

**STUDENT VERSION**

**Tiling an** $n×3$ **Hallway with** $1×2$ **Tiles**

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**GOAL**

In this exercise, you will explore difference equations in the context of tiling hallways. You will calculate solutions to your difference equation by iteration and become familiar with a helpful tool known as the shift operator.

**STATEMENT**

Work has been slow for Kyle the Tile Guy lately. With all the new wood and laminate flooring options, Kyle is not in demand. To pass the time, Kyle, a double major in math and construction engineeringwas reading one of his old math textbooks and ran across the following problem:

How many ways can the floor of a hallway that is $n$ units long and $3$ units wide be tiled with tiles, each of which is 2 units by 1 unit? Tiles can be placed vertically or horizontally. Kyle was intrigued…how about you? [1, p. 89]

**Part 1**

1. Draw a few different tiling pictures of $n×3$ hallways for $n=6$.

Hint: Draw a big “X” to represent the two by one tiles.

1. Draw a tiling picture of $n×3$ hallways for $n=3$ in the figure to the right.

What happened?

1. How many different patterns are possible for a $2×3$ hallway?



1. How many different patterns are possible for a $4×3$ hallway? Hint: Think about symmetry and the patterns developed in the $n=2$ case so you don’t have to draw each one.
2. Define $y\left(i\right), i\geq 2,$ (for even $i$) to be the number of ways that an $i×3$ hallway can be tiled with the $1×2$ tiles. Find: $y\left(2\right)=\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_$ and $y\left(4\right)=\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_$.
3. If we had an $2×3$ hallway, how many ways could the first row be covered by the tiles?



**Part 2**

1. Define the function $z$ such that $z\left(n+2\right) $is the number of ways to tile the middle hallway in Figure 1. How many ways could the third hallway be tiled? Hint: Read carefully and remember symmetry!

*Figure 1: Initial Tiling Options*

1. Recall that $y(n+2)$ would be the number of ways that an $(n+2)×3$ hallway can be tiled with the $1×2$ tiles. Write an equation using the three pictures in Figure 1 for
$y(n+2)$.
2. How many patterns are possible for the next step of the middle hallway in Figure 1? Draw the pictures.



1. Write an equation for $z\left(n+2\right)$ using the pictures you drew in (9) above.

**Part 3**
We wish to calculate several values of $y(n)$ by iteration using the equations for$ y\left(n+2\right)$ and $z(n+2)$.

1. Combine the two equations in (8) and (10) into one by eliminating the $z(n)$ terms.
2. Given we know $y\left(2\right)=3 and y\left(4\right)=11$, find $y\left(6\right), y\left(8\right), and y\left(10\right)$.

**Part 4**

We wish to calculate a closed form expression of $y(n)$ using the equations for$ y\left(n+2\right)$ and $z(n+2)$ and something called the **shift operator**. [1, p. 17]

1. Define the **shift operator** $E$ such that $Ey\left(n\right)=y(n+1)$. Similarly, $y\left(n+2\right)=Ey\left(n+1\right)=E(Ey\left(n\right))=E^{2}y(n)$ where we define $E^{2}$ to be $E$ composed with $E$. Use the shift operator to reduce the two equations in (8) and (10) into expressions of $y\left(n\right)$ and $z(n)$.

1. Use the elimination method of linear equations to eliminate $z(n)$. What is the result?
2. This “characteristic” equation can be solved just like differential equations, but the solutions are in the form $E^{n}$ instead of $e^{rt}$.

The solutions of $E^{4}-4E^{2}+1=0$ are $E=\pm \left(2+\sqrt{3}\right)^{\frac{1}{2}}$ and $E=\pm \left(2-\sqrt{3}\right)^{\frac{1}{2}}$. Recalling $n$ is even, we obtain the general solution:

$$y\left(n\right)=A\left(2+\sqrt{3}\right)^{\frac{n}{2}}+B\left(2-\sqrt{3}\right)^{\frac{n}{2}}.$$

where A and B are some constants. Using the initial conditions $y\left(2\right)=3 and y\left(4\right)=11$, find the particular solution:

1. Find $y\left(6\right), y\left(8\right), and y\left(10\right)$ and compare with solutions in (12).

**REFERENCES**

[1] Peterson, A., and W. Kelley. 1991. *Difference Equations: An Introduction with Applications.* San Diego: Academic Press.