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STUDENT VERSION

Analyzing an Efficient Wireless Power Transmission System

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Abstract: This activity presents an engineering application that is modelled by a coupled system of two linear second-order differential equations with constant coefficients. One of the equations is homogeneous while the other one is non-homogeneous. The application is called wireless power transmission (WPT), which belongs to a bigger class of engineering technology that does not utilize wires or cables to transfer power across space. The students will be guided in modelling WPT and the resulting system is most efficiently solved using Laplace transforms method.

The Story

You woke up this morning with a terrible headache. You have been studying for your final unit exam in your Differential Equations class where you have been learning about Laplace transforms. The theory is fine and neat and mumbled to yourself "I am sure glad I listened to my Calculus II professor when we talked about improper integrals..." but you still feel uneasy because they all seem so ...abstract. "Well, my professor said that we will talk about an interesting engineering application today so I better get ready!" You hopped off your bed because you get excited every time your professor talks about electric circuits. As a future engineer, you like applications that involve tinkering with the components of a circuit or designing circuit diagrams and simulating them with online circuit simulators. You are secretly hoping that your professor will somehow make connections between Laplace transforms and electric circuits. "Hmmm ...maybe I can make an Honors project about differential equations and circuits ..." You went to the bathroom and grabbed your electric toothbrush that has been sitting overnight in its charging base...

SCENARIO DESCRIPTION

Wireless power transmission (WPT) happens when electrical energy is transmitted without the use of cables or wires. A wireless charger, the most popular application of WPT, allows one to charge

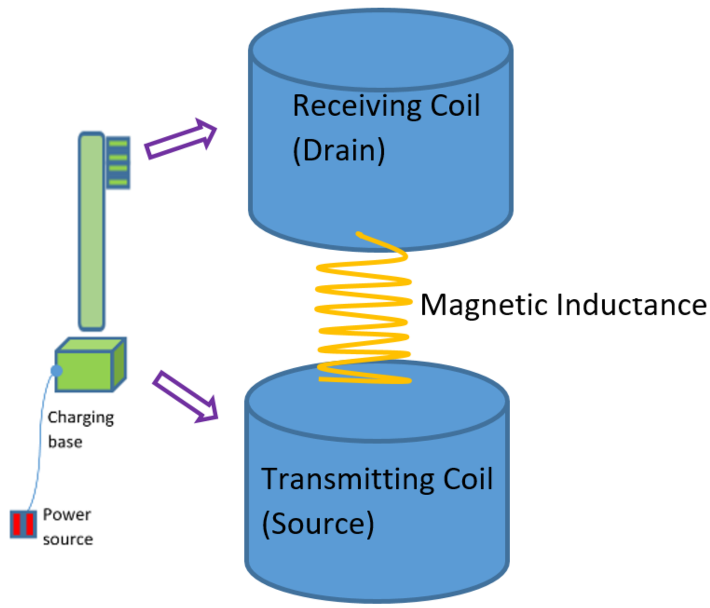

Figure 1. A schematic diagram of an electric toothbrush as a WPT engineering application.

cellphones or toothbrushes without the use of any cable or plug. Do you know that your training in differential equations up to this point gives you the power to model the current flowing from the power source to your device, through the charging base? See Figure 1 for a simplified diagram of an electric toothbrush, viewing it with modelling in mind.

In this learning activity, you will model and explore a WPT system by coupling two second-order differential equations. This activity is interesting because it gives you a chance to solve an engineering application via the Laplace transform method. Although current research analyzes systems of differential equations using sophisticated computer simulations, it is still important to analyze such systems of equations step-by-step. As a bonus, this application is something that you actually see in your home.

An equivalent circuit for a WPT system is illustrated in Figure 2. An equivalent circuit is a theoretical, highly idealized circuit that captures and retains the most important electrical components and characteristics of a circuit. Actual circuits for an electric toothbrush and its charging base are more complicated. The electric circuit on the left-hand side of Figure 2 corresponds to the electric circuit of the transmitting coil (source) of Figure 1 while the electric circuit on the right-hand side of Figure 2 corresponds to the electric circuit on the receiving coil (drain) Figure 1.

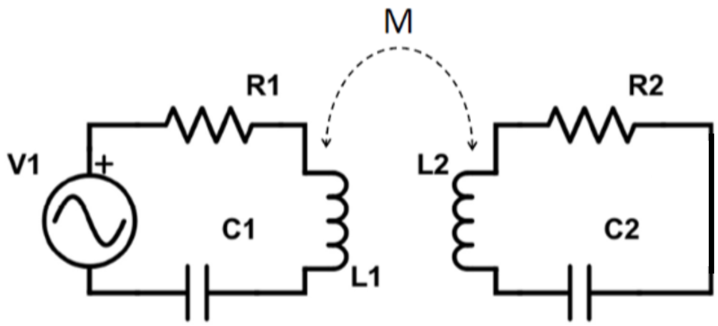

Figure 2. An equivalent circuit of a WPT system

Part 1. Creating the mathematical model.

The diagrams in Figure 1 and Figure 2 should help in writing the systems of differential equations for WPT. Observe that these two pictures have visually led you from the physical realm (the electric toothbrush) to the electrical world (the equivalent circuit) and now it is time to go the mathematical world! Towards the goal of writing the model, let i_1, i_2 be the currents in the first and second loops; L_1, L_2 be the inductors in the first and second loops; R_1, R_2 be the resistors in the first and second loops; C_1, C_2 be the capacitors in the first and second loops; and V_1 be the voltage power source. In the meantime, do not worry about the broken arrow in Figure 2 which is labelled M.

Assume that the transmitting coil is an RLC-series electrical circuit. Recall from your modeling experience (for example, Chapter 5 in textbook [6]), that an RLC-series electric circuit may be written as a linear second-order differential equation with constant coefficients. Then the left circuit in Figure 2 gives us the equation

$$L_1 \frac{d^2 i_1(t)}{dt^2} + R_1 \frac{di_1(t)}{dt} + \frac{1}{C_1} i_1(t) = \frac{dV_1(t)}{dt}. \tag{1}$$

Now, since V_1 is an electric power source (also called alternating-current (AC) source), V_1 is typically of the form $A\cos(mt) + B\sin(mt)$ and so $\dfrac{dV_1(t)}{dt}$ is nonzero and hence (1) is a non-homogeneous differential equation.

Next, let us look at the right circuit in Figure 2. Analogously, we have

$$L_2 \frac{d^2 i_2(t)}{dt^2} + R_2 \frac{di_2(t)}{dt} + \frac{1}{C_2} i_2(t) = 0 \tag{2}$$

and unlike the first equation, this is a homogeneous linear second-order differential equation with constant coefficients. In other words, the transmitting coil (source) is non-homogeneous while the receiving coil (drain) is homogeneous.

It is time to couple our two equations to form our system. Observe that the two electrical circuits are linked by the mutual inductance M. By Ampere's Law, the flow of the electric current in the left circuit will create a magnetic field. Since the two electric circuits are linked by M and that the field is time-varying, Faraday's Law predicts the creation of a voltage within the second circuit. Although this occurs to some extent in all circuits, the effect is magnified in coils because the circuit geometry amplifies the linkage effect.

Mathematically, we capture the effect of magnetic inductance into the two circuits by adding the term $M\dfrac{d^2 i_2(t)}{dt^2}$ to the left-hand side of (1) and adding the term $M\dfrac{d^2 i_1(t)}{dt^2}$ to the left-hand side of (2). This magnetic inductance is the key to the engineering and mathematics behind wireless power transmission.

The wireless power transmission system can be modelled by:

$$\begin{cases} L_1\dfrac{d^2 i_1(t)}{dt^2} + M\dfrac{d^2 i_2(t)}{dt^2} + R_1\dfrac{di_1(t)}{dt} + \frac{1}{C_1}i_1(t) = \dfrac{dV_1(t)}{dt} \\ L_2\dfrac{d^2 i_2(t)}{dt^2} + M\dfrac{d^2 i_1(t)}{dt^2} + R_2\dfrac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) = 0 \end{cases} \tag{3}$$

subject to the initial conditions $i_1(0), i_2(0), i_1'(0), i_2'(0)$. What we have presented here is a slightly different perspective of modelling the WPT circuit; please refer to [1] for another way to derive the system (3) via first principles, using Kirchoff's Voltage Law.

You have learned how to solve homogeneous and non-homogeneous, linear differential equations with constant coefficients such as (1) and (2), respectively. For example, you may want to review the method of undetermined coefficients and variations of parameters [6]. Also, refer to [2] for a review on how to analyze a second-order differential equation arising from an RLC-series electric circuit.

Let us consider a simpler version of the system by assuming that we have a loss-less circuit, that is, let us assume that $R_1 = R_2 = 0$. This is a simple way to analyze the efficiency of a circuit because there is no resistance component. For example, the discussion in [5] starts with a similar analysis and investigates the efficiency of a WPT via transitionless quantum driving.

$$\begin{cases} L_1\dfrac{d^2 i_1(t)}{dt^2} + M\dfrac{d^2 i_2(t)}{dt^2} + \frac{1}{C_1}i_1(t) = \dfrac{dV_1(t)}{dt} \\ L_2\dfrac{d^2 i_2(t)}{dt^2} + M\dfrac{d^2 i_1(t)}{dt^2} + \frac{1}{C_2}i_2(t) = 0 \end{cases} \tag{4}$$

The voltage power source function $V_1(t)$ is usually assumed to be of the form $A\cos(mt) + B\sin(mt)$ to account for the alternating current AC voltage source (connected to a power outlet). Note that for the sake of practice, one of the exercises below assumes that $V_1(t)$ is a constant.

Upon taking the Laplace transforms of the system (4), we get

$$\begin{pmatrix} L_1 s^2 + \frac{1}{C_1} & Ms^2 \\ Ms^2 & L_2 s^2 + \frac{1}{C_2} \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \mathcal{L}\{V_1'(t)\} \\ 0 \end{pmatrix}. \tag{5}$$

Then, using Cramer's Rule on (5) to solve for $I_1(s)$ and $I_2(s)$, we have

$$\begin{aligned} I_1(s) &= \frac{(L_2 s^2 + \frac{1}{C_2})\mathcal{L}\{V_1'(t)\}}{(L_1 s^2 + \frac{1}{C_1})(L_2 s^2 + \frac{1}{C_2}) - M^2 s^4}, \\ I_2(s) &= \frac{-Ms^2\mathcal{L}\{V_1'(t)\}}{(L_1 s^2 + \frac{1}{C_1})(L_2 s^2 + \frac{1}{C_2}) - M^2 s^4} \end{aligned} \tag{6}$$

In one of the exercises below, you will be asked to provide the details in deriving the system of linear equations (6) using Cramer's Rule. Finally, the solutions are given by the inverse Laplace transforms of these functions.

Analyzing an Efficient Wireless Power transmission: Activity

The goal of the first two exercises below is to provide you with some opportunities to learn more about WPT. The next three exercises require that you analyze WPT using your knowledge of Laplace transforms. To solve the problems, make sure that you have a table of Laplace transforms next to you. Suppose $I_k(s) = \mathcal{L}\{i_k(t)\}$ for $k = 1, 2$ and $s > 0$. In engineering courses, the functions $i_k(t)$ and $I_k(s)$ are usually called currents expressed in the time and frequency domains, respectively. Assume that $i_1(0) = i_2(0) = i_1'(0) = i_2'(0) = 0$.

1. *Relating WPT to another electromagnetic induction application.* Use the internet to learn about electricity transformers. In your exploration, you will see that a transformer is based on a very simple fact about electricity: when a time-varying power source enters a circuit, it generates an oscillating magnetic field which, in turn, generates an electric current that enters another circuit. Observe that this concept is similar to the basic principle of a WPT. Find out what the difference is between a WPT and an electricity transformer. What are the equations that govern an electricity transformer?
2. *Relating WPT to TET.* Another engineering wonder that uses the same principle as WPTs and transformers is transcutaneous energy transmission (TET). This concept is used to recharge biomedical prosthetic devices that are implanted in the human body, for example, artificial hearts, pacemakers, and insulin pumps. Read the paper [6] to see that TETs are governed by electromagnetic induction, too. Moreover, based on your reading, what are some practical issues regarding TETs when used by patients?
3. *Analytical solution of the WPT system.* Provide the details to show how the matrix equation (5) is obtained from (4). Then, write out the details to show how (6) is obtained from (5).
4. *Thinking about a mathematical WPT system.* For a first experience in analyzing the system (4), assume that $V_1(t)$ is a constant value K. In electrical engineering, this assumption happens when the voltage power or input is a DC source. (Note, however, that a real WPT system requires a time-varying source by virtue of Faraday's Law.)
   (a) Assuming $V_1(t) = K$ for some constant K, write the two equations in the WPT system.
   (b) Write the solutions to the system (4) using the equations in (6) by taking the inverse Laplace transforms.
   (c) Observe that the inverse Laplace Transform of zero does not appear in your table of Laplace transforms. What property of the inverse Laplace transforms guarantees that the inverse Laplace of zero is also zero?
5. *Computing and graphing solutions to a WPT example.* Analyze the wireless power trans-

Table 1. Table of values for Exercise 5.

Voltage Source $V_1(t)$	$\sin(t)V$
Inductor L_1	$1H$
Capacitor C_1	$1F$
Inductor L_2	$1H$
Capacitor C_2	$1F$
Mutual Inductor M	$1H$

mission system whose component values are given in Table 1.

(a) Sketch a circuit diagram that corresponds to this WPT system. Write the corresponding system of differential equations.

(b) Write the solutions of the system you developed in (a). You may find [8] useful in checking your work.

(c) Sketch the solutions to the system on one coordinate plane. You may use [4] or other graphing software.

(d) Analyze the graph of the solutions that you obtained in (c) using the following guide questions. For exploration purposes, choose any closed interval that you see in your graph. For example, the interval $[0, 4\pi]$.

i. What is the maximum amplitude of $i_1(t)$ and $i_2(t)$ in this interval ?

ii. Find the points of intersection between $i_1(t)$ and $i_2(t)$ for $t \in [0, 4\pi]$.

iii. What are the zeros of $i_1(t)$ and $i_2(t)$?

iv. From the electrical circuit perspective, why does it make sense that the current $i_1(t)$ is a linear combination of two cosines while the the current $i_2(t)$ is a single cosine function?

Now that you have learned about wireless power transmission system, you realize that mathematics is indeed everywhere, even in your bathroom! Every time you use your electric toothbrush or put your cellphone on its charging base, you now know that the underlying principle for such technology is a system of second-order differential equations that you can solve using Laplace transforms methods. Back to your terrible headache this morning, be sure to visit simiode.org to find that activity that lets you model how a medicinal pill is dissolved into your bloodstream. Yes, there are applications of differential equations in your first-aid box!

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