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STUDENT VERSION

Thermometer in Varying Temperature Stream

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Abstract: We present a first order differential equation model for the temperature of a mercury thermometer which is sitting in a stream of water whose temperature oscillates. We suggest a solving strategy which uses Laplace Transforms.

SCENARIO DESCRIPTION

An Application of Laplace Transform

Consider a thermometer which is located in a flowing stream of water for which the temperature x varies with time. We would like to calculate the response or the time variation of the thermometer reading, y, for a given change in x, the temperature of the fluid in the stream in which the thermometer sits under the following assumptions:

1. All the resistance to heat transfer resides in the film surrounding the bulb (we are neglecting the resistance of the glass and mercury).
2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.
3. The glass wall containing the mercury does not expand or contract during the transient response.
4. The thermometer is initially at steady state. Therefore, before time zero, there is no change in temperature with time.
5. At time $t = 0$ the thermometer will be subjected to some change in the surrounding temperature, $x(t)$.

Now if we apply the unsteady-state energy balance:

$$\text{Rate of Input} \;-\; \text{Rate of Output} \;=\; \text{Rate of Generation} \tag{1}$$

we obtain

$$hA(x-y) - 0 = mC\frac{dy}{dt}\,. \tag{2}$$

where

A is surface area of the bulb for heat transfer, ft^2,

C is the heat capacity of mercury, $\text{BTU}/(\text{lb}_m)^o\text{F}$,

m is the weight of mercury in bulb, lb_m,

t is time, in hr, and

h is the film coefficient of heat transfer, $\text{BTU}/(\text{hr})(\text{ft}^2\ )(^0\text{F})$.

We note the following:

1. The coefficient h will depend on the flow rate and properties of the surrounding fluid and the dimensions of the bulb. Also, h is constant for a particular installation of the thermometer.
2. Prior to the change in x, the thermometer is at steady state and the derivative, dy/dt, is zero. Therefore the steady state conditions, x_s and y_s, render (2) to be:

$$hA(x_s - y_s) = 0, \;\; \text{for } y < 0\,. \tag{3}$$

Upon subtracting (3) from (2) we get

$$hA[(x-x_s) - (y-y_s)] = mC\frac{d(x-y_s)}{dt}\,, \tag{4}$$

or

$$hA(X-Y) = mC\frac{dY}{dt}\,, \tag{5}$$

using this change of variables, $X_s = x - x_s$ and $Y_s = y - y_s$, for deviation variables X_s and Y_s.

We can rewrite (5) as

$$(X_s - Y_s) = \frac{mC}{hA}\frac{dY_s}{dt} = \tau\frac{dY_s}{dt}\,, \tag{6}$$

using $\tau = \frac{mC}{hA}$, the time constant.

If we take the Laplace Transform of both sides of (6) and rearrange the results we can determine, in terms of the transfer functions, $X(s)$ for $X_s(t)$ and $Y(s)$ for $Y_s(t)$, the equation

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1} \tag{7}$$

or

$$Y(s) = X(s)\frac{1}{\tau s + 1} \tag{8}$$

which facilitates a response through the Inverse Laplace Transform of the deviation variable transforms $Y(s)$ to an input $X(s)$. Here $\frac{1}{\tau s+1}$ is called the *transfer function* which takes an input described by $X(s)$ and converts it into an output described by $Y(s)$.

ACTIVITY

A mercury thermometer having a time constant of $\tau = 0.1$ min, is placed in a temperature bath at 100^oF and allowed to come to equilibrium with the bath. At time $t = 0$, the temperature of the bath begins to vary sinusoidally about its average temperature of 100^oF with an amplitude of 2^oF. If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading's temperature as a function of time. What is the phase lag?

Data

We are given the data: $\tau = 0.1$, $x_s = 100^o$F, f (frequency) $= (10/\pi)$ cycles min, $\omega = 2\pi f = 20$ radians/min, $A = 2^o$F, with input: $x(t) = x_s + A\sin(\omega t)$.

We can solve for $Y(s)$ as in (8) and then take the Inverse Laplace Transform to obtain $y(t)$.

1. Show that
$$Y(s) = \frac{A\omega}{s^2+\omega^2} \cdot \frac{1/\tau}{s+1/\tau}\,. \tag{9}$$

2. Take the Inverse Laplace Transforms of all the parts of (9) to obtain $y(t)$. Show your steps. You should have
$$y(t) = \overbrace{\frac{A\tau\omega e^{-\frac{t}{\tau}}}{\tau^2\omega^2+1}}^{\text{transient}} + \underbrace{\frac{A\sin(t\omega)}{\tau^2\omega^2+1} - \frac{A\tau\omega\cos(t\omega)}{\tau^2\omega^2+1}}_{\text{steady state}}, \tag{10}$$
where the first term is the *transient* solution which goes to zero shortly. While the second and third terms form the *steady state* solution which remains and continues to oscillate in response to the varying temperature of the water flowing around the thermometer.

3. Now determine that the amplitude of the steady state response portion of the function, $y(t)$, is $\frac{A}{\sqrt{\tau^2\omega^2+1}}$ and the phase angle of the thermometer's temperature response is $\phi = \tan^{-1}(-\omega t)$. Hint: Use the trigonometric identity
$$\sin(\omega t - \phi) = \sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)\,.$$

4. Finally, determine the response of the thermometer to be $Y_s(t) = 0.896\sin(20t - 63.5)$ or equivalently, $y(t) = 100 + 0.896\sin(20t - 63.5)$.