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STUDENT VERSION
Exploring SIR Modeling

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Abstract: Students will transform, solve, and interpret Susceptible Infected Recovered (SIR) models using systems of differential equation models. The project is progressively divided into three parts to understand, to apply, and to develop SIR models. Part one focuses on understanding and interpreting SIR models using the spread of the swine flu at West Point. Part two focuses on transforming, solving, and interpreting by applying student knowledge of SIR modeling. Students are asked to modify the base model established in part one to model the spread of a rumor. Part three asks students to transform a real world problem of their choosing into an SIR model.

SCENARIO DESCRIPTION

This is an INDIVIDUAL assignment. Your primary submission is a hard copy report that should **not exceed five (5) pages**. To support your primary report, you may include annotated and referenced appendices.

Include any key graphs and equations in your report and include any additional supporting graphs, equations, and computations in the appendices. If you think that your code is important, you may include it in an appendix, but you should ensure that it includes annotations describing what you are doing. If you think a computation or equation is important for the reader, then ensure it is in the report or an appendix and not just in your digital file.

Ensure all work is logical, neat, and organized. Properly document any assistance you receive. Doing the mathematics correctly is important, but it is also critical to be able to analyze and effectively communicate your mathematical results, as well as reflect on the relevance of your results in the real world.

Goals:

1. Clearly communicate your mathematical models and solution methods in a written form.

2. Design and solve a mathematical model for a problem using differential equations.
3. Confront the ambiguity of problem solving and understand the necessity of making assumptions in formulating a mathematical model and the impact these assumptions have on their solutions.
4. Apply technology learned in the course to analyze and to solve a differential equations model.
5. Develop an appreciation for how technology enhances problem solving capabilities.

Helpful Tools:

1. Mathematical computation software.
2. Differential equations textbook.

PROJECT SCENARIO: The Final Exam is Cancelled!

After a particularly challenging lecture, Cadet Sleuth Quidnunc found himself outside of his instructor's office with lots of questions. The office door was partially opened and Cadet Quidnunc could hear his instructor, LTC Boucher, speaking with the course director, Dr. Dungan. Rather than interrupt the conversation, Cadet Quidnunc patiently waited outside for the conversation to be completed. He thought he heard LTC Boucher say, "Well, perhaps if that is the case, we need to cancel the Final Exam." Cadet Quidnunc's heart stopped, as his mind raced with the implications of having a cancelled Final Exam. "This is fantastic news!" he thought to himself. "What am I doing here? I need to let everyone else in the course know!"

PART I: SIR Models

The spread of a rumor behaves similarly to the spread of a communicable disease [2]. During a flu epidemic we are interested in how many people are infected and how many people have been infected and have recovered. Mathematical models help doctors and researchers understand how epidemics spread and formulate predictions for the spread of similar diseases in the future. Figure 1 displays the influence of social networks on the outbreak of the swine flu for a school in Pennsylvania in 2009.

SIR models define a system of nonlinear differential equations to model the spread of disease through the interaction of three population groups. These groups include susceptible individuals, infected individuals, and recovered individuals. Smith *et al.* [3] define these populations as a function of time, t, (days) in the following manner:

$$\begin{aligned}
S(t) &\text{ is the number of \textit{susceptible} individuals at time } t,\\
I(t) &\text{ is the number of \textit{infected} individuals at time } t, \text{and}\\
R(t) &\text{ is the number of \textit{recovered} individuals at time } t.
\end{aligned}$$

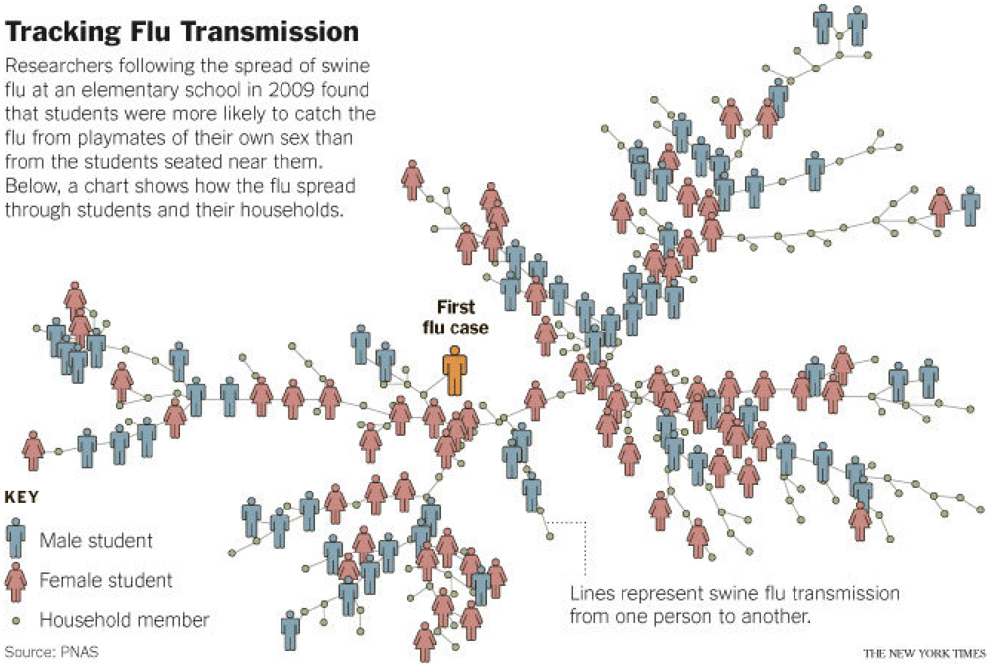
Tracking Flu Transmission
Researchers following the spread of swine flu at an elementary school in 2009 found that students were more likely to catch the flu from playmates of their own sex than from the students seated near them. Below, a chart shows how the flu spread through students and their households.
First flu case
KEY
Male student
Female student
Household member
Source: PNAS
Lines represent swine flu transmission from one person to another.
THE NEW YORK TIMES

Figure 1. Example of Disease Spread [1].

To model this system we need to discuss how these three populations interact. Assume an infected person interacts with a fixed fraction, b, of individuals from the susceptible population each day. This means that every day one infected person creates $bS(t)$ more infected individuals. We can express the susceptible population change using the following differential equation:

$$\frac{dS}{dt} = -bS(t)I(t) \tag{1}$$

Assume that a fixed fraction, k, of individuals from the infected group recover every day. We can express the recovery population change as the following differential equation:

$$\frac{dR}{dt} = kI(t) \tag{2}$$

We can describe the infected population change by combining our formulation of the susceptible and recovered population rates of change over time.

$$\frac{dI}{dt} = bS(t)I(t) - kI(t) \tag{3}$$

Figure 2 summarizes the population interactions of our SIR model as a schematic diagram.

PART I TASKS: Understand the SIR Model

We have created the following base model to describe the spread of disease for a given population.

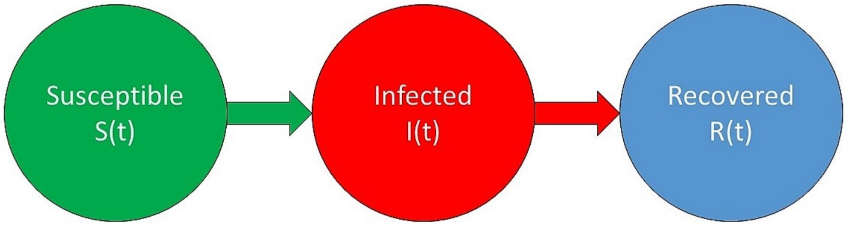

Figure 2. Flu Schematic Diagram.

$$
\begin{aligned}
\frac{dS}{dt} &= -bSI, & S(0) &= S_0 \\
\frac{dI}{dt} &= bSI - kI, & I(0) &= I_0 \\
\frac{dR}{dt} &= kI, & R(0) &= R_0
\end{aligned}
$$

$S(t)$ = number of *susceptible* individuals at time t.

$I(t)$ = number of *infected* individuals at time t.

$R(t)$ = number of *recovered* individuals at time t.

S_0 = initial population at $t = 0$ (Susceptible)

I_0 = initial population at $t = 0$ (Infected)

R_0 = initial population at $t = 0$ (Recovered)

t = time (days)

b = infection rate constant

k = recovery rate constant

1. Explain why $\frac{dS}{dt}$ is negative.
2. What additional assumption concerning the recovered population does our model make?
3. Explain why $I(t) + R(t) + S(t) =$ Total Population.
4. Assume there are $7,138$ people living on West Point. Initially, one person contracts the swine flu. Define your initial conditions and solve your system of differential equations defined above. Assume $b = \frac{1}{7138}$, $k = 0.40$.
5. How many people are infected on day 6?
6. Plot your solution curves, $S(t)$, $I(t)$, and $R(t)$ verses time (days) for the first 40 days on one graph. Be sure to label your graph appropriately.
7. During the first 40 day period, when will the infected population be highest?

8. Will everyone be infected? Explain why or why not.

PART II: Rumor SIR Model

Apply your knowledge of the spread of diseases to develop a system of ordinary differential equations to model the spread of a rumor. Instead of a "recovered" population, we can introduce a new population of "rational" individuals who do not believe the rumor and have the potential to convince rumor spreaders that the rumor is not true. In a sense, "rational" individuals can infect the "rumor spreaders" with "reason" to prevent further spread of the rumor. Assume k infected $I(t)$ people interact with rational $R(t)$ people everyday. Assume your starting susceptible population includes 266 cadets from the MA153 course. Additionally, assume that eight MA153 instructors comprise your initial population of "rational" individuals and that one cadet is the initial "rumor spreader."

9. Using the SIR model from PART I, develop a new initial value problem to describe the spread of a rumor that the MA153 TEE is cancelled. Clearly define your dependant and independent variables and all initial conditions (with units) for your model.
10. Using $b = 0.004$, solve the IVP first with $k = 0.01$ and again with $k = 0.02$. Plot your solution curves, $S(t)$, $I(t)$, and $R(t)$ verses time (days) on one graph for the first 15 days on one graph for both values of k. Be sure to label your graph appropriately.
11. For $k = .01$ and again for $k = .02$, answer the following questions regarding the first 15 days of the rumor outbreak:
    (a) When did the greatest number of cadets believe the rumor?
    (b) What percentage of the total population believed the rumor during the first 15 days?
    (c) What real world factors effect the choices for the constants b and k?
12. If instead of 1 initial rumor spreader, there were 8 initial rumor spreaders, would this significantly change your results? Explain.
13. If instead of 8 initial rational individuals, there was only 1 initial rational individual, would this significantly change your results? Explain.
14. Discuss how rumor spreading in a regular Army unit might be different than rumor spreading at West Point.

PART III: Develop a Different SIR Model

Develop your own scenario for which an SIR model could be appropriate (excluding the spread of a disease or rumor). Be sure to define all variables (with units), include initial conditions, and discuss any key assumptions for your new model. Draw a schematic diagram to illustrate how the model works. Make reasonable assumptions on values for your parameters and render plots of solutions, showing the effects of changes in the parameter values.

REFERENCES

[1] Bakalar, Nikolas. 2011. What's a Little Swine Flu Among Friends? Diagram. *Tracking Flu Transmission*. 8 February 2011. Accessed 25 October 2017. https://www.nytimes.com/2011/02/08/health/research/08flu.html?_r=1&ref=science.

[2] Fedewa, Nick *et al*. 2013. Spread of a Rumor. *Society for Industrial and Applied Mathematics, (SIAM) Undergraduate Reasearch Online (SIURO)*. 6. Web. Accessed 17 October 2017. https://www.siam.org/students/siuro/vol6/S01182.pdf.

[3] Smith, David *et al*. 2000. The SIR model for spread of disease. *AMC*. 10: 1-12.