Modeling Bubbles of Beer

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Overview

- The goal of this talk is to look at how to model the behavior of CO_2 bubbles in a glass of beer.
- Note that the following arguments carry over to gas bubbles in liquids in general!

Physical Observations

- Pour a glass of beer, wait a few moments, and you will observe the following:
  - Bubbles form at the bottom of the glass and rise to the top.
  - Bubbles accelerate as they rise.
  - Bubble diameter at top is twice the initial diameter.

What do we want to Model?

- We want to model the height $x(t)$ and radius $r(t)$ at time $t \geq 0$ of a CO_2 bubble that starts to rise at time $t = 0$ seconds.
- Parameters in our model include:
  - x_0:= initial depth of bubble below surface [m]
  - r_0:= initial bubble radius [m]
  - P_{atm}:= atmospheric pressure [N/m^2]
  - T:= temperature of beer [^oK]
  - K:= “freshness of beer” [moles/(m^2 s)] (rate at which CO_2 passes from beer into bubble)

Assumptions for our Model

- In our model we will make the following assumptions:
- (a) The bubble is a sphere.
- (b) Rate at which CO_2 enters bubble ~ Surface area of bubble.
- (c) The CO_2 in the bubble obeys the *Ideal Gas Law*: $PV = nRT$, where
  - P is pressure
  - V is volume
  - n is amount of CO_2
  - T is temperature
  - R is the *universal gas constant.*

Amount of CO_2 in Bubble

- We first find an equation for the amount of CO_2 in a bubble of beer.
- Let n(t):= number of moles of CO_2 in a bubble at depth x(t).
- Then assumptions (a) and (b) can be expressed mathematically as:

$$n'(t) = K4\pi[r(t)]^2. \qquad (1)$$

Amount of CO_2 in Bubble

- Using the equation for the volume of a sphere to relate the radius of a bubble to its volume and applying the Ideal Gas Law assumption, we see that

$$r(t) = \left[\frac{3V(t)}{4\pi}\right]^{1/3} = \left[\frac{3n(t)RT}{4\pi P(t)}\right]^{1/3}, \qquad (2)$$

so we can rewrite equation (1) as

Amount of CO_2 in Bubble

$$n'(t) = K4\pi \left[\frac{3n(t)RT}{4\pi P(t)}\right]^{2/3} .$$

- From college physics, we know that the pressure on a submerged bubble at a depth x(t) is given by:

$$P(t) = P_{atm} + P_{fluid} = P_{atm} - g\rho_{beer}x(t).$$

(The fluid pressure is the weight of a column of beer of height |x(t)| with unit cross--sectional area.)

- It follows that

$$n'(t) = K4\pi \left[\frac{3n(t)RT}{4\pi[P_{atm} - g\rho_{beer}x(t)]}\right]^{2/3} . \qquad (3)$$

Motion of Bubble

- Next we find an equation that describes the acceleration of a beer bubble.
- Recall *Newton's Law*: $F(t) = m(t)\, x''(t)$.
- There are three forces acting on the bubble at time t:
  - (a) Buoyancy Force: $F_{buoy} = g\, \rho_{beer}\, V(t)$
    (Weight of beer displaced by bubble. Archimedes, 250 B.C.)
  - (b) Fluid drag: $F_{drag} = 4\pi\eta\, r(t)\, x'(t)$
    (Hadamard, 1911)
  - (c) Weight: $F_{weight} = m(t)g = g\, \rho_{CO2}(t)\, V(t)$
    (Newton?, 1665)

Motion of Bubble

- Therefore the total force on the bubble is:

$$\begin{aligned} F(t) &= F_{buoy} - F_{weight} - F_{drag} \\ &= g\rho_{beer}V(t) - g\rho_{CO_2}(t)V(t) - 4\pi\eta r(t)x'(t) \\ &= g\rho_{CO_2}(t)V(t)\left[\frac{\rho_{beer}}{\rho_{CO_2}(t)} - 1\right] - 4\pi\eta r(t)x'(t) \end{aligned}$$

and Newton's Law becomes:

$$m(t)x''(t) = g\rho_{CO_2}(t)V(t)\left[\frac{\rho_{beer}}{\rho_{CO_2}(t)} - 1\right] - 4\pi\eta r(t)x'(t).$$

Motion of Bubble

- Dividing through by the mass of the CO_2 bubble, m(t), gives an equation relating x''(t) to ρ_{CO2}(t), r(t), x'(t), and m(t):

$$x''(t) = g\left[\frac{\rho_{beer}}{\rho_{CO_2}(t)} - 1\right] - \frac{4\pi\eta r(t)x'(t)}{m(t)}. \qquad (4)$$

- We'd like an equation with just x(t), n(t), and x'(t).

Motion of Bubble

- Note that

$$\rho_{CO_2}(t) = \frac{m(t)}{V(t)} = \frac{m_{CO_2} n(t)}{V(t)},$$

where m_{CO2} is the (fixed) mass/mole of CO_2.

- Using the Ideal Gas Law to write the volume in terms of pressure and moles of CO_2, we see that

$$\rho_{CO_2}(t) = \frac{m_{CO_2} n(t)}{n(t)RT/P(t)} = \frac{m_{CO_2}[P_{atm} - g\rho_{beer}]}{RT}. \tag{5}$$

Motion of Bubble

- Then, writing m(t) = m_{CO2} n(t), and using (2) and (5), (4) becomes

$$x''(t) = g\left[\frac{\rho_{beer}RT}{m_{CO_2}[P_{atm} - g\rho_{beer}x(t)]} - 1\right] - \frac{4\pi\eta x'(t)}{m_{CO_2}n(t)}\left[\frac{3n(t)RT}{4\pi[P_{atm} - g\rho_{beer}x(t)]}\right]^{1/3}.$$

- Letting x'(t) = y(t), we get a first order system of three differential equations in the three unknown functions x(t), y(t), and n(t):

Mathematical Model

$$
\begin{aligned}
n'(t) &= 4\pi K \left[\frac{3n(t)RT}{4\pi[P_{atm} - g\rho_{beer}x(t)]} \right]^{2/3}, && (6) \\
x'(t) &= y(t), && (7) \\
y'(t) &= g \left[\frac{\rho_{beer}RT}{m_{CO_2}[P_{atm} - g\rho_{beer}x(t)]} - 1 \right] \\
&\quad - \frac{4\pi\eta y(t)}{m_{CO_2}n(t)} \left[\frac{3n(t)RT}{4\pi[P_{atm} - g\rho_{beer}x(t)]} \right]^{1/3} && (8)
\end{aligned}
$$

Parameters and Initial Data

- To actually check that this model works, we can put in measured values for the parameters and initial data:
  - $g = 9.807$ [m/s^2]
  - $P_{atm} = 1.013 \times 10^5$ [N/m^2]
  - $R = 8.3145$ [Nm/(mole ^{o}K)]
  - $m_{CO2} = 4.401 \times 10^{-2}$ [kg/mole]
  - $\rho_{beer} = 1.010 \times 10^3$ [kg/m^3]
  - $\eta = 1.3 \times 10^{-3}$ [N s/m^2] (viscosity of beer at 8^{o}C = 46.4^{o}F)
  - $T = 281$ [^{o}K]
  - K = “fudge factor” (freshness of beer) [moles/(m^2 s)]

Parameters and Initial Data

- Initial Data:
  - x_0 = -.156 m = -15.6 cm
  - $r_0 = 1.7 \times 10^{-4}$ m = 0.17 mm
  - $n_0 = P(0)V(0)/(RT) = 9.05893 \times 10^{-10}$ moles
- Finally, assume $y(0) = x'(0) = 0$ m/sec.

Testing our Model

- To check our model, we can use Mathematica to solve ODE system (6)–(8) numerically and compare to actual data!

time (sec)	depth (cm)	radius/r0
0	-15.6	1
0.54	-14	1.15
1.08	-12.2	1.29
1.62	-10.4	1.44
2.16	-8.6	1.58
2.7	-6	1.73
3.24	-3.2	1.88
3.78	0	2.03

References

- This talk is based on the paper “Quenching a Thirst with Differential Equations” by Martin Ehrismann which appeared in the College Mathematics Journal: Volume 25, Number 5, pp. 413-418.