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TEACHER VERSION
COOL IT

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Abstract: We offer data on the temperature of water in a beaker which resides in a room of constant temperature and then in an environment of nonconstant temperature. Students are encouraged to consider both empirical and analytic modeling approaches. We offer additional data sets in Excel spreadsheets for further work.

SCENARIO DESCRIPTION

We offer you the opportunity to build mathematical models which describe the temperature of hot water in a beaker as it cools in a room environment of constant and then imn a room of nonconstant temperature. Using both empirical and analytic models you will compare your results and decide which is the best model on criteria you select.

Data collection and data set

Figure 1 shows a set up we used for measuring the changing temperature of a beaker of water in a constant temperature environment while Figure 2 shows the screen output for a typical data collection run using Vernier's LoggerPro software and a Stainless Steel Temperature Probe.

Table 1 offers the temperature of the water in the beaker at equal time intervals (sampled from a larger dataset found in 1-031-CoolItData.xls) as collected by the probe. More data from this run and from other runs are offered in the Excel file 1-031-CoolItData.xls.

Modeling opportunities

We ask you to build a mathematical model of the temperature of water in a beaker (in degrees Fahrenheit) as a function of time (in minutes), i.e. $T(t)$ is the temperature of the water in the beaker in degrees Fahrenheit and t is time in minutes.

There are several approaches we can take in building a model:

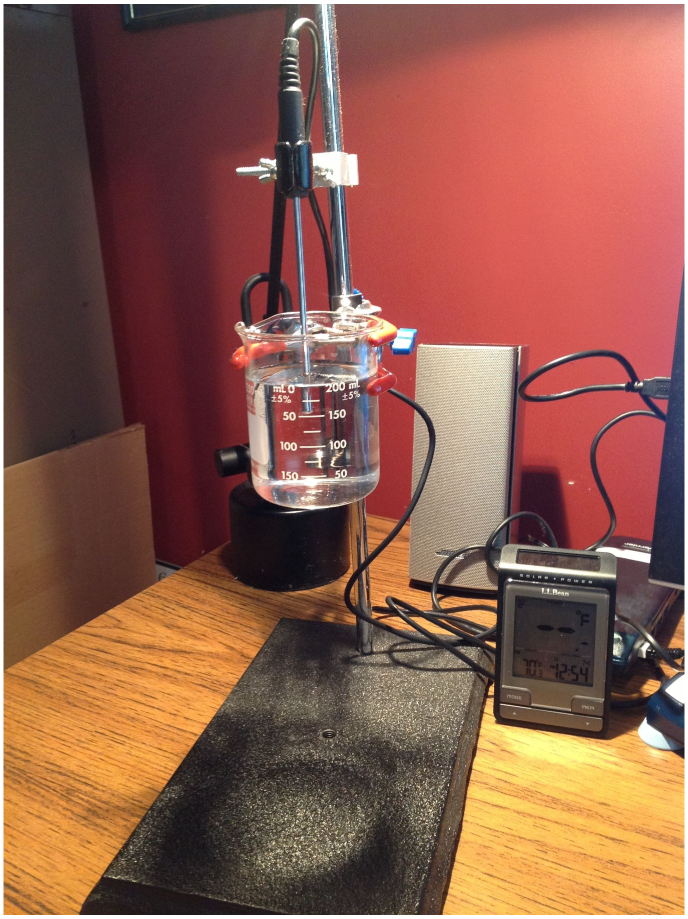


Figure 1. Shown is the apparatus used for collecting data on cooling of water in a beaker. The beaker, containing 200 mL of tap water and the Vernier stainless steel temperature probe are held in place on a stand. A room thermometer sits at the lower right for room temperature corroboration.

- *Empirical*, in which we attempt to fit the data to some mathematical function, perhaps with little or no understanding of the underlying physical phenomenon.
- *Analytic*, in which we attempt to offer assumptions about the underlying physical process which would lead to a mathematical model.

The analytic approach is more desirable, as we are attempting to use our modeling skills to better understand the phenomenon as well as mathematically predict the behavior of the phenomenon, while determining some significant physical parameters which have meaning. However, the empirical model might serve predictive purposes and interpolating values not observed would be easy from this model.

Time (min)	Temp °F	Time (min)	Temp °F
0	104.69	50	82.78
5	100.41	55	81.69
10	97.53	60	80.69
15	94.83	65	79.73
20	92.38	70	78.89
25	90.41	75	78.06
30	88.37	80	77.31
35	86.81	85	76.72
40	85.34	90	76.10
45	84.00		

Table 1. Sample of data from the temperature of water (200 mL) in a beaker where the environmental temperature is a constant 72.0 °F. Complete data set is found in the Excel file 1-031-CoolItData.xls.

Empirical modeling

Let us examine some possible empirical models (see Table 2) and ask why we might question or accept each one.

1. For each function model (in Table 2) for $T(t)$, the temperature of the water in the beaker in degrees Fahrenheit at time t in minutes, offer a critique as to why the function might be appropriate or might not be appropriate. Discuss your rationale for each of your critiques with colleagues.

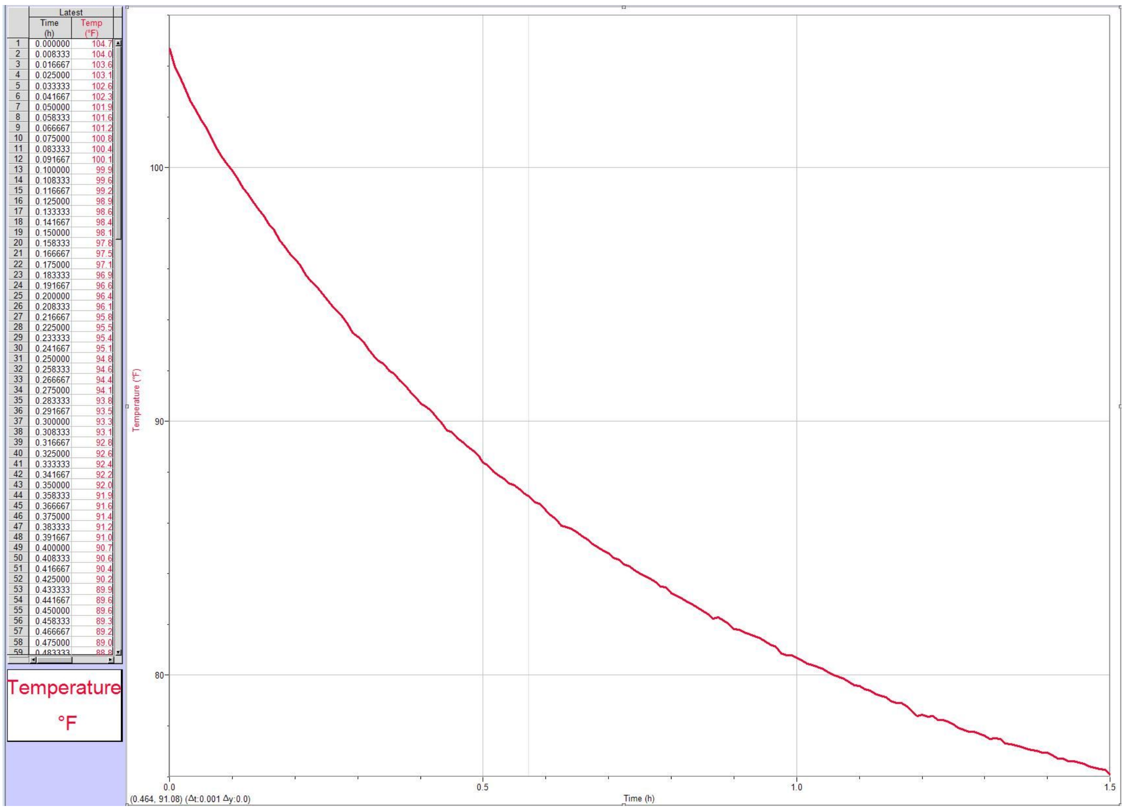
In fact, we could "slam" a polynomial through any set of data to actually "go through" each data point. Consider this fact.

> Given a set of $n+1$ observations $\mathcal{S} = \{(t_i, T_i)|$ no two t_i's are the same, $i = 1, 2, 3, \ldots, n+1\}$ there is a unique polynomial of degree n, $S_n(t)$, which passes through each point of $\mathcal{S}$, i.e. we can uniquely determine numbers $a_0, a_1, \ldots, a_n$ such that $S_n(t) = \sum_{i=0}^{n} a_i t^i = a_0 + a_1 t + \ldots a_n t^n$.

2. Explain why this fact about "slamming polynomials" of sufficiently high enough degree through any finite data set may not be good for modeling. See your second entry in Table 2, namely, $T(t) = a + bt + ct^2$, and your collective critiques.

Testing an empirical model against observed data

One needs to develop criteria for testing just how good a model is. First, one has to estimate parameters in some manner and then compare the resulting model to the data.



	Latest	
	Time (h)	Temp (°F)
1	0.000000	104.7
2	0.008333	104.0
3	0.016667	103.6
4	0.025000	103.1
5	0.033333	102.6
6	0.041667	102.3
7	0.050000	101.9
8	0.058333	101.6
9	0.066667	101.2
10	0.075000	100.8
11	0.083333	100.4
12	0.091667	100.1
13	0.100000	99.9
14	0.108333	99.6
15	0.116667	99.2
16	0.125000	98.9
17	0.133333	98.6
18	0.141667	98.4
19	0.150000	98.1
20	0.158333	97.8
21	0.166667	97.5
22	0.175000	97.1
23	0.183333	96.9
24	0.191667	96.6
25	0.200000	96.4
26	0.208333	96.1
27	0.216667	95.8
28	0.225000	95.5
29	0.233333	95.4
30	0.241667	95.1
31	0.250000	94.8
32	0.258333	94.6
33	0.266667	94.4
34	0.275000	94.1
35	0.283333	93.8
36	0.291667	93.5
37	0.300000	93.3
38	0.308333	93.1
39	0.316667	92.8
40	0.325000	92.6
41	0.333333	92.4
42	0.341667	92.2
43	0.350000	92.0
44	0.358333	91.9
45	0.366667	91.6
46	0.375000	91.4
47	0.383333	91.2
48	0.391667	91.0
49	0.400000	90.7
50	0.408333	90.6
51	0.416667	90.4
52	0.425000	90.2
53	0.433333	89.9
54	0.441667	89.6
55	0.450000	89.6
56	0.458333	89.3
57	0.466667	89.2
58	0.475000	89.0
59	0.483333	88.8

Figure 2. Vernier Software data collection screen output. On the left are the time and temperature observations with a plot on the right. In this situation the room or environmental temperature is 70.3 °F and the temperature of the water in the beaker is initially 104.7 °F.

3. Once you get a good empirical model with which you are comfortable, then ask yourself how will you determine the parameters and test the model against the data? Discuss this and develop some criteria for judging your model's appropriateness for this data.
4. Develop your criteria and carry it out, determining your best empirical model of choice.

Analytic modeling

One approach for analytic modeling is to try to understand all the mechanisms for heat loss, conduction, convection, and radiation, in great detail and build a model, perhaps even a stochastic model. For the small atomic level actions are often stochastic, but collectively appear to be deterministic. This is certainly too deep for us to consider now. However, we can make some assumptions about how the temperature is changing.

Function Model	Critique
$T(t) = a + bt$	
$T(t) = a + bt + ct^2$	
$T(t) = a \arctan(bt + c)$	
$T(t) = ae^{bt}$	
$T(t) = ae^{bt} + c$	
$T(t) = a\cos(bt + c) + d$	

Table 2. Some empirical models offered with critiques as to why they may or may not be a good model for our phenomenon.

5. Write down some things that you notice about the temperature, $T(t)$, as time goes on. Next, translate these observations into mathematical statements about the rate of change of the temperature, $T'(t)$, as time goes on. It is from this latter activity that we will draw up a reasoned differential equation model for $T'(t)$.
6. Consider the following candidates (see Table 3) for a differential equation model, i.e. an attempted analytic model, for $T(t)$, the temperature of the water in the beaker at time t. Offer critiques as to why each may or may not be a good model for our phenomenon.
7. After completing the activity (6) you should have a differential equation model with some parameters, parameters which have units and meaning. Be sure to determine the parameters' units and try to offer significance or meaning to the parameters in your differential equation.
8. Finally, using the data in Table 1 use your criteria developed above to find best estimates of your parameters, determine your final model, compare the model with the data and other models you and your classmates developed already, and offer your observations on the process and result.

Function Model	Critique
$T'(t) = a$	
$T'(t) = a + bt$	
$T'(t) = \frac{A}{B+Ct}$	
$T'(t) = k(T(t) - T_{\text{Env}})$	
$T'(t) = -kT(t)$	
$T'(t) = Ae^{-kt}$	

Table 3. Some analytic models offered with critiques as to why they may or may not be a good model for our phenomenon.

Cooling in a changing environment

We did several sessions to collect data on water cooling in our office environment using the apparatus shown in Figure 1. Table 1 was one of our first runs. One morning we got up early and began another data collection run. We had our trusty little room thermometer (see lower right side of Figure 1) and as we began to collect the data we noticed that the temperature of the room was changing as well. It is a small room, the collector is a big person, the monitor is a large hot running device, the heat was coming on, as the sun was rising. So we were not surprised that the heat was rising in the room. In Table 4 we sample the time, temperature of the water, *and* temperature of the room or environment.

9. Model the temperature in the environment, $T_E(t)$, as a function of time, t.

10. Build a complete model for the rate of change of the temperature of the water ($T(t)$) in terms of the temperature in the room ($T_E(t)$), solve the model, estimate parameters in the model, and validate your model by comparing its predictions to the data.

Time (min)	Temp Room °F	Temp Water °F
0.0	-	103.1
5.0	58.3	99.
9.5	59.7	96.
15.0	61.	92.8
20.5	62.1	89.9
27.0	62.8	86.9
35.0	63.7	83.9
44.0	64.2	81.0
54.0	64.8	78.4
58.5	64.9	77.5
70.0	65.5	75.3
75.	65.7	74.6
80.0	65.8	73.8
88.5	66.	72.7
102.0	66.2	71.4
106.5	66.2	70.9
112.5	66.4	70.5
117.0	66.4	70.2
126.	66.6	69.7
130.5	66.7	69.4
139.5	66.7	69.0

Table 4. Sample of data from the temperature of water (200 mL) in a beaker where the environmental temperature is changing. Complete data set is found in the Excel file 1-031-CoolItData.xls.

NOTES FOR TEACHERS

This data can be collected over a shorter period of time using common thermometers and visual stop watch calibrations for additional data sets. Students can also access the extended data sets in the accompanying Excel file 1-031-CoolItData.xls.

Empirical modeling is a good way to start this modeling activity, for the obvious weaknesses in some of the models presented in Table 2 evoke notions and ideas which will help in building an analytic differential equation model and some of these (with flaws) are also offered in Table 3.

We encourage models of the form $T'(t)$ = something, i.e. differential equations. Somehow the environment's temperature (be it constant or changing) needs to be in such an equation. Some students have seen Newton's Law of Cooling, but we rarely offer those words, lest we tip our hand as to a route they will eventually find for themselves.

We offer complete analyses in the Mathematica notebook 1-031-Mma-CoolIt-TeacherVersion.nb and a pdf version of the same executed file for readers using other computer algebra systems.

Activities such as this should be initiated in class, although using the modern flipped classroom, it is even better to ask students to read the introductory material on empirical models and perhaps try building some type of differential equation model before class. For students familiar with a computer algebra system and a modicum of commands this would be an overnight assignment, taking about an hour or two, at most.