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STUDENT VERSION
Floating Box

J. B. Thoo
Yuba College
Marysville CA 95901 USA
jthoo@yccd.edu

Abstract: In this scenario, we lead students through the process of building a mathematical model for a floating rectangular box that is bobbing up and down.

SCENARIO DESCRIPTION

A rectangular box (parallelepiped) floats in water as shown in Figure 1. We are interested in the up-and-down motion of the floating box when it is raised or lowered from its equilibrium (floating) position and released. The goal is to produce a mathematical model of the motion (equation of motion for the vertical position) of the box as a function of time.

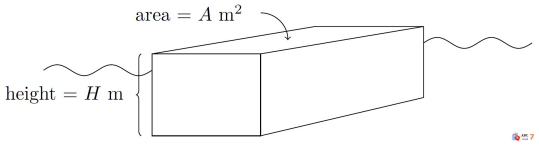

Figure 1. A rectangular box (parallelepiped) of mass M kilograms, height H meters, and horizontal surface area A square meters floats in water.

The principal forces acting on the box are its weight (the force due to gravity) and the buoyancy force (that makes it float). By Archimedes' Principle, the buoyancy force acting on an object equals the weight of the liquid that is displaced by the object.

If an object that is in water is in equilibrium (neither sinking nor rising), then the buoyancy force that is acting on the object is equal in magnitude and in the opposite direction to the force due to the weight of the object. If the object is submerged below its equilibrium (floating) position, then the buoyancy force will cause the object to rise. The upward motion would move the object above

its equilibrium position, and then the weight of the object would cause it to fall. The downward motion would move the object below its equilibrium position, and then the buoyancy force would cause the object to rise again, and the cycle would repeat indefinitely (bobbing up and down) if there were no friction (damping, drag).

1 Activity 1

The goal is to come up with a mathematical model for a floating rectangular box that bobs up and down in water. The box has mass M kg, height H m, and horizontal surface area A m^2 as shown in Figure 1. We wish to develop the simplest possible model. What are some of the simplest assumptions that we can make? Here are some considerations. Can we rule out some or add others?

- still water,
- humidity,
- temperature (air or water),
- time of the year, month, or day,
- friction or damping or drag, and
- wind.

Let us assume that the box is floating in still water, and that the only forces that are acting on the box are its weight and the buoyancy. (No wind, no friction or damping or drag, and so on.) Figure 2 depicts a free-body diagram for this situation. Note that we fix a vertical axis with positive downward, 0 at the water surface where the box is at rest or static equilibrium, and x meters measuring the distance from the surface to the bottom of the box for convenience.

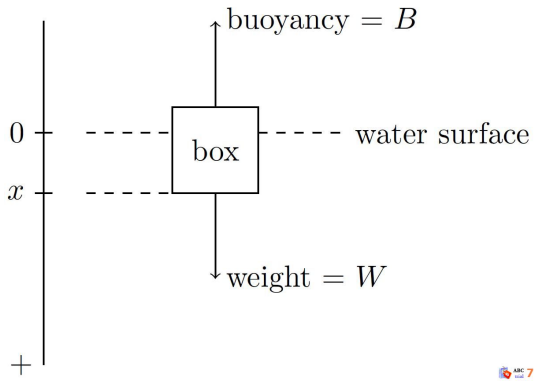

Figure 2. A free-body diagram for a box that is floating in still water, where the only forces that are acting on the box are its weight and the buoyancy.

What is the weight of the box? What is the buoyancy? What are the units?

Sketch a graph of the position x versus time t that captures what you believe would be the qualitative behavior of the bobbing box. How would adding friction or damping or drag affect the graph?

Now, suppose that the mass of the box is M kg in kilograms. The weight of the box is the product of its mass and its acceleration due to gravity, so here

$$\text{weight of the box} = M \text{ kg} \cdot g \ \frac{\text{m}}{\text{s}^2} = Mg \ \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = Mg \text{ N},$$

where g m/s^2 is the acceleration due to gravity in meters per second squared, and N $=$ kg $\cdot$ m/s^2 represents the unit of force in newtons. The buoyancy is given by Archimedes' Principle—the weight of the water that is displaced by the (partially) submerged box—so here we have,

$$B = \begin{cases} 0 \text{ N} & \text{if } x < 0, \\ xA\rho g \text{ N} & \text{if } 0 \le x \le H, \\ HA\rho g \text{ N} & \text{if } H < x, \end{cases}$$

where ρ kg/m^3 in kilograms per cubic meter is the mass density of the water.

The buoyancy is defined as a piecewise function. Explain the three different pieces?

Let $W = Mg$, so that the weight of the box is W newtons. The net force F that is acting on the box is then

$$F = W - B.$$

Now, recall Newton's second law of motion:[1]

> The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object.

This is often remembered by the phrase, "Force equals mass times acceleration" or "$F = ma$." Thus, by Newton's Second Law, we have

$$Mx''(t) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \quad \text{for } 0 \le x \le H.$$

Attempt to fill in the blank, i.e. use the forces we outlined above.
If we let $k = A\rho g$, then rearranging the equation yields the equation of motion

$$Mx'' + kx = Mg \quad \text{or} \quad x'' + \omega^2 x = g \quad \text{for } 0 \le x \le H,$$

where $\omega^2 = k/M$.

Write an equation of motion for $x < 0$ (the box is entirely out of the water), and also for $H < x$ (the box is entirely submerged).

[1] http://www.physicsclassroom.com/class/newtlaws/Lesson-3/Newton-s-Second-Law

Look up the motion of a mass hanging on a spring that is governed by Hooke's law. How does the equation of motion for a mass-spring system compare to the equation of motion for a floating box that we developed above?

2 Activity 2

Suppose that the box described above develops a leak in its bottom, and that water is flowing into the box at a rate of L ℓ/s (L liters per second).

What would change in the analysis above? How would this change the equation of motion $Mx'' + kx = Mg$? Continue to assume that the only forces acting on the box are its weight and the buoyancy.

Sketch a graph of the position x versus time t that captures what you believe would be the qualitative behavior of the bobbing box in this situation.

Only the mass of the box is changing (increasing) by the mass of the water that it is taking on. The extra mass μ kg is

$$\mu \text{ kg} = L \frac{\ell}{\text{s}} \left(\frac{1 \text{ m}^3}{1000 \ \ell} \right) \cdot t \text{ s} \cdot \rho \frac{\text{kg}}{\text{m}^3} = 0.001Lt \text{ kg}.$$

Hence, the weight of our box with water in it now depends on time. Determine the weight $W(t)$.

$$W(t) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Consequently, by Newtons' Second Law, we have

$$(M + 0.001Lt)x''(t) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \quad \text{for } 0 \leq x \leq H.$$

If we let $k = A\rho g$, then rearranging the equation yields the equation of motion

$$(M + 0.001Lt)x'' + kx = (M + 0.001Lt)g \quad \text{or} \quad x'' + \omega(t)^2 x = g,$$

where $\omega(t)^2 = k/(M + 0.001Lt)$.

3 Activity 3

In Activity 1, we assumed that the box is floating in still water, and that the only forces that are acting on the box are its weight and the buoyancy. We now add the assumption that a third force, friction or drag, is also acting on the box. The simplest assumption that we can make about the drag that is acting on an object is that it is proportional to the velocity of the object—the faster an object is moving, the greater the drag it experiences. (Think about tossing a balloon that is filled with air. How well does the balloon move when you toss it hard compared with when you toss it softly?) Incidentally, another assumption that we can make about the drag that is acting on an object is that it is proportional to the square of the velocity of the object. However, we use a drag force which is proportional to the velocity here.

What might affect the drag on the bobbing box?

Now, the floating box is in two fluids: air and water, each causing a different drag. To keep our analysis as simple as possible, however, we neglect this, as well as the shape of the box, and combine all the drag into one term.

If we assume that the drag is only proportional to the velocity of the box, then the drag is bx', where x meters is the distance from the surface of the water to the bottom of the box (as in Activity 1), and b is the drag coefficient. Since drag is a force, it has units of newtons. What are the units of the drag coefficient, b?

Draw a free-body diagram that is similar to the one in Figure 2 for a box that is floating in still water, where the forces that are acting on the box are its weight, buoyancy, and drag due to the water.

Sketch a graph of the position x versus time t that captures what you believe would be the qualitative behavior of the bobbing box in this situation.

Refer to your free-body diagram. By Newton's Second Law, we have

$$Mx''(t) = \_\_\_\_\_ \quad \text{for } 0 \le x \le H.$$

If we let $k = A\rho g$, then rearranging the equation yields the equation of motion

$$Mx'' + bx' + kx = Mg \quad \text{or} \quad x'' + \beta^2 x' + \omega^2 x = g \quad \text{for } 0 \le x \le H,$$

where $\beta^2 = b/M$ and $\omega^2 = k/M$.

Write an equation of motion for $x < 0$ (the box is entirely out of the water), and also for $H < x$ (the box is entirely submerged).

4 Activity 4

In Activity 1, suppose that

- the box is a cube that measures 1 m on a side;
- the mass of the box is 41 kg;
- the box is floating in the ocean, and the density of sea water is 1025 kg/m^3;
- the acceleration due to gravity is 9.8 m/s^2; and
- the box is held with its bottom 0.8 m below the surface and then released from rest.

Then, the equation of motion of the floating box is

$$41x'' + 10\,045x = 401.8 \quad \text{or} \quad x'' + 245x = 9.8$$

with initial conditions

$$x(0) = 0.8, \quad x'(0) = 0.$$

Solve the equation of motion (by hand or using technology) and plot the solution. Compare this plot with your sketch of the graph of the position x versus time t in Activity 1.

In Activity 2, use the same values given for Activity 1 above, but now suppose that water is flowing into the box through its bottom at the rate of 1 ℓ/s. Then, the equation of motion of the floating box is

$$(41 + 0.001t)x'' + 10\,045x = 9.8(41 + 0.001t) \quad \text{or} \quad x'' + \frac{10\,045x}{41 + 0.001t} = 9.8$$

with initial conditions

$$x(0) = 0.8, \quad x'(0) = 0.$$

Solve the equation of motion and plot the solution. Compare this plot with your sketch of the graph of the position x versus time t in Activity 2.

In Activity 3, use the same values given for Activity 1 above, but now suppose that the drag coefficient is $b = 1.025$. Then, the equation of motion of the floating box is

$$41x'' + 1.025x' + 10\,045x = 401.8 \quad \text{or} \quad x'' + 0.025x' + 245x = 9.8$$

with initial conditions

$$x(0) = 0.8, \quad x'(0) = 0.$$

Solve the equation of motion and plot the solution. Compare this plot with your sketch of the graph of the position x versus time t in Activity 3.

Compare Activities 1, 2, and 3. Write a summary of the activities, and offer your observations and any conclusions that you have drawn from them.

REFERENCES

[1] Wikipedia. Archimedes' Principle. 2015. https://en.wikipedia.org/wiki/Archimedes%27_principle. Accessed August 25, 2015.

[2] Wikipedia. Hooke's Law. 2015. https://en.wikipedia.org/wiki/Hooke%27s_law. Accessed July 17, 2015.