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STUDENT VERSION

Frequency Response

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Abstract: We describe the frequency response to a second order differential equation with a driving function as the maximum steady state solution amplitude and perform some analyses in this regard.

SCENARIO DESCRIPTION

Frequency Response - Maximum Steady State Amplitude

Engineers are often interested in the response of a system to inputs. Consider an RLC circuit in which a voltage input drivers the circuit at a certain frequency. What is the gain or system response to a particular driver's frequency? Or consider a mathematical model consisting of a second order differential equation for a spring-mass model for a building structure which is driven by an input force with low frequency, think earthquake at 10 Hz. What is the amplitude of the structures response? A ride over a rough rippled dirt road drives the shock absorber of the vehicle differently at different frequencies of bumps in the road? Can this hurt the vehicle or make the ride very uncomfortable?

We offer the steps to compute the frequency response or *maximum steady state amplitude* in a spring mass dashpot system as an imagery example.

In a spring mass dashpot system described in (1) the solution will have two parts: (1) *transient state* involving exponential decay terms which will go to zero, perhaps rather quickly, but nevertheless goes to zero and (2) the *steady state* which persists due to the driver function, in this case $F_0 \sin(\omega t)$.

The steady state solution in the case of (1) will be a combination of $\sin(\omega t)$ s and $\cos(\omega t)$'s.

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = F_0 \sin(\omega t)\,, \quad y(0) = y_0 \quad \text{and} \quad y'(0) = v_0\,. \tag{1}$$

We solve the differential (1) and we grab the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, and determine the steady state's amplitude (ssAmp(ω))) as a function of the input frequency ω and our parameters.

We are using Mathematica to reduce the tedium of the analysis, but any computer algebra system would serve as well, e.g., Maple or SAGE. Indeed, by hand analysis would work also, but would demand utmost rigor in manipulating and collecting terms. In Mathematica we use the Coefficient command to collect the coefficients of the respective terms $\sin(\omega t)$ and $\cos(\omega t)$ and we take the amplitude of the response by square rooting the sum of square terms, respectively:

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ssAmp[ω_] = Sqrt[Coefficient[ys[t, ω], Cos[t ω]]^2
                + Coefficient[ys[t, ω], Sin[t ω]]^2] // Simplify
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When we finish this computation we have a nice closed form solution for the amplitude of our steady state frequency response as a function of the input frequency ω with all the other parameters of our system present as well.

$$\texttt{ssAmp}(\omega) = \sqrt{\frac{F_0^2}{c^2\omega^2 + k^2 - 2km\omega^2 + m^2\omega^4}} \tag{2}$$

As an example, consider the following parameters, $m = 1$ kg, $k = 15$ N/m, $c = 3$ N/(m/s), and $F_0 = 1$ N. We plot the curve of Maximum Steady State Amplitude vs. Input Frequency ω in Figure 1.

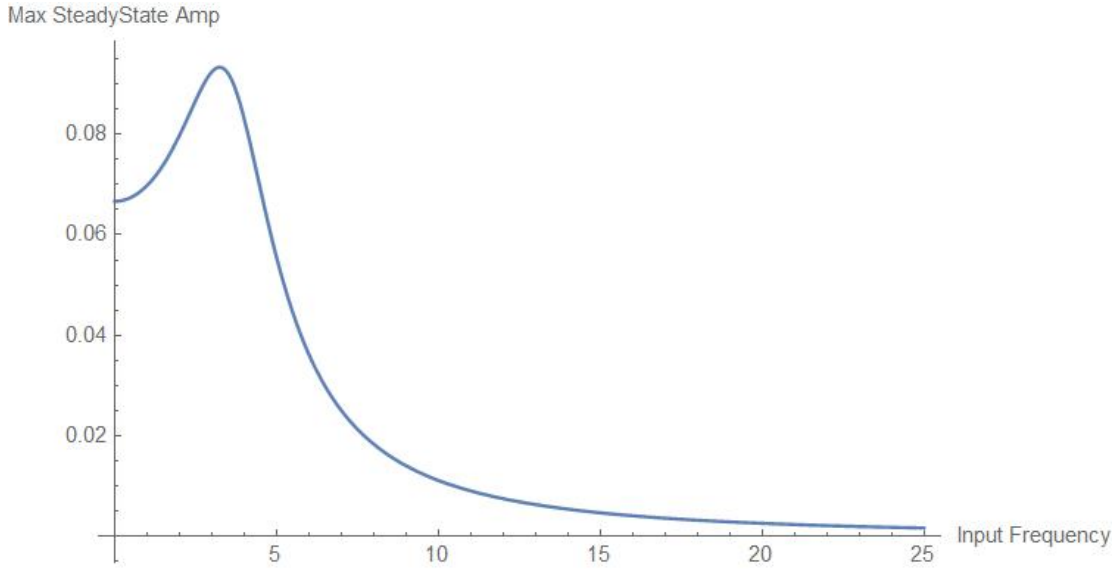


Figure 1. Maximum Steady State Amplitude vs. Input Frequency.

Figure 1 is a very powerful plot as it tells us what frequency will produce maximum steady state response for our system in differential equation (1). Indeed, for the parameters here we find that $\omega_{\max} = 3.24037$ radians/second and the maximum steady state amplitude is 0.093352 m.

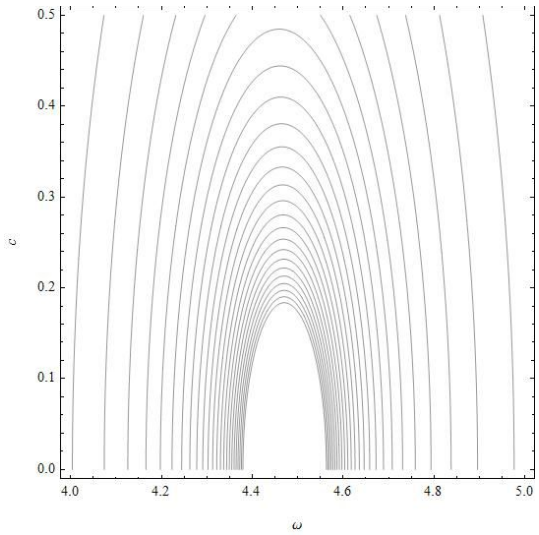
a) Find the range $[\omega_{\text{Low}}, \omega_{\text{High}}]$ surrounding $\omega_{\max}$ such that all responses are at or above 90% of the maximum steady state amplitude of 0.093352 m.

b) Find the first ω such that all responses from input at that value or higher are only 10% of the maximum steady state amplitude of 0.093352 m.

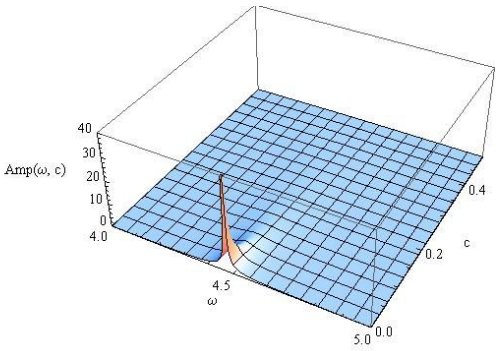
c) Determine the value of ω at which the maximum steady state amplitude increases fastest.

d) Create the plots in Figure 2 (a), (b), (c) and (d) and explain what each pair (a) and (b) and then (c) and (d) display. In Figure 2 (b) what is the spike really representing? Hint: think $c = 0$ and resonance.

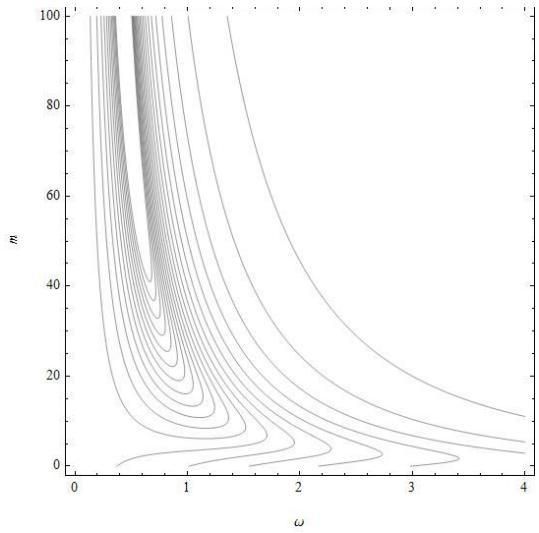
e) Change one of the paramaters from the set give in the material above, $m = 1$ kg, $k = 15$ N/m, $c = 3$ N/(m/s), and $F_0 = 1$ N, and see how the plot in Figure 1 changes and what the relationship is between maximum frequency response and parameter change.



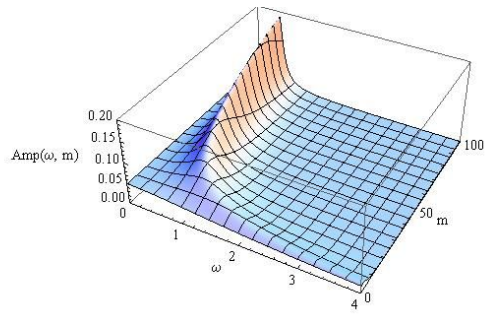
(a) Contour plot of steady state amplitude as a function of input frequency ω and damping constant c for parameters $m = 1$, $k = 20$, $F_0 = 1$ in differential equation (1).



(b) Plot of steady state amplitude as a function of input frequency ω and damping constant c for parameters $m = 1$, $k = 20$, $F_0 = 1$ in differential equation (1).



(c) Contour plot of steady state amplitude as a function of input frequency ω and spring constant k for parameters $m = 1$, $c = 12$, $F_0 = 1$ in differential equation (1).



(d) Plot of steady state amplitude as a function of input frequency ω and spring constant k for parameters $m = 1$, $c = 12$, $F_0 = 1$ in differential equation (1).

Figure 2. Plots of steady state amplitude as a function of input frequency ω and damping constant c ((a) and (b)) and input frequency ω and spring constant k ((c) and (d)) in differential equation (1).