ln(A) + \ln(B) \\
&= \ln(16) + rt\ln(e) && \text{since } \ln(A^C) = C\ln(A) \\
&= \ln(16) + rt && \text{since } \ln(e) = 1
\end{aligned}$$

So the slope is $r$ and the intercept is $\ln(16) \approx 2.772589$.

**2e.** 0.02 or anything near it would be a reasonable guess. It should be similar to their earlier estimate of $r$.

## Part 3: Fitting the Linear Model to the Data

**3a.** 0.02 or anything near it would be a reasonable guess. It should be similar to their earlier estimates of $r$.

**3b.** A candidate for the global minimum occurs at a critical number, so we can solve for when $\frac{dS}{dm}$ is zero.

$$\begin{aligned}
0 = \frac{dS}{dm} &= 2(mx_1 + b - y_1)x_1 + 2(mx_2 + b - y_2)x_2 + 2(mx_3 + b - y_3)x_3 \\
&= 2mx_1^2 + 2bx_1 - 2y_1x_1 + 2mx_2^2 + 2bx_2 - 2y_2x_2 + 2mx_3^2 + 2bx_3 - 2y_3x_3 \\
&= 2(x_1^2 + x_2^2 + x_3^2)m + 2(bx_1 + bx_2 + bx_3) - 2(x_1y_1 + x_2y_2 + x_3y_3)
\end{aligned}$$

$$\begin{aligned}
2(x_1y_1 + x_2y_2 + x_3y_3) - 2(bx_1 + bx_2 + bx_3) &= 2(x_1^2 + x_2^2 + x_3^2)m \\
\frac{(x_1y_1 + x_2y_2 + x_3y_3) - (bx_1 + bx_2 + bx_3)}{x_1^2 + x_2^2 + x_3^2} &= m
\end{aligned}$$

To verify that this is a local minimum we can apply the second derivative test.

$$\frac{d^2S}{dm^2} = 2x_1^2 + 2x_2^2 + 2x_3^2$$

As a sum of squared terms this is always nonnegative, meaning that $S(m)$ is always concave up, and that the critical point found earlier is a local minimum.

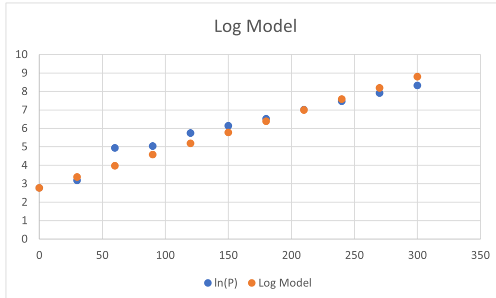
**3c.** (This question just requires following the given instructions in the spreadsheet.)

**3d.** (This question just requires following the given instructions in the spreadsheet.)

**3e.** The computed slope should be $r = 0.020082$, which should be similar to their earlier estimates of $r$.

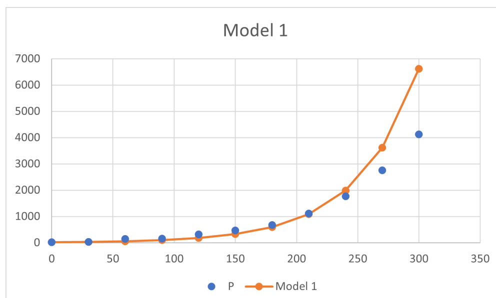
**3f.** (This question just requires following the given instructions in the spreadsheet.)

**3g.** The students should get the following plot.



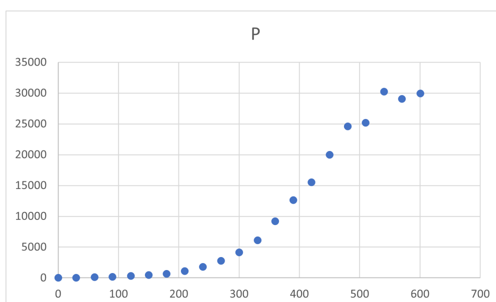

**3h.** (This question just requires following the given instructions in the spreadsheet.)

**3i.** The students should get the following plot.




## Part 4: Extending to a Logistic Growth Model

**4a.** The students should get the following plot.




**4b.** Noting that $e^{-rt} \to 0$ as $t \to \infty$, the limit is

$$\lim_{t\to\infty} \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} = \frac{KP_0}{P_0} = K$$

This indicates that the population will approach $K$ as time moves forward.

**4c.** $\frac{dP}{dt}$ is zero when $P = 0$ or $P = K$. It is positive when $0 < P < K$. It is negative when $P > K$. This implies that, as long as the initial population is positive, the population will approach $K$ as time moves forward.

**4d.** 30,000 or anything near it would be a reasonable guess.

**4e.** The derivation is:

$$\begin{aligned}\frac{dP}{dt} &= 0.02P\left(1-\frac{P}{K}\right)\\ \frac{dP/dt}{P} &= 0.02\left(1-\frac{P}{K}\right)\\ &= -\frac{0.02}{K}P+0.02\end{aligned}$$

This is a linear function of $P$ with a slope of $-\frac{0.02}{K}$ and an intercept of 0.02.

**4f.** Dividing the overall population growth rate $\frac{dP}{dt}$ by the current population $P$ gives the *per capita* growth rate, or the average growth rate per individual.

## Part 5: Fitting the Logistic Growth Model to the Data

**5a.** This question just requires following the given instructions in the spreadsheet, though there is some choice in how to compute the numerical derivatives. Forward differences ($\frac{P_{i+1}-P_i}{t_{i+1}-t_i}$) can be used for all times except the last, backward differences ($\frac{P_i-P_{i-1}}{t_i-t_{i-1}}$) can be used for all times except the first, and central differences ($\frac{P_{i+1}-P_{i-1}}{t_{i+1}-t_{i-1}}$) can be used for all times except the first and last.

**5b.** $-7\cdot 10^{-7}$ or anything near it would be a reasonable guess. It should be approximately $-0.02$ divided by their earlier estimate of $K$.

**5c.** (This question just requires following the given instructions in the spreadsheet.)

**5d.** (This question just requires following the given instructions in the spreadsheet.)

**5e.** The slope that the student obtains depends on whether they numerically approximated $\frac{dP}{dt}$ using forward, backward, or central differences. The exact values are:

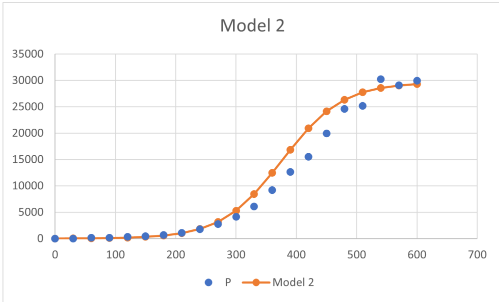
$$\begin{aligned}&-6.71683\cdot 10^{-7} && \text{from forward differences}\\ &-6.77596\cdot 10^{-7} && \text{from backward differences}\\ &-6.74639\cdot 10^{-7} && \text{from central differences}\end{aligned}$$

**5f.** The value of $K$ also depends on the numerical approximation used. The exact values are:

| | |
|---|---|
| 29,776 | from forward differences |
| 29,516 | from backward differences |
| 29,645 | from central differences |

**5g.** (This question just requires following the given instructions in the spreadsheet.)

**5h.** The students should get the following plot.




The precise plot depends on which estimate of $K$ was used, but all are similar enough that the resulting plots look nearly identical.