STUDENT VERSION
M&M Attrition Warfare

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Abstract: Students will model attrition between two opposing forces using M&M candies and discover a system of linear differential equations of order one, often called the Lanchester equations. Through experiment and data collection, students will determine the parameters describing the fighting strength of each force, solve the system, arrive at Lanchester's N-square Law and use it to analyze a historical battle.

SCENARIO DESCRIPTION

In his 1916 work [4], Frederick William Lanchester proposed a model of how two opposing forces of aircraft would attrit one another in combat. Lanchester's model came two years before the establishment of the first air force as an independent branch of any nation's armed forces. Since then, Lanchester's model continues to find applications in operations research (see [1, 2, 3]). In this scenario we will investigate mock military engagements between opposing forces of M&M candies and derive Lanchester's model.

To proceed, consider two opposing forces. Suppose that the opposing units engage one another in repeated single combat rounds until one opponent is defeated. Modern weaponry, capable of inflicting mass casualties, is not considered in this model, so for example, imagine a duel between two knights, two fighter airplanes in a dog fight, or two pirate ships attempting to sink one another. In each situation one ultimately will defeat the other. We will use M&M's to simulate such combat all the while recognizing the severe constrains and heavy assumptions implicit in the model. For a discussion of the obstructions to implementing Lanchester's model in practice see [3, Chapter VI].

The experiment

Begin with M&M's of two different colors in equal numbers. Say, 12 blue and 12 red pieces. Record the initial size of each force as Battle 0 in a table. Subsequent rows (see Table 1) track the number of casualties on each side and each row represents a single battle.

A typical round - of sequential battles

- Toss all the M&M's onto a plate and correct any ambiguities by slightly jostling the plate until each piece is resting properly with either the M facing up, or facing down.
- Any piece with the M facing up is vulnerable to defeat, but defeat occurs only if a piece of the opposing color with the M facing down is available. So, group opposing colors in pairs with exactly one M facing up, then remove the piece with the M facing up from the plate – it is a casualty – and place it off to the side. The other piece survives until the next battle, but until then it must be removed into the cup.
- Repeat the process until all pieces with M facing up are removed, or until no more pairs remain.
- Record the casualties on each side in a table.
- Collect all pieces in the cup. The round is over.

We will return to this sequential battle modeling, but first let us consider just several simulations of a single battle, in each case starting with the same Red and Blue configuration.

A closer look at a single battle simulations, each with same initial forces

Let us first consider just one battle simulation, rather many many such battle simulations, but in each case we always use the same starting values of Red forces and Blue forces.

1. Make a hypothesis on the attrition *rates* between two opposing forces of equal numbers for a single battle. How do you expect the rate at which the number of Red units decreases is related to the number of Blue units?

   Initial hypothesis:

2. Experiment and collect your data in Table 1, Table 2, and Table 3. You may divide the labor between groups. Each entry, e.g., "Battle 1" in Table 1, indicates the number of casualties suffered by respective forces at the hands of the other when the forces were at strength 12 for Red and 12 for Blue.

   (*) Now conduct a single battle simulation, by tossing the M&M's representing the initial forces of Red and Blue as represented by M&M's onto a plate. Recall, any piece with the M facing up is vulnerable to defeat, but defeat occurs only if a piece of the opposing color with the M facing down is available. So, after grouping opposing colors in pairs with exactly one M facing up, remove the piece with the M facing up from the plate – it is a casualty – and place it off to the side. The other piece survives until the next battle, but until then it must be removed into the cup. So for example, if after one battle (a single toss of your initial number of Red and Blue M&M's) you obtain 7 Red casualties and 5 Blue casualties then enter 7 under Battle

1 in column denoted Red and under Battle 1 enter 5 in column denoted Blue. We have thus recorded the outcome of a single battle in which we had 7 casualties for Red and 5 casualties for Blue.

Repeat the simulation of a single battle, by first restoring the forces to full strength, in the case of Table 1,12 Red and 12 Blue, and then return to (*) above and conduct another single battle simulation. Continue this process reporting casualties for Red and Blue for additional battles until you have conducted 10 battles from initial value (each) of 12 Red and 12 Blue.

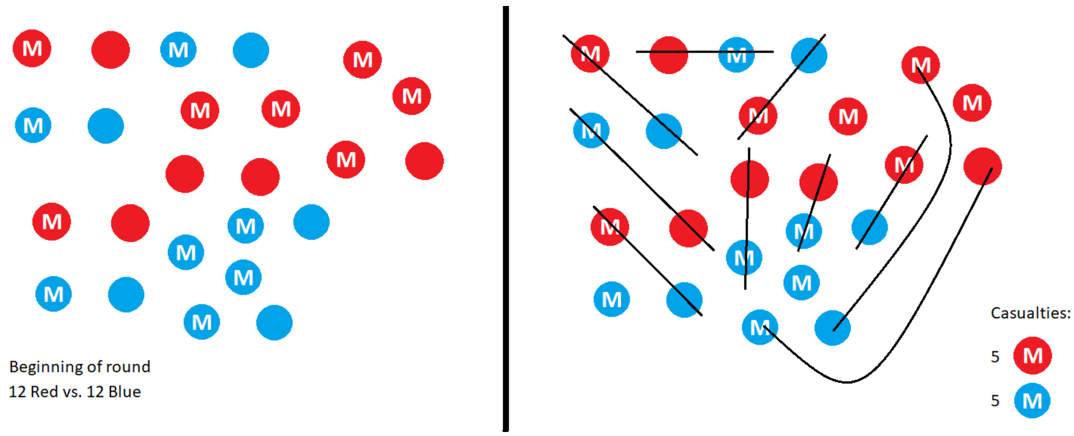

Figure 1. Illustration of casualties of "Battle 1" in Table 1

Table 2 and Table 3 suggest doing the same thing as done in generating entries in Table 1, only with different starting values for each battle, namely 16 Red and 16 Blue in Table 2 with 10 Red and 20 Blue in Table 3.

3. Average each column. Do the results appear significant? Offer an explanation for any anomalous results. Alter your initial hypothesis if necessary.
4. For each Red and each Blue column, compute

$$\rho = \frac{\text{Mean Blue casualties}}{\#\text{ Red units}} \text{ and } \beta = \frac{\text{Mean Red casualties}}{\#\text{ Blue units}}$$

5. Incorporate the constants ρ and β into your initial hypothesis to make an updated hypothesis.

   Updated hypothesis:

6. Average the values of ρ and β.

   - The parameter $\rho =$ represents the rate at which the number of Blue units is decreasing per unit of Red. It is the *fighting strength* of a typical ______ unit.

Battle	Red	Blue
0	12	12
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Mean		
	$\rho =$	$\beta =$

Table 1. Starting with 12 R and 12 B forces.

Battle	Red	Blue
0	16	16
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Mean		
	$\rho =$	$\beta =$

Table 2. Starting with 16 R and 16 B forces.

Round	Red	Blue
0	20	20
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Mean		
	$\rho =$	$\beta =$

Table 3. Starting with 20 R and 20 B forces.

- The parameter $\beta =$ represents the rate at which the number of Red units is decreasing per unit of Blue. It is the *fighting strength* of a typical ______ unit.
- The rate of change of the number of Red units is proportional to the number of the Blue units, or in the language of mathematics

$$\frac{dB}{dt} =$$

- The rate of change of the number of Blue units is proportional to the number of the Blue units, or in the language of mathematics

$$\frac{dR}{dt} =$$

Together, these are called the *Lanchester equations*.

Solving the system

Let $R(0) = R_0$ and $B(0) = B_0$ be the initial strengths of each force, respectively . Solve the system

$$\begin{cases} \frac{dR}{dt} = -\beta B \\ \frac{dB}{dt} = -\rho R \end{cases}$$

by observing that the opposing forces would be equally matched if the attrition rates were equal on both sides; i.e. set $\frac{dR}{dt} = \frac{dB}{dt}$ and integrate both sides. The solution,

$$\rho \left(R^2 - R_0^2 \right) = \beta \left(B^2 - B_0^2 \right)$$

gives rise to Lanchester's N-square Law:

> *The fighting strengths of two forces are equally matched when the ______ of the _________ strength multiplied by the fighting value of the _________ for the respective forces are equal.* [5, p. 2145]

A Lanchester model

We will analyze the battle of Trafalgar in terms of Lanchester's equations. For this exercise, assume that $\rho = \beta = 1$.

In a naval battle in 1805 off of Cape Trafalgar, Spain, 40 British vessels faced a combined French and Spanish force of 46 vessels. The British force was outgunned, yet under the brilliant command of Admiral Horatio Nelson the British devastated the opposing force. Nelson's plan was to split the enemy line into two halves by sending a vanguard of 8 vessels (see [5, pp. 2154-2157] or [1, pp. 25-27]).

- According to the N-square law, what is the expected outcome of 8 units fighting against 23?
- Answer:

- What advantage does a force of 32 have against 23?
- Answer:

- With this strategy, the British fighting strength of ____________ slightly tipped the French and Spanish strength of ____________ . Research the Battle of Trafalgar to find out whether adopting this strategy paid off.

M&M wargame scenario

Using the Lanchester model with one of the sets of parameters ρ and β you obtained from the averages of entries in Table 1, Table 2, or Table 3, analyze a scenario in which 20 Red M&M's face a stronger army of 26 Blue M&M's.

- Determine how Blue should engage Red to at least in part enjoy a 2 to 1 fighting strength advantage.
- Is it possible for Red to engage Blue in such a way to at least neutralize Blue's advantage? Explain in detail the optimal attack strategy for the Red force.

Duration of battle

The Lanchester equations may be solved for $B = B(t)$ and $R = R(t)$ through various methods. Applying methods of linear algebra, the solution to the matrix differential equation

$$\begin{pmatrix} R' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ -\rho & 0 \end{pmatrix} \begin{pmatrix} R \\ B \end{pmatrix}$$

may be obtained through the fundamental matrix. Thus, we have

$$\begin{pmatrix} R \\ B \end{pmatrix} = \begin{pmatrix} \cosh \sqrt{\beta\rho}\; t & -\frac{\beta}{\sqrt{\beta\rho}} \sinh \sqrt{\beta\rho}\; t \\ -\frac{\beta}{\sqrt{\beta\rho}} \sinh \sqrt{\beta\rho}\; t & \cosh \sqrt{\beta\rho}\; t \end{pmatrix} \begin{pmatrix} R_0 \\ B_0 \end{pmatrix}.$$

It follows that the solution to the Lanchester equations is

$$\begin{cases} R(t) = \cosh\left(\sqrt{\beta\rho}\; t\right) R_0 - \frac{\beta}{\sqrt{\beta\rho}} \sinh\left(\sqrt{\beta\rho}\; t\right) B_0 \\ B(t) = -\frac{\beta}{\sqrt{\beta\rho}} \sinh\left(\sqrt{\beta\rho}\; t\right) R_0 + \cosh\left(\sqrt{\beta\rho}\; t\right) B_0 \end{cases}$$

Then, if it is determined that, for example, Red will prevail over Blue, the duration of battle is the solution to the equation $B(t) = 0$, because the number of Blue units will be reduced to zero.

Returning to a round of sequential battles until one side loses

1. Use one of the sets of parameters ρ and β you obtained from the averages of entries in Table 1, Table 2, or Table 3 and the corresponding set of differential equations to model one round of a set of battles until the size of one side, Red or Blue, is reduced to 0. How many battles did it take until the size of one side, Red or Blue, is reduced to 0?
2. Fix ρ from one of the average sets of parameters in Table 1, Table 2, or Table 3 and let the value of β vary ($\rho < \beta$, $\rho = \beta$, or $\rho > \beta$. Using your differential equations, model one round of a set of battles until the size of one side, Red or Blue, is reduced to 0. How many battles did it take until the size of one side, Red or Blue, is reduced to 0. How does the duration of the round of battles relate to the value of β? Recalling the meaning of β as the fighting strength of a typical Blue unit, do the results you have obtained make sense? Explain.
3. For fixed ρ and given initial sizes for Red and Blue forces can you determine the value(s) of β for which neither side wins, or that both sides annihilate themselves.

> The concept of parity in combat modeling is important. We define *parity* as a fight to finish that ends in a draw—neither side wins. We assume that under the conditions of parity that both sides annihilate themselves (mathematically they both go to zero). No side wins and both sides lose." [1, p. 21]

4. In the explicit formulas $R = R(t)$ and $B = B(t)$ given above, what are the units of t? Does t represent time, the number of battles, or another parameter? Explain.

REFERENCES

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