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STUDENT VERSION
Modeling Populations with Death and Immigration Using $m\&m$s

Brian Winkel, Director SIMIODE
Chardon OH 44024 USA
BrianWinkel@simiode.org

Abstract: We describe a classroom activity in which students use $m\&m$ candies to simulate changes in a population due to death and immigration. Students collect data, build a mathematical model — usually a linear, first-order, difference or differential equation, estimate parameters, and compare their model prediction with their actual data.

SCENARIO DESCRIPTION

For this experiment you will need a regular size bag of $m\&m$ candies, not peanut, just regular. There are usually about 55 pieces in each bag. The candies are of different colors, but (almost) every piece has an "m" pressed on one side and not the other, and so we can distinguish between the two sides of each piece of candy. Let us conduct two experiments (life and death experiments) on the $m\&m$s.

In each experiment, we will shake the $m\&m$s from a paper cup onto a paper plate. Those that land m side up die and are set aside; for the others, there is life! They will go back into the cup for the next iteration.

A population with death but no birth or immigration

For this simulation in which $m\&m$s either live or die, open the bag of candy put up to 50 of the 55 or so pieces (using 20-25 $m\&m$s will make counting faster and get across the concepts) into a small container; a paper cup will do.

a) Describe what you think will happen in each toss.
b) Offer some assumptions in support of your description in (a) as well as some which might not have anything to do with what you expect to happen. Often, assumptions are run into each other, so break the assumptions down to their simplest form. Do not use words like "and."

The simulation (death only) Select a starting number of $m\&m$s, say 20, and place that number of $m\&m$s in a container and also write that number in Table 1 as the number of $m\&m$s at start of Iteration 0.

c) In Table 1 record what happened and compare this information with what you thought and described would happen in (a) above.

Modeling Death of $m\&m$s	
Iteration	# $m\&m$s at start of iteration
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Table 1. Modeling death of $m\&m$s.

Now, gently shake the $m\&m$s out onto the desk (you might want to use a paper plate to catch the $m\&m$s and keep them clean as well). Any m&m with an m showing on top has died. Remove them and save them to the side for later. The remaining $m\&m$s have survived. Count the survivors and write down that number of $m\&m$s who survive to the next iteration in Table 1 to start the next generation and place the surviving $m\&m$s back into the container to begin the next iteration. Repeat ten times and record your results in Table 1.

d) Indicate which of your assumptions from (b) were reasonable and played a relevant role in the experiment.

e) Make a predication as to what will happen in the future; long term.

Building a mathematical model of the population (death only)

Let us build a mathematical model of this situation. Suppose the number of $m\&m$s alive at the start of iteration n is $a(n)$, where n is a whole number between 0 and 10. What values would n

take? What is $a(0)$? What was $a(1)$ in your simulation?

Based on your observations and your assumptions, produce a reasonable formula for $a(n)$, with $n = 0, 1, 2, \ldots$.

f) In what ways is your function model for $a(n)$ unreasonable and in what ways is it reasonable?
g) How will you measure whetcher your model of this situation is successful? (What does it mean for a model to be successful?)
h) Compare your model with others in the class.
i) Just how good is your model at predicting the results of your experiment. Defend your rationale.

A population with death and immigration

Each of us will conduct this simulation in which $m\&m$s either live or die, but this time we will permit immigration to recover the damage to the m&m population from death.

The Experiment: Gently shake the $m\&m$s out onto the desk. You might want to use a paper plate to catch the $m\&m$s and keep them clean as well. Then determine for each $m\&m$ if it lives or dies. Remember, if the m shows on top this $m\&m$ dies, otherwise there is life for this $m\&m$. Upon death you should remove the m&m from the population (set these aside as we will need these for another experiment).

And now for this very important change:

> **VERY IMPORTANT**: After returning the surviving $m\&m$s to the cup, add 10 immigrants from your "death pile". Repeat this death and immigration step for ten iterations and record your results in Table 2.

j) State your assumptions about the physical activity.
k) Offer up a description of what should happen.
l) In Table 2 record what happened and compare this with what you thought would happen.
m) Compare your description/prediction with what actually happened.
n) Indicate which of your assumptions were reasonable and played a role in the experiment.

Building a mathematical model of the population (death and immigration)

Let us build a mathematical model of this situation. Suppose the number of $m\&m$s alive at the start of iteration n is $b(n)$, where n is a whole number between 0 and 10. What values would n take? What is $b(0)$? What was $b(1)$ in your simulation?

Based on your observations and your assumptions, produce a reasonable formula for $b(n)$, with $n = 0, 1, 2, \ldots$.

Modeling Death of $m\&m$s	
Iteration	# $m\&m$s at start of iteration
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Table 2. Modeling death and immigration of $m\&m$s.

o) In what ways is your function model for $b(n)$ reasonable? In what ways is it unreasonable?

p) How will you measure whether your model of this situation is successful? (What does it mean for a model to be successful?)

q) Compare your model with others in the class.

r) Just how good is your model at predicting the results of your experiment. Defend your rationale.

s) Compare the two processes and the results of the two simulations: (1) death and (2) death with immigration. What was different about these models? Explain.

Modeling change

Finding a closed-form solution for the death and immigration model, $b(n)$, is difficult. (In a closed-form solution we can substitute the independent variable, n, into a forumla that depends only on n and determine $b(n)$.) Instead, let us think of how we get from generation to generation. At each iteration we lose about one-half of the $m\&m$s (why?), but then 10 $m\&m$s immigrate into the population as immigrants. What does the $(n+1)^{\text{th}}$ generation, $b(n+1)$, look like in terms of the n^{th} generation, $b(n)$? Try to predict $b(n+1)$ in terms of $b(n)$ and immigration with a formula:

$$b(n+1) = \underline{\hspace{6cm}}.$$

What was easy and what was hard in deciding what to put on the right hand side of the expression above? It is worth pausing here to savor the experiences of discovery and frustration in this modeling activity and to learn from them. Reflect for a moment on what you have accomplished and what it took to build a mathematical model.

Now that you have spent time reflecting on your model let us ask ourselves: is it the only reasonable possibility?

Here are several mathemaical models that *do not* work. Explain what does not work in each one; is not good about each one. Put each of these ill-conceived models into words and test them against your understanding of the physical process of m&m tossing and migration.

$$\begin{aligned}
b(n+1) &= 10 \cdot b(n) + .5 \\
b(n+1) &= 0.5 \cdot 10 \cdot b(n) \\
b(n+1) &= 0.5 \cdot b(n+1) + 10 \\
b(n+1) &= -0.5 \cdot b(n) + 10 \\
b(n+1) &= b(n+1)^n + 10 \\
b(n+1) &= (b(n+1) + 10)^n + 10
\end{aligned}$$

Let us say out loud to ourselves what we want the function to do: "The value of $(n+1)^{\text{st}}$ iteration, $b(n+1)$, is one-half that of $b(n)$ (due to death) plus 10 (due to immigration)." Does that sentence translate directly into mathematics for you? Does (1) look like a mathematical translation of that sentence?

$$b(n+1) = 0.5 \cdot b(n) + 10\,. \tag{1}$$

Another way to describe the population is to model the *change* at each generation rather than the new population itself. We can do this by subtracting $b(n)$ from each side of (1) to obtain the equation:

$$b(n+1) - b(n) = 0.5 \cdot b(n) - b(n) + 10 = -0.5b(n) + 10\,. \tag{2}$$

Equation (2) is called a *difference equation*. Now translate (2) into a sentence.

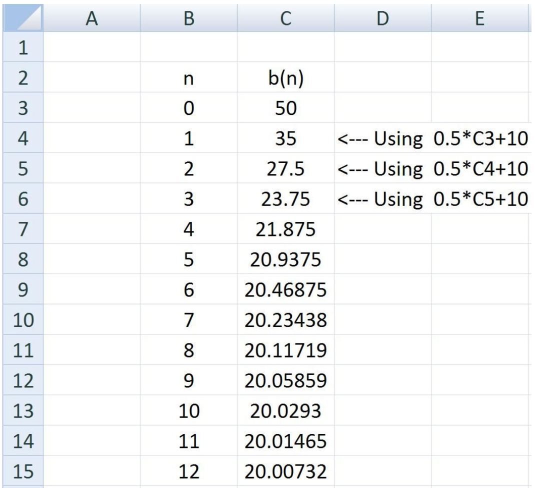
Often we use the symbols $\Delta b(n)$, read "change in $b(n)$", for $b(n+1) - b(n)$. Thus (2) could be written as:

$$\Delta b(n) = -0.5b(n) + 10\,. \tag{3}$$

In all three equations, (1), (2), and (3), we need to tell our model where to start. In these cases we might have $b(0) = 50$ or, say, $b(0) = 25$, the number of $m\&m$s we had in the cup at the beginning of the experiment. We can consider $b(0) = 50$ to be part of the model, and view (1) and $b(0) = 50$ as a pair of equations called, a *difference equation with initial condition.* We can do the same for (2) and (3). The expression in $b(n)$ on the right hand side of (3) is linear, suggesting that we call such a difference equation *linear*. So, our death and immigration activity is a *first-order, linear difference equation with initial condition.*

Solving our first-order, linear difference equation with initial condition

Before we try to solve the difference equation, let us think about the long-term behavior of our model. One way to this is to use a spreadsheet program, such as *Excel*, to iterate (1). In Figure 1 what can you see about the long term behavior of the model? Does it match what we saw in our experiment?



	A	B	C	D	E
1					
2		n	b(n)		
3		0	50		
4		1	35	<--- Using	0.5*C3+10
5		2	27.5	<--- Using	0.5*C4+10
6		3	23.75	<--- Using	0.5*C5+10
7		4	21.875		
8		5	20.9375		
9		6	20.46875		
10		7	20.23438		
11		8	20.11719		
12		9	20.05859		
13		10	20.0293		
14		11	20.01465		
15		12	20.00732		

Figure 1. Using *Excel* spreadsheet to iterate (1).

The powerful `RSolve` command in *Mathematica*

```
RSolve[{b[n + 1] == 1/2 b[n] + 10, b[0] == 50}, b[n], n]
```

,

provides a solution for $b(n)$, namely,

$$b(n) = 20 + 15 \cdot 2^{(1-n)} .$$

We note that the long term behavior seen in our Table 3 agrees with the iterations of (1) using *Excel* displayed in Figure 1.

Another approach is to conjecture a solution, say of the form $b(n) = k_1 + k_2 2^{(1-n)}$, and then use the initial condition and the first iteration to determine k_1 and k_2.

Moving to a differential equation model

Equation (2) describes the change for discrete steps. From calculus, we know quite a bit about how to model change in continuous functions using derivatives which is repeated below.

$$b(n+1) - b(n) = -0.5 \cdot b(n) + 10 . \tag{4}$$

In the simulation we used step sizes of 1 by going from n to $n+1$. Here 1 could be 1 generation or 1 year; it would certainly make sense to measure the population more frequently. Suppose we

`RSolve` solution of (1)	
n	$b(n)$
0.	50.0000
1.	35.0000
2.	27.5000
3.	23.7500
4.	21.8750
5.	20.9375
6.	20.4688
7.	20.2344
8.	20.1172
9.	20.0586
10.	20.0293

Table 3. Using *Mathematica*'s `RSolve` command to solve (1).

measured in step sizes of 0.01. Then the change in n, Δn, is 0.01. In this case we would not lose half of our population. Instead, we would lose a fraction of that half of our population, namely $\Delta n \cdot 0.5 = 0.01 \cdot 0.5 = 0.005$, every step. Similarly, our population would increase due to immigration by $\Delta n \cdot 10 = 0.01 \cdot 10 = 0.1$, every step.

What assumptions are we making about the process when we say these things? These adjustments yield a new equation, where again $b(n)$ is the size of the population at time n:

$$b(n + 0.01) - b(n) = -0.005 \cdot b(n) + 0.1\,. \tag{5}$$

More genereally, for a time step size of Δn we get:

$$b(n + \Delta n) - b(n) = -(\Delta n) \cdot 0.5 \cdot b(n) + \Delta n \cdot 10\,. \tag{6}$$

As in calculus, we could measure our population changes over smaller and smaller time intervals, Δn. Indeed, by dividing both sides of in (6) by Δn, we obtain

$$\frac{b(n + \Delta n) - b(n)}{\Delta n} = -0.5 \cdot b(n) + 10\,. \tag{7}$$

Look familiar?

Letting Δn approach 0 on both sides of (7), we get the definition of the derivative of $b(n)$ with respect to n, namely,

$$\frac{db}{dn} = \lim_{\Delta n \to 0} \frac{b(n + \Delta n) - b(n)}{\Delta n} = -0.5 \cdot b(n) + 10\,. \tag{8}$$

Equation (8) is called a *differential equation*. If we also include an initial condition, in this case, $b(0) = 50$, we have in (9) what is called an *initial value problem (IVP)*:

$$\frac{db}{dn} = \lim_{\Delta n \to 0} \frac{b(n + \Delta n) - b(n)}{\Delta n} = -0.5 \cdot b(n) + 10\,, \quad b(0) = 50\,. \tag{9}$$

It is conventional to use t for time-dependent differential equations. Rewriting our differential equation as an IVP in variable t we get (10):

$$b'(t) = \frac{db}{dt} = \lim_{\Delta t \to 0} \frac{b(t + \Delta t) - b(t)}{\Delta t} = -0.5 \cdot b(t) + 10\,, \quad b(0) = 50\,. \tag{10}$$

Using standard notation the *first-order, linear, differential equation with initial condition* is written as (11).

$$b'(t) = -0.5 \cdot b(t) + 10\,, \quad b(0) = 50\,. \tag{11}$$

Now what are we supposed to do with this? Well, this, in part, is what our effort is about, namely solving differential equations, or coming up with a $b(t)$. We saw how hard it was to "guess" a solution in the discrete case for $b(n)$ from (3) and it will be as difficult to guess what $b(t)$ is from (11).

Closed-form solutions to IVPs

Well . . . at least for IVP (11).

Even before learning to solve IVP's by hand, we can explore their solutions usign a computer algebra system such as *Mathematica* in which the command to solve the differential equation is `DSolve`. In this case we would input:

```
DSolve[{b'[t] == -1/2 b[t] + 10, b[0] == 50}, b[t], t]
```

which gives

$$b(t) = 20 + 30e^{-t/2}\,. \tag{12}$$

In Table 4 we compare the iterated solution to the discrete difference equation model we found in *Excel* with the *Mathematica* solution to the continuous differential equation model.

There are slight differences in the early values for of $b(n)$ and $b(t)$ $n = t = 1, 2, 3, 4, \text{and}\, 5$, but then both models appear to "settle down" and head toward 20. What does this tell us?

One model, many applications

Suppose we are studying a population of microorganisms in a Petri dish. Half the population dies each hour, but 10 microorganisms enter our Petri dish every hour. Model this situation with (a) a discrete difference equation model and (b) a continuous differential equation model. Be sure to

n or t	$b(n)$	$b(t)$
0	50.0000	50.0000
1	35.0000	38.1959
2	27.5000	31.0364
3	23.7500	26.6939
4	21.8750	24.0601
5	20.9375	22.4625
6	20.4688	21.4936
7	20.2344	20.9059
8	20.1172	20.5495
9	20.0586	20.3333
10	20.0293	20.2021

Table 4. Comparing the results of our discrete difference equation model $b(n)$ for $n = 0, 1, 2, \ldots, 10$ given in (2) with the results of our continuous differential equation model $b(t)$, $t \geq 0)$ (11) .

define the variables and identify their units. What are the pros and cons of each model in this situation?