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STUDENT VERSION

How High?! Modeling Free Fall with Air Drag

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Abstract: Most projectile motion and free fall models are based on the assumption that gravity is the only force acting on the object. Here we develop, solve, and analyze a second order nonhomogeneous differential equation model for free fall which incorporates air resistance. Students will solve the model using two different methods – reduction of order and separation of variables, and method of undetermined coefficients. Using the solution, students derive an expression for the terminal velocity of the object as well as a prediction of the maximum height of the object (assuming is is fired directly upwards). The model is then parameterized for a Nerf dart by an experiment performed by students. The terminal velocity and muzzle velocity of the dart is measured using video frames of the dart's motion. Finally, the model is validated by an experiment wherein students fire their darts upward and measure the assent time for comparisons with their predictions.

1 INTRODUCTION AND SCENARIO DESCRIPTION

Free fall refers to any motion of a body where gravity is the only force acting upon it. Galileo's law of free fall states that, in the absence of air resistance, all bodies fall with the same acceleration, independent of their

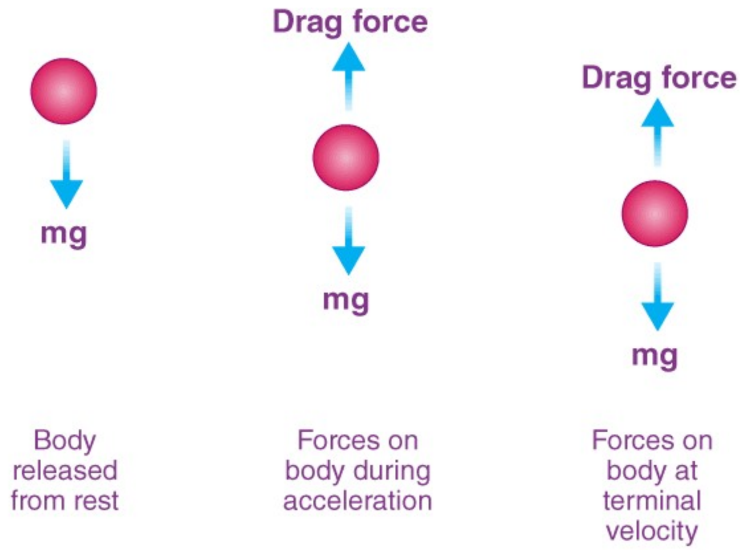


Figure 1. Force diagram of an object in free fall with air drag.

mass. See the following videos illustrating this concept:

– Hammer vs Feather: Physics on the Moon
  `https://www.youtube.com/watch?v=KDp1tiUsZw8`

– Brian Cox visits the world's biggest vacuum
  `https://www.youtube.com/watch?v=E43-CfukEgs`

Objects falling in a gas medium will experience what is called a *drag force*. Air drag, a type of friction, is a force acting opposite to the relative motion of any object moving with respect to the surrounding gas. Furthermore, the magnitude of the drag force increases in proportion (approximately) to the object's speed. Hence, for an object in free fall in Earth's atmosphere, for example, there are two forces acting on the object – gravity and air drag. If the motion is essentially straight down, then these two forces act in opposite directions (Figure 1).

For an object dropped from rest, gravity accelerates the objects causing it to pick up speed. However, as its speed increases, so does the drag force. Eventually, the magnitude of the drag force approaches that of the force of gravity. Given a sufficient amount of time, the two forces are essentially equal but in opposite directions resulting in zero net force. Hence, the object will continue to fall at a constant speed – its *terminal velocity*.

In this project we will predict the maximum height of a projectile (Nerf dart) subject to air resistance fired directly upwards. We first develop a differential equation governing the motion of such a projectile. We then solve the differential equation which contains parameters for air drag and muzzle velocity. Muzzle velocity of the firing apparatus (dart gun) is estimated experimentally using video frames of the dart being fired with a distance measure in the background. Similarly, we estimate the terminal velocity of the dart by firing it directly upwards and measuring its speed on its descent. The terminal velocity is then used to estimate the drag parameter. Using the parameterized model, we predict the time required for the dart to reach its maximum height. Finally, we test our prediction experimentally using a stopwatch to time the ascent of the dart.

2 Materials and Setup

- Several dart guns (one for each student group) with at least 3 darts each
- Tape measure
- Video camera
- VLC video editing software (free):
  `https://www.videolan.org/vlc/`

To parameterize the free fall model students must estimate the muzzle velocity of their dart gun and the terminal velocity of a dart using video footage of the dart fired horizontally (for muzzle velocity) and vertically (for terminal velocity). Videos should be shot with some way to measure distance travelled by the dart in the background. This can be accomplished with a tape measure affixed to a wall (horizontally for muzzle velocity and vertically for terminal velocity) or with a brick wall in the background. Video files can be analyzed with VLC video editing software wherein each frame is advanced by pausing the video and pressing the 'E' key. If using a brick wall for measuring distance, students will need to measure the width (for muzzle velocity) and thickness/height (for terminal velocity) of a brick.

Each group will need to determine the frame rate, i.e., frames per second (fps) for their video recording device. This can be found in the manual for the devise. A common frame rate is 30fps.

3 Differential Equation Model

Using Figure 1 we see that the net force on an object in free fall in the atmosphere is the sum of the force due to gravity and the force due to air resistance,

$$F_{\text{net}} = F_{\text{gravity}} + F_{\text{drag}}.$$

Using Newton's Second Law of motion and the assumption that a drag force is proportional to velocity (and in the opposite direction of gravity), we have

$$F_{\text{gravity}} = -mg$$

(assuming that up is the positive direction) and

$$F_{\text{drag}} = -\rho v$$

where m is the mass of the object, $g > 0$ is the acceleration due to gravity, v is the object's velocity, and ρ is the drag coefficient. Hence, we have

$$F_{\text{net}} = -mg - \rho v$$

Using Newton's Second Law on the left side of the equation, together with $v(t) = \frac{dy}{dt} = y'$ and $a(t) = \frac{d^2y}{dt^2} = y''$, where $y(t)$ represents the height of the object at time t, we have

$$my'' = -mg - \rho y'.$$

With $k = \frac{\rho}{m}$, we have

$$y'' = -g - ky'. \tag{1}$$

4 Practice Problem

Before proceeding to solve (1), we will solve a similar problem without parameters:

$$y'' = -5 - 2y'$$

- **Method 1**:

  1. Let $v = y'$ and rewrite the equation in terms of v and v'.
  2. Find the general solution of the resulting differential equation for $v(t)$. Note that $v' = \frac{dv}{dt}$.
  3. Use $v(0) = 10$ to find the particular solution.
  4. Find $\lim_{t\to\infty} v(t)$.
  5. Graph the solution and verify your answer from part 4.
  6. Using $y' = v$ and your solution for $v(t)$ (from part 3), find the general solution $y(t)$.
  7. Find the particular solution for $y(t)$ if $y(0) = 0$.
  8. Graph the particular solution $y(t)$.
  9. Determine the maximum value of $y(t)$ on $[0, \infty)$ and verify your answer graphically.

- **Method 2**:

  1. Find the general solution of $y'' = -5 - 2y'$ using the method of undetermined coefficients (UC). (Hint: Use a linear function for the particular solution of the UC method.)
  2. Using $y(0) = 0$ and $y'(0) = 10$, find the particular solution of $y'' = -5 - 2y'$.
  3. Determine the maximum value of $y(t)$ on $[0, \infty)$ and verify your answer graphically.
  4. Let $v = y'$ and find $\lim_{t\to\infty} v(t)$.
  5. Graph $v(t)$ and verify your answer from part 4.

5 Solving the Free-fall with Air Drag Problem

We now use the technique(s) of the previous section to solve Equation (1).

- **Method 1**:

  1. Let $v = y'$ and rewrite (1) in terms of v and v'.
  2. Find the general solution of the resulting differential equation for $v(t)$. Note that $v' = \frac{dv}{dt}$.
  3. Use $v(0) = v_0$ to find the particular solution.
  4. Find $\lim_{t\to\infty} v(t)$ (in terms of model parameters). Denote this value by v_T. What does this value represent in the context of free-fall?
  5. Graph the solution (use $g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2}$). Verify your answer from part 4.
  6. Using $y' = v$ and your solution for $v(t)$ (from Part 3), find the general solution $y(t)$.
  7. Find the particular solution for $y(t)$ if $y(0) = 0$.
  8. Graph the particular solution $y(t)$.

- **Method 2**:

  1. Find the general solution of (1) using the method of undetermined coefficients (UC). (Hint: Use a linear function for the particular solution of the UC method.)
  2. Using $y(0) = 0$ and $y'(0) = v_0$, find the particular solution of Equation (1).
  3. Let $v = y'$. Graph $v(t)$.
  4. Find $\lim_{t\to\infty} v(t)$ (in terms of model parameters). Denote this value by v_T. What does this value represent in the context of free-fall? Verify your prediction using the graph of $v(t)$.

6 Estimating Terminal and Muzzle Velocity (Model Parameterization)

- **Terminal Velocity**: We experimentally estimate the terminal velocity of the dart using video frames of the dart being fired vertically with a distance measure in the background. We measure the dart's velocity on its descent and assume this is its terminal velocity. The example and images below illustrate this process.

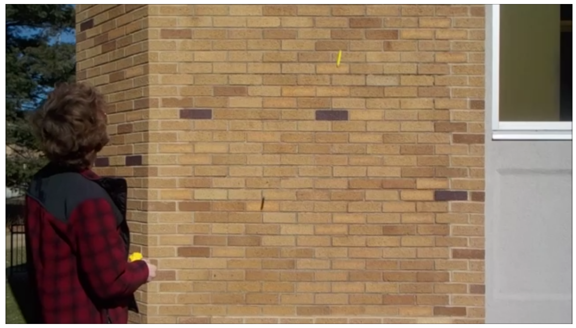


Figure 2. Video frame of Nerf dart on its descent. We note its position relative to the brick wall in the background.

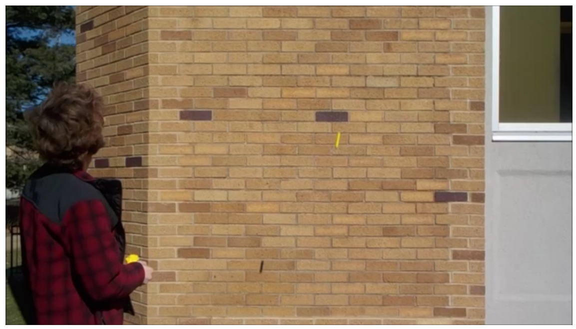


Figure 3. Next video frame of Nerf dart on its descent. We note its position relative to the brick wall in the background. We see the dart has travelled approximately 7 bricks since the last frame. Since the frame rate for the video was 30fps, we note that the time between frames is 1/30 seconds.

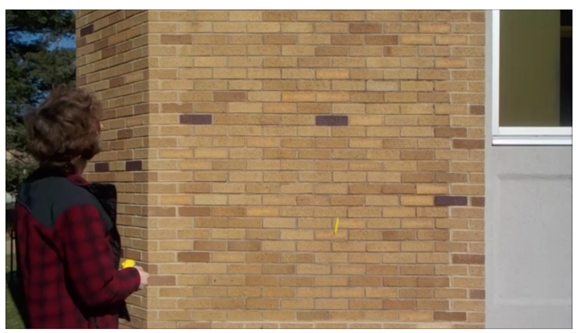


Figure 4. Next video frame of Nerf dart on its descent. We note its position relative to the brick wall in the background. We see the dart has travelled approximately 7 more bricks since the last frame indicating that the dart has reached its terminal velocity.

In each frame, the dart fell approximately 7 brick heights (75 mm including mortar). The frame rate of the video was 30fps. Hence, the dart fell $7 \times 75\text{mm} = 525\text{mm}$ in $\frac{1}{30}$ second. This results in a terminal velocity estimate of $v_T \approx \frac{525}{\frac{1}{30}} = 15,750\frac{\text{mm}}{\text{sec}} = 15.75\frac{\text{m}}{\text{sec}}$. See the following video for an illustration of this process:

– `https://www.youtube.com/watch?v=H2A2I_UbvKA`

1. Position a video camera (with vertical frame) approximately 6 to 8 feet from a wall with a tape

measure affixed to it vertically.

*Alternatively, use a brick wall with known brick height for measuring distance.

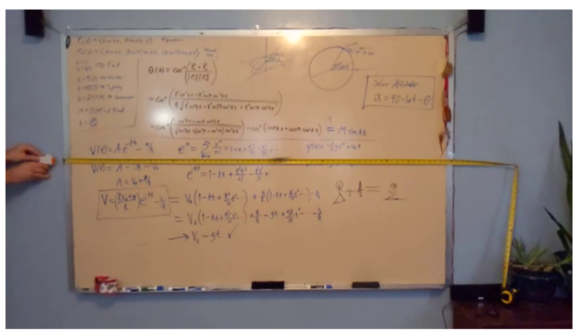
2. While recording video, fire the dart vertically upwards. Make sure the flight of the dart is fairly smooth and that its descent is captured in the frame of the video.

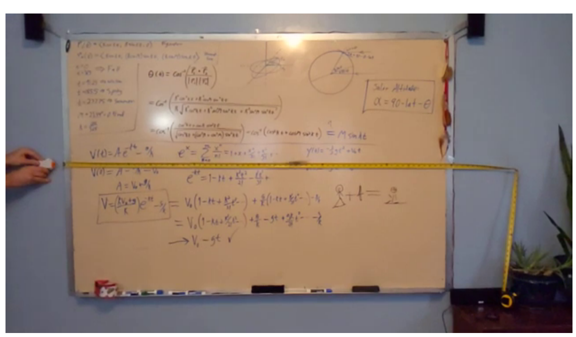
   *Alternatively, drop the dart from a 3rd or fourth story window so that it falls along the brick wall and into the frame of the video at the bottom.

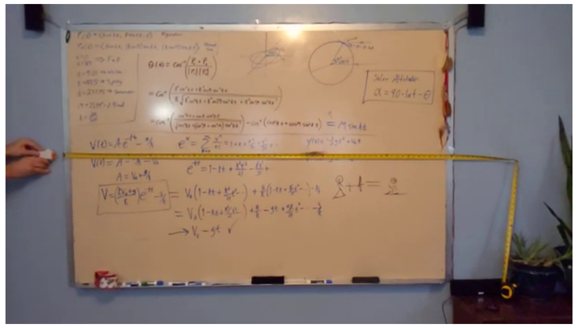
3. Play the video back to determine how far the dart fell between each frame (for the first 2-3 frames of the video). This distance should be roughly constant. Average the distances. Using the frame rate of the video camera and the average distance, calculate the velocity of the dart – this is the terminal velocity, i.e., v_T.

4. Use your experimental estimate of v_T (from the previous part) in your prediction of v_T (from part 4 of section 4) to approximate the value of k.

- **Muzzle Velocity**: We experimentally estimate the muzzle velocity of the firing apparatus (dart gun) using video frames of the dart being fired horizontally with a distance measure in the background. The example and images below illustrate this process.







In each frame, the dart travelled approximately 7.6 cm. The frame rate of the slow motion video was 240fps. Hence, the muzzle velocity of the dart is approximately $18.25\frac{\text{m}}{\text{sec}}$. See the following video for an illustration of this process:

– `https://www.dropbox.com/s/ufd1nb4i6io1vqs/Muzzle2.mp4?dl=0`

1. Position a video camera (with horizontal frame) approximately 6 to 8 feet from a wall with a tape measure affixed to it horizontally. Alternatively, use a brick wall with known brick length for measuring distance.
2. While recording video, fire the dart horizontally from the zero mark on the tape measure. Make sure the gun is on the left side of the frame and is fired to the right.
3. Play the video back to determine how far the dart travelled in the first 2-3 frames of the video (this should be roughly constant). Average the distances. Using the frame rate of the video camera and the average distance, calculate the velocity of the dart – this is the muzzle velocity, i.e., v_0.

7 Predicting Ascent Time (Model Validation)

1. Use your solution from section 4 to determine the ascent time of the dart (time required for the dart to reach maximum height) in terms of the model parameters g, k, and v_0.
2. Use your formula for v_T (from section 4) to write the ascent time prediction in terms of g, v_0, and v_T.
3. Verify your ascent time prediction graphically using the determined values for v_0 and v_T (use $g = 9.8\frac{\text{m}}{\text{s}^2}$).
4. Test your prediction of ascent time by timing (with a stopwatch) the actual ascent of the dart. This should be done outside, in a stairwell, or a room with a very high ceiling.
5. Calculate the relative error of your prediction:

$$\text{relative error} = \frac{\text{observed value} - \text{predicted value}}{\text{observed value}} \times 100\%$$

8 Equivalent Model with no Air Resistance

If we neglect air drag in our model, Equation 1 simplifies to

$$y'' = -g \tag{2}$$

Here we show that if the effect of air drag is less and less, i.e., $k \to 0$, the solution to Equation (1) becomes the solution to Equation (2).

1. Solve Equation (2) using $v(0) = v_0$ and $y(0) = 0$.
2. Graph the solutions (v and y) of both Equations (1) and (2) on the same coordinate axis. Let k approach 0 using a slider. What is the interpretation of this limit? What do you observe in the graphs?
3. To show that the solution of Equation (1) is equivalent to the solution of Equation (2) if we neglect air resistance, we can let k approach 0 in the solution of Equation (1). What do you observe upon attempting this limit?
4. Use the Taylor series expansion for e^x to expand any exonential terms in your solution of Equation (1) for velocity $v(t)$ from section 4.
5. Substitute this expansion into the velocity solution of Equation (1) and simplify.
6. Now let $k \to 0$.
7. Compare the result to your velocity solution of Equation (2).
8. Graph