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STUDENT VERSION

LC-Circuits, Beats, and Wireless Telegraphy

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Abstract: This project has three parts, the first part to be done at home, the last two parts in class. In the home part we recall how to solve second order, linear, differential equations with constant coefficients and simple source functions. In the first in-class part we analyze the solutions found at home and try to understand what is a resonant solution, the beats phenomenon, and how both arise. The last in-class part is to apply the beats phenomenon to send an audible tone across the Atlantic Ocean using radio waves.

SCENARIO DESCRIPTION

This project has three parts, the first part is done at home and the other two parts are done in class. The home part is purely computational—we solve second order linear differential equations having constant coefficients with and without sources. In the second part, done in class, we analyze the solutions found at home. We try to visualize in an interactive graph the resonant solution. We also try to understand what is the beats phenomenon and how it arises. In the last part, also done in class, we apply what we have learned about the beats phenomenon to solve a new problem—to use radio waves to send an audible tone across the Atlantic Ocean.

Part 1 (at home): Finding the Electric Current in an LC-Circuit

This system is described by an integro-differential equation found by Kirchhoff, now called Kirchhoff's voltage law,

$$L\,I'(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = V(t). \tag{1}$$

If we take one time derivative in the equation above we obtain a second order differential equation for the electric current,

$$L\,I''(t) + \frac{1}{C}\,I(t) = V'(t).$$

In this project we study the current oscillations in an LC-series circuit, which is an electric circuit containing a inductor L and a capacitor C. We assume that there is no resistance in the circuit, i.e. $R = 0$. We also introduce in the circuit a time-dependent input voltage $V(t)$. See Figure 1.

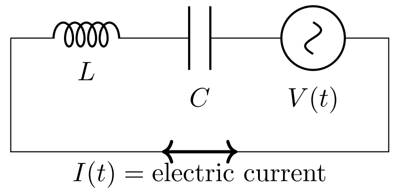

Figure 1. An LC circuit.

This equation is usually rewritten as

$$I''(t) + \frac{1}{LC}\, I(t) = \frac{V'(t)}{L}.$$

If we introduce the *natural frequency* $\omega_\circ = \dfrac{1}{\sqrt{LC}}$, then Kirchhoff's law can be expressed as

$$I'' + \omega_\circ^2\, I = \frac{V'(t)}{L}. \tag{2}$$

We are now ready to solve the following example.

Problem 1: Homogeneous Case (No Source).
Consider an LC-series circuit with capacitor C, inductor L, but no voltage source, i.e. $V(t) = 0$ for all t.

(1a) Write the Kirchhoff's equation for the electric current I of this circuit.

(1b) Find all possible solutions for the electric current.

Problem 2: Non-Homogeneous Resonant Case.
Consider an LC-series circuit with capacitor C, inductor L and voltage source

$$V(t) = V_0\, \sin(\omega_\circ t), \qquad V_0 \neq 0, \qquad \omega_\circ = \frac{1}{\sqrt{LC}}.$$

If the electric current I flowing in this circuit satisfies the initial conditions

$$I(0) = 0, \qquad I'(0) = 0,$$

then show that such current is given by a solution of equation (2),

$$I(t) = \frac{v_\circ\, t}{2}\, \sin(\omega_\circ t), \qquad v_\circ = \frac{V_\circ}{L}.$$

Note: The frequency in the Voltage source $V(t)$ is the natural frequency of the circuit, $\omega_\circ$.

Problem 3: Non-Homogeneous Non-Resonant Case.

Consider an LC-series circuit with capacitor C, inductor L and voltage source

$$V(t) = V_0 \sin(\nu t), \qquad \nu \neq \omega_0 = \frac{1}{\sqrt{LC}}.$$

If the electric current I flowing in the electric circuit satisfies the initial conditions

$$I(0) = 0, \qquad I'(0) = 0,$$

then show that such current is given by

$$I(t) = \frac{v_0 \nu}{(\omega_0^2 - \nu^2)} \big(\cos(\nu t) - \cos(\omega_0 t)\big), \qquad v_0 = \frac{V_0}{L}.$$

Note: The frequency ν in the Voltage source $V(t)$ is different from the natural frequency ω_0 of the circuit.

Part 2 (in class): Graphical Analysis of the Solutions, Resonance, and Beats

Consider an LC-circuit where

$$v_0 = 1, \qquad \omega_0 = 5, \qquad \nu \neq 5.$$

In this case we have that the solution of Problem 2, called the **Resonant** solution, is

$$I_R(t) = \frac{t}{2} \sin(5t), \tag{3}$$

while the solution of Problem 3, called the **Non-Resonant** solution, is

$$I_{NR}(t) = \frac{\nu}{(25 - \nu^2)} \big(\cos(\nu t) - \cos(5t)\big). \tag{4}$$

Notice that the Non-Resonant solution is the **sum** of two solutions, $I_{NR} = I_p + I_h$, where

$$I_p(t) = \frac{\nu}{(25 - \nu^2)} \cos(\nu t), \qquad I_h(t) = -\frac{\nu}{(25 - \nu^2)} \cos(5t). \tag{5}$$

We call I_p the **Particular** solution and I_h the **Homogeneous** solution.

Problem 4.
An oscillatory function has **beats** when the function has a periodic modulation in amplitude with frequency smaller than the function frequency, as shown in the Figure 2.

These solution names appear at the bottom of the interactive graph, which at the start will look as in Figure 3. When you click on these buttons in the interactive graph, the corresponding solution will show up, **Resonant** in blue, **Non-Resonant** in purple, **Particular** in red, and **Homogeneous** in green. The last button on the bottom right in Figure 3 is a slider that changes the value of the driver frequency ν in the interval $[0, 5]$, where 5 is the natural frequency of the system.

Now click on the following Interactive Graph Link:

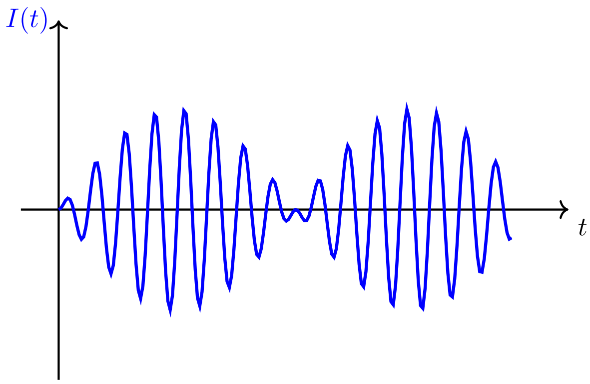

Figure 2. The I for ν close to ω_0, showing the beats phenomena.

In order to answer the questions below, recall the solutions names given above:

- I_R in (3) is the **Resonant** solution;
- I_{NR} in (4) is the **Non-Resonant** solution;
- I_p and I_h in (5) are the **Particular** and **Homogeneous** solutions respectively.

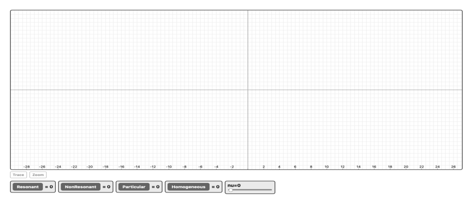

Figure 3. Picture of the interactive graph.

Beats Phenomenon.

(4a) In the **interactive graph link** above turn **on** the **Resonant** solution.

> **(4ai)** Look at the graph of the resonant solution I_R, and describe what happens with the amplitude of the solution as time grows.
>
> **(4aii)** Identify the precise part in the formula for I_R that produces the behavior you observed in the graph.

(4b) Let's go back at the behavior of the Resonant solution as t grows.

> **(4bi)** What does that behavior in I_R mean for the current in the circuit? Are the electrons in the circuit moving at the same speed, or slower, or faster as t grows?
>
> **(4bii)** Do you think that this mathematical model predicts accurately the behavior of an actual physical circuit as $t \to \infty$? Why?
>
> **(4biii)** How would you improve the mathematical model to get a better description of an electrical circuit for large t?

(4c) Go back to the **interactive graph link** above. Keep the Resonant solution on and now turn **on** the **Non-Resonant** solution. Vary the driving frequency **nu** from zero to five. Describe what you see. Why is **nu** equal five so special?

(4d) Go back to the **interactive graph link** above. Turn off the Resonant solution and turn **on** the **Non-Resonant** solution. Vary the driving frequency **nu** from 0 to 5. Find the minimum value of the driving frequency **nu** such that the solution I_{NR} displays beating.

(4e) Go back to the **interactive graph link** above. Keep the Non-Resonant solution on, and now turn **on** both the **Particular** and the **Homogeneous** solutions. Vary the driving frequency **nu**. Based on what you see, describe in your own words why the beating phenomenon occurs.

Note: Feel free to experiment with the graph. Turn on and off different functions, move the driving frequency. Play around with the graph.

(4f) The beating phenomena occurs when there is a sinusoidal behavior that satisfies the superposition property. This may happen with waves propagating in a medium; for example sound—air pressure waves propagating through the air. If there is beating in a sound wave, what would it sound like?

(4g) Watch the short video "Beat Frequency", from SMUPhysics,

`https://www.youtube.com/watch?v=V8W4Djz6jnY`.

What are the frequencies of the two tuning forks? Explain why beating occurs.

Part 3 (in class): Application of Beating: Wireless Transmission of an Audible Tone

Imagine this is 1887. Heinrich Hertz has just shown that radio waves are real, not something that James Maxwell had imagined in 1861 when he modified Michael Faraday's equations of electromagnetism. Maxwell modified Faraday's laws so that the electric charge would be conserved. Faraday's laws predicted electric charge creation from nothing. Maxwell equations fixed that. But many people doubted Maxwell, because he changed the equations without any experimental data. Even more, this modification also predicted the existence of radio waves and hypothesized that light is a radio wave. Maxwell's modification was a matter of heated debate, until Hertz silenced everybody when he showed that radio waves actually exist. Maxwell became a hero overnight; unfortunately he didn't live to see it.

Hertz transmitted information without the need of any cables. Up to then, communications used wires. People even laid submarine cables across the Atlantic ocean as early as 1858. However, no communications with ships at sea were possible. As soon as Hertz transmitted radio noise without cables, people wanted to use radio waves to transmit Morse code as audible sounds without cables across long distances. In this way you wouldn't need expensive submarine cables and you could reach ships at sea. We now summarize a few facts needed for the last questions.

- Humans are able to hear frequencies in the range 20 Hz to 20,000 Hz.
- Radio waves are electromagnetic waves that travel at the speed of light, c, given by

$$c = 300,000 \text{ km/sec}.$$

- The frequency γ and the wavelength λ of a radio wave satisfy the relation

$$\gamma\lambda = c.$$

  So, given the frequency one can compute the wavelength, and vice versa.
- To capture a radio signal of wavelength λ one needs an antenna of length no smaller than $\lambda/3$.

Problem 5:

(5a) It turns out that antennae that capture radio waves in the audible wavelength range are too long to be built. Find out how long an antenna must be so that we can capture radio waves of 1 kHz.

(5b) Radio waves that can be transmitted and captured by small antennae do not have audible frequencies. Find the minimum frequency γ_1 of a radio wave that can be captured by an antenna of $\ell_1 = 1$ meter.

(5c) Use what you have learned about beating to send an audible tone using radio waves that can be captured by a 1 meter antenna. How many radio waves do you need to transmit? What should their frequencies be?

(5d) Can you think of a way to transmit an audible tone by transmitting only one radio wave?

Appendix: MathStudio Code

We reproduce here the MathStudio code used to create the interactive graph **Beats Phenomenon** in Part 2. If the link shown in Part 2 does not work, use any browser to go to MathStudio, **http://mathstud.io** and then type the code below.

```
/*
No Damping
Beats for nu close to omega
And Resonance for nu = omega
*/

Button(Resonant,[0,2,4],0)
Button(NonResonant, [0,2,4],0)
Button(Particular,[0,2,4],0)
Button(Homogeneous,[0,2,4],0)

slider(nu,0..4.999->0.001,0)
omega = 5

yp(x) = nu/(omega^2-nu^2)*cos(nu*x)
yh(x) = -nu/(omega^2-nu^2)*cos(omega*x)
yt(x) = yh(x) + yp(x)

yr(x)=(x/2)*sin(omega*x)

Plot(yp(x),color=red,linewidth=Particular)
Plot(yh(x),color=green,linewidth=Homogeneous)

Plot(yt(x),color=purple,linewidth=NonResonant)
Plot(yr(x),color=blue, x=[-30,30],y=[-4,4], height=600, width=1400, linewidth=Resonant)
```