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STUDENT VERSION

Neutral Buoyancy Point for a Weight/Lift Bag System

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SCENARIO DESCRIPTION

An object may hang suspended at, say, ten foot depth in a column of water if at ten feet underwater the density of the object equals the density of water. We often think of density (mass per unit volume) as a fixed property of matter, but there are many common examples of systems that under pressure can change their volume and thus their density. A SCUBA diver with wet suit and buoyancy compensator is an example of such a system.

According to Archimedes' Principle [1] a body immersed in a fluid will weigh as much as the weight of the fluid it displaces. This principle accounts for the tendency of boats to float and hot air balloons to rise into the atmosphere. But it also accounts for the ability of a SCUBA diver to float motionless at depth. For this exercise let's think of the fluid as fresh water and for a human diver we will substitute a three pound lead weight coupled with 0.05 ft^3 of air in a flexible container which could be a rubber balloon or rubber or canvas lift bag. Let's call it a *lift bag*. The weight and lift bag form a greatly simplified model of a SCUBA diver in that it is composed of components that will lose volume as pressure is increased (thus increasing density) and parts that will maintain constant volume and density with increasing pressure.

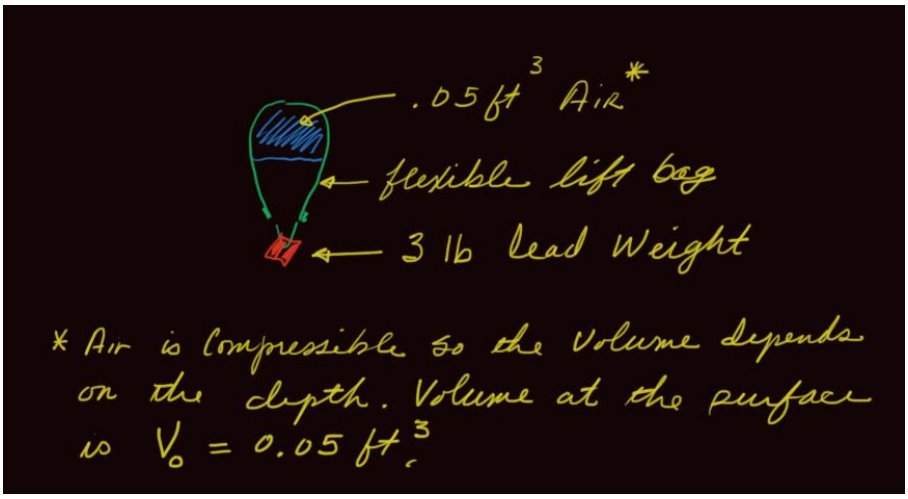

Figure 1. Lift bag attached to lead weight.

Assumptions:

The experiment takes place in fresh water, as opposed to salt water. We start with 0.05 $ft^3$of air (V_o) and a three pound lead weight. Air is compressible and its volume depends on both temperature and pressure. In this exercise we will start with 0.05 ft^3 volume measured at 70°F and one atmosphere of pressure.

Throughout this exercise we will be modeling changes with depth (change in volume, pressure, buoyancy, etc.). It is a bit awkward to write "d depth" or even worse "dd" for the depth differential so let us agree to us the symbol "x" for depth in feet.

A first step will be gathering values for three physical constants. A Google search will quickly find these values:

Density of lead: _____
Density of fresh water: _____
Density of air at 70.00 °F and 1 ATA: _____

ATA is the atmospheric pressure at elevations near sea level, which is 14.7 psi which we call one atmosphere absolute (ATA).

Now you can calculate the volume of the lead weight: ____

With knowledge of the density and volume of your lead you can calculate the weight of the lead when immersed in fresh water. This value is often called the *wet weight* of the lead and it is computed by subtracting from the dry weight the weight of the displaced water. This is an application of Archimedes' Principle.

Wet Weight of three pound lead weight: _____

Here is another way to think about "wet weight." When we put an object in water we talk about its buoyancy which can be positive, negative, or neutral. An object that is positively buoyant will rise in the water column, a negatively buoyant object will sink,

and a neutral buoyant will neither rise nor sink. Using V(x) as the volume of our weight at a depth x and ρ as a symbol for density we can calculate the buoyancy of the lead weight in the preceding paragraph through the formula:

$$B_{lead}(x) = \rho_{H_2O} * V(x) - \rho_{lead} * V_0 \, .$$

Of course for a lead weight the volume doesn't change with depth so $V(x)=V_0$ for all values of depth x and using the parameters above we calculate $B_{lead}(x)$ = -2.736 lbs.

This is the value that earlier we called "wet weight" for lead, but with a negative sign attached, which tells us that lead immersed in water will sink.

Before we calculate the buoyance of the lift bag we need to detour a bit, since unlike the lead, the volume of the air in the lift bag changes with depth.

A little background

Atmospheric pressure at elevations near sea level is 14.7 psi or one atmosphere absolute (ATA). As you descend into the depths the pressure increases due to the weight of water above you. For fresh water a column of water 34 foot high exerts a pressure equivalent to one ATA. Thus when one descends to a depth of 34 feet the absolute pressure is two ATA: one ATA from the air above plus one ATA due to the weight of water above. This change is linear. Each time you increase your depth by 34 feet the pressure increases by one ATA. This gives us our first differential equation in P(x) the pressure in ATA units:

$$\frac{d\ pressure}{d\ depth} = \frac{dP}{dx} = \frac{1}{34} \, .$$

Solve this equation to obtain a function that gives the pressure as a function of depth.

P(x) = _____

Do you know an initial condition for this equation? _____
HINT: what is the pressure when the depth is zero? _____

Use the initial condition to solve for a particular solution (pressure as a function of depth without an arbitrary constant.) The particular solution is: _____

The pressure function will prove very useful as we will now see.

The density of a system is mass/unit volume. For our lift bag/lead weight system we can change the density of the system by changing the pressure on the system, e.g., moving the system to a given depth. We can't do much about the mass of the system, but we can change the volume of the system. This is because as we push or pull the system downward the increase in pressure will compress the air and decrease its volume. As our system moves deeper the volume of air changes according to the formula

$$\frac{dV}{dP} = \frac{-V_o}{P^2}$$

where pressure is measured in ATA. From this differential equation and the chain rule we see that:

$$\frac{dV}{dx} = \frac{dV}{dP} * \frac{dP}{dx} = \frac{-V_0}{P^2} * \frac{1}{34} = \frac{-V_0}{34} * \frac{34^2}{(x+34)^2} = \frac{-34V_0}{(x+34)^2}.$$

Solving for V(x) we have

$$V(x) = \frac{34V_0}{x+34}.$$

Thus, for example, if a container of air has a volume at the surface of 1 ft^3, at depth x = 34 feet the pressure will be two ATA and the volume will be 0.5 ft^3.

We're almost at our goal to analyze our system for neutral buoyancy. As stated above, the amount of lift from our bag, by Archimedes' Principle, will be the volume of the bag times the density of water. Thus we see that

$$B_{bag}(x) = \rho_{H_2O} * V(x) - \rho_{air} * V_0$$

In the paragraph above we found V(x) so you can now determine, $B_{bag}(x)$:

$$B_{bag}(x) =$$

When the buoyancy of the lift bag is equal to the absolute value of the buoyancy of the attached lead the system will be neutrally buoyant and will hang suspended unless acted upon by an outside force. This suggests we solve for x in the equation,

$$B_{bag}(x) = 2.736$$

since we found the buoyancy of the lead to be -2.736 lbs. Under the stated assumptions at what depth will our system be neutrally buoyant? _____

REFERENCES

(1) Wikipedia contributors. *Archimedes' principle*. Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Special:CiteThisPage&page=Archimedes%27_principle&id=721257448. Accessed 3 June 2016.