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SIMIODE
A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
SECOND-ORDER INTRODUCTION

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SCENARIO DESCRIPTION

Equations of the type (1) and (2)

$$ay''(t) + by'(t) + cy(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,, \tag{1}$$

$$ay''(t) + by'(t) + cy(t) = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,, \tag{2}$$

will arise, as we shall see, quite often in any study of differential equations and their applications. There are a number of ways to solve them and thus gain insight into the application under study. The most common and traditional way makes use of the quadratic formula for the roots of a quadratic equation. That is right. The formula we all learned in early algebra in junior or senior high school is the key to solving the broadest class of differential equations. We can and do say more.

The world runs on second-order differential equations

. . . and the solutions depend on the quadratic equation!!! It is true. Motion requires forces and the forces acting on a body can be accounted for in two ways: (1) the mass of the body times the acceleration it possesses and (2) the sum of all the external forces acting on the mass. Condition (1) can be modeled by $m \cdot y''(t)$ where m is the mass of a body in some reasonable and consistent unit (kilograms, grams, slugs, etc.) and $y(t)$ is the one dimensional displacement from some equilibrium or fixed point in some reasonable unit (meters, centimeters, feet, etc.) so that $y''(t)$ is the acceleration the mass is experiencing. Condition (2) refers to $\sum_{i=1}^{n} (\text{External Force})_i$. Newton's Second Law says these two accountings must give the same result, i.e. the product of mass and the acceleration applied to that mass is equal to the sum of all the external forces acting on t hat mass. This gives us the mathematical formulation of (3):

$$m \cdot y''(t) = \sum_{i=1}^{n} (\text{External Force})_i \,. \tag{3}$$

If we can identify, formulate, and account for the External Forces acting on our body then we have built a differential equation, albeit one with a second derivative, not just a first derivative as we have previously studied. This will be a challenge, but if our section title is correct, "The world runs on second-order differential equations," then it will be a very rewarding challenge.

Springs

Consider a spring suspended from a bar with a mass attached to the bottom of the spring. Furthermore, the mass and a good portion of the attached spring is dipped into a cylinder of liquid. The cylinder of fluid is often referred to as a *dashpot*. We take a thin rod and reach into the cylinder and push down on the mass and withdraw the rod, thus stretching the spring.

a) Describe what happens and give some reasons for why you think your description reflects reality.

What kind of forces might the External Forces acting on the mass be? One obvious force is the force due to gravity and it is simply $m \cdot g$ where m is our mass and g is the acceleration due to gravity usually measured in units like m/s^2 (MKS system for meter-kilogram-second) or cm/s^2 (CGS system for centimeter-gram-second) where here m is meters while s is seconds and cm is centimeters.

Let us work in the MKS system and thus $m \cdot g$ has units $kg \cdot m/s^2$. If we accelerate one kilogram at a rate of one meter per second each second then we have a force whose magnitude is 1 and in this case we call this 1 unit of force a *Newton*. If we were in the CGS system we would have a comparable notion of force, for if we accelerate one gram at a rate of one centimeter per second each second then we have a force whose magnitude is 1 and in this case we call this 1 unit of force a *dyne*.

So, let's go back to our spring. What we often do is construct a Free Body Diagram (FBD) in which we simply isolate the body or mass we wish to analyze and in picture form draw arrows indicating direction for the forces acting on this body. Force, of course, is a vector quantity and has magnitude and direction. Both will be very important in our analysis. So we see our mass as having a downward (agree??) force due to gravity of magnitude $m \cdot g$. We show this in Figure 1 with more detail for a number of possible cases and other forces we will encounter.

Now picture this. We place the mass on the end of the spring and gently support it as it stretches the spring and comes to rest. This resting position is often called the *static equilibrium*. IF gravity is pulling the mass down and it is at rest (agree??) then there must be a counter force pushing up! What could it be? Well, think about what happens when you pull on a spring or stretch it. What does it want to do? It wants to restore itself to its original position. "Leave me at rest," it says to us. Thus there is a force we call a *restoring force*. How might we calculate this force? Well we could go into the lab with our spring stretch it 0.1 m and measure the force tugging on us as we

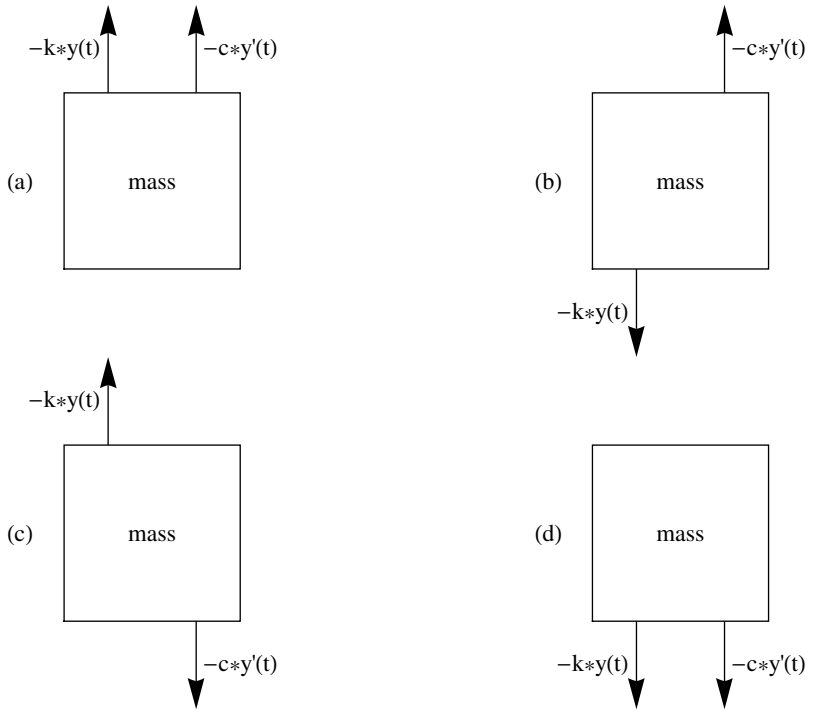

Figure 1. All possible variations of (1) restoration force $(k \cdot y(t))$ and (2) resistance force due to velocity $c \cdot y'(t)$.

hold the mass in place; then stretch the spring 0.2 m and measure the force tugging on us as we hold the mass in place; then 0.2 m; then 0.4 m, etc. If we record all these forces we will find a very (VERY!) interesting relationship between the force (F in Newtons) to displace a spring x meters. We look to some history to find out what this relationship might be.

> In physics, Hooke's Law of elasticity states that the extension of an elastic spring is linearly proportional to its tension.
>
> The law holds up to a limit, called the *elastic limit*, or the *limit of elasticity*, after which springs suffer plastic deformation up to the plastic limit or limit of plasticity, after which they break down. Have you ever stretched a Slinky toy so far it has no restoring force? Indeed, this ruins the fun of the Slinky!
>
> It is named after the 17^{th} century physicist Robert Hooke, who initially published it as an anagram *ceiiinosssttuv*, which he later revealed to mean *ut tensio sic vis*, or "as the extension, the force."

"Linearly proportional," eh? That simply means $F = k \cdot x$ where k is some constant of proportionality with units N/m for Newton per meter. This means that for every meter we wish to stretch

our spring we will have to exert k Newtons. Many of us did a laboratory in high school physics in which we hung successive weights or masses on a spring and measured the displacement. It was amazing, for it came out to be a straight line plot of $F = k \cdot x$.

Again, let us get back to our spring. We find that when we hang a mass of m kg on the end of the vertically hanging spring we subject the spring to a force of $m \cdot g$ N and if we know our spring constant k (which is often offered by the manufacturer or we can get a simple collection of a few data points to determine the linear relationship $F = k \cdot x$ and hence the value of k) then we can pencil in another force acting on our spring, namely $k \cdot y(t)$ where k is the spring constant with units N/m and $y(t)$ is the vertical displacement with units m. (See this force in Figure 1.)

This is VERY IMPORTANT. Notice that when we place our mass on the spring and gently let the spring stretch and come to rest or an equilibrium at an extended length of y_e we have two equal and opposite forces, namely $m \cdot g$ due to gravity, downward, and $k \cdot y_e$ (in this case when $t = 0$) due to the restoring force of the spring, upward. They cancel because we see the mass is at rest - so no motion means the two forces are equal and opposite.

Let us denote positive distance and direction as downward (completely arbitrary, but we need to be consistent). This means in (3) for our differential equation we now have

$$\begin{aligned} m \cdot y''(t) &= \sum_{i=1}^{n} (\text{External Force})_i = m \cdot g - k \cdot y_e + \text{other External Forces} \\ &= 0 + \text{other External Forces} = \text{other External Forces}\,, \end{aligned} \tag{4}$$

where $m \cdot g - k \cdot y_e = 0$. So we are on our way. Now let us continue to develop our differential equation (4) by finding out what these other forces might be.

If we find our spring extended downward (beyond length y_e) at time t, say to an extended length of $y(t) > 0$, then there will be a restoring force due to Hooke's Law upward of size $k \cdot y(t)$. Similarly, if we find our spring compressed upward (above length y_e) at time t, say to an extended length of $y(t) < 0$, then there will be a restoring force due to Hooke's Law downward of size $k \cdot y(t)$, which we denote as $-k \cdot y(t)$.

For various phenomena the notion of resistance due to the media or device itself has been studied. In low velocity motion in liquid empirical and theoretical considerations support the conclusion that the force of resistance is proportional to the velocity of the object, i.e. $c \cdot y'(t)$, while in higher velocity motions such as parachutes, bullets, airplane fuselages, etc. the force of resistance is proportional to the square of the velocity of the object, i.e. $c \cdot (y'(t))^2$. Thus for certain types of physical situation we still have a linear term, while others it might appear to be a quadratic term.

Now we need to discuss exactly what the spring is doing at the start, $t = 0$ s, of our experiment. Suppose we push our spring down (could be up as well), i.e. we initially extend the spring beyond its static equilibrium position, to say $y(0) = y_0$ m ($y_0 > 0$ if downward and $y_0 < 0$ if upward). Further, we decide to impart an initial velocity of $y'(0) = v_0$ ($v_0 > 0$ if moving downward and $v_0 < 0$ if moving upward). We have four possible cases, two each for $y(t)$ and $y'(t)$ being positive or negative.

We outline them below and then depict them in Figure 1.

(a) $y(t) > 0$ and $y'(t) > 0$ (b) $y(t) < 0$ and $y'(t) > 0$.
(c) $y(t) > 0$ and $y'(t) < 0$ (d) $y(t) < 0$ and $y'(t) < 0$.

Remember upward arrows are in the negative direction and downward arrows are in the positive direction. Thus in case (a) since $y(t) > 0$ and $y'(t) > 0$ and both $k, c > 0$ we have upward forces applied to our mass since $y(t) > 0$ means the mass has been extended or stretched downward and the restoring force is upward, while since $y'(t) > 0$ means the motion is downward and the resistance force is upward. You should check out each case and make sure the accounting is correct as well as the diagrams. This is illustrated in Figure 1. Can you draw the Free Body Diagrams on your own?

We are ready to sum up our "other External Forces." No matter which of the four cases (a) - (d) we consider, the "other External Forces" sum this same way, $-k \cdot y(t) - c \cdot y'(t)$. Hence our differential equation describing the position of the spring's mass is evolving with

$$\begin{aligned} m \cdot y''(t) &= \sum_{i=1}^{n} (\text{External Force})_i = \underbrace{m \cdot g - k \cdot y_e}_{0 \text{ at static equilibrium}} + \text{other External Forces} \\ &= \text{other External Forces} \\ &= -k \cdot y(t) - c \cdot y'(t)\,, \end{aligned}$$

to (5) with our initial conditions $y(0) = y_0$ and $y'(0) = v_0$:

$$m \cdot y''(t) = -k \cdot y(t) - c \cdot y'(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{5}$$

Equation (5) is called the *Spring Mass Dashpot* differential equation for an oscillating spring with mass attached and if we can solve it we will be able to predict how our spring behaves by obtaining a function which represents $y(t)$, the displacement in meters of the mass at the end of the spring at time t in seconds. Are you ready to solve? Well, let's go for it!!

First, we point out that (5) is often written as differential equation (6):

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{6}$$

Quite often in reality $v_0 = 0$ as we would really have trouble consistently imparting a set velocity to our spring mass configuration under water, say!

This equation is an example of a *second-order, constant coefficient, linear, homogeneous differential equation with initial conditions*; second-order because is has a second derivative; constant coefficients because all the function terms have constant coefficients; linear because we have all derivatives and the function for which we are solving to first power; and homogeneous because there are no terms other than terms with our function, $y(t)$ and its derivatives, involved.

Before attempting a general solution of (6) let us consider the situation were $c = 0$, i.e. where there is no resistance due to the media in which the spring mass configuration sits. This is unreal, because all moving spring mass objects will meet with resistance either do to internal friction or

external media resistance. We know this because they do not oscillate forever. Nevertheless, let us study this spring mass system, presumably purchased from our local "Ideal Physics Store" where such things as perpetual motion machines are offered on sale at marked down prices, no doubt! If $c = 0$ then we have the following differential equation (7):

$$m \cdot y''(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{7}$$

a) Suppose we had a mass of $m = 1$ kg and our spring constant was $k = 1$ N/m. Further suppose we displaced our mass by $y(0) = 1$ m and gave it an original velocity of $y'(0) = 0$ m/s, i.e. can we find a solution of the differential equation (8)?

$$y''(t) + y(t) = 0\,, \quad y(0) = 1\,, \quad y'(0) = 0\,. \tag{8}$$

Think hard and ask if you know any functions $y(t)$ which have the property that when you add the function and its second derivative you get 0. We asked that you think hard, but not too hard!! Write down two such function. Hint: They are in the same family of functions.

$y_1(t) =$ _____ and $y_2(t) =$ _____.

By the way, the function $y(t) = 0$ does not count as it is what we refer to as the *trivial solution* and will not give us anything useful in our model solution. Check with your neighbors to see what they got.

Here is one of the beauties of linear homogeneous differential equations. If you have two solutions, say $y_1(t)$ and $y_2(t)$, you can add them to get a new function $y_{\text{new}}(t) = y_1(t) + y_2(t)$ and you will find that $y_{\text{new}}(t)$ still solves the same differential equation. Go ahead and take time to check and see if when you add your two candidate solutions that the sum of them is also a solution. This means directly substituting $y_{\text{new}}(t) = y_1(t) + y_2(t)$ into $y''(t) + y(t) = 0$, and seeing if the equation is satisfied, i.e. take the second derivative of your new y_{new} and add it to y_{new} and check that it comes out 0. Do it!

In fact if you take any multiple of a solution to such a differential equation, say a times $y_1(t)$ to obtain $y_{\text{multiple}}(t) = a \cdot y_1(t)$ then $y_{\text{multiple}}(t)$ will also be a solution. Go ahead and try it with one of your candidate solutions. Use the same process as we did in checking that $y_{\text{new}}(t)$ satisfies the same differential equation.

Now comes a really neat fact. There is always a great deal of theory that supports applied mathematics and one of the results of theory says roughly that if we could produce two solutions of a linear homogeneous differential equation, say like your two candidates, $y_1(t)$ and $y_2(t)$, and if these solutions are not scalar multiples of each other (there is no constant c such that $y_1(t) = c \cdot y_2(t)$ - this condition is called *linear independence*, meaning in this case $y_1(t)$ and $y_2(t)$ are *linearly independent*) then the *complete solution*, called the *general* solution to (8), will always be of the form $y_{\text{general}}(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$. Ah, there is a key word *form*, a rather amorphous term. But take the time now to check that for your candidate solutions $y_1(t)$

and $y_2(t)$ the $y_{\text{general}}(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ function IS, in fact, a solution to differential equation (8). Do it!

Now here is a wonderful piece of good news. In an effort to determine just what c_1 an d c_2 are in our case of our (8) what information could we use to determine c_1 and c_2? We will let you ponder that for a bit. Go ahead and do so and actually determine what the c_1 and c_2 are so that the general solution $c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ is actually the unique solution to our differential equation(8) *with* initial conditions (the last two words are a hint!)

N.B.: Engineers refer to this addition of scalar multiples of two solutions to obtain a general solution *superposition*. We call it the addition of scalar multiples of two solutions to obtain a general solution!

Report your conclusions in words AND in mathematical symbols and be sure to check that your final unique solution IS really a solution.

b) Now, look at your solution from (a) above and ask, "Is it reasonable that this function should be a solution to our spring mass (albeit ideal) differential equation(8)?" Go ahead and do just that and write up your defense as to why it IS reasonable or unreasonable. Take time out to celebrate this news. High Five your room mate! Treat yourself to a candy bar! Take a break and play a game of ping-pong in the dorm rec room! Go ahead and do it, you deserve it, but come back and reflect upon what you just did and be prepared to move on.

Now differential equations come in all sizes and variations and some model reality and some do not. Consider the following differential equation which might not have any basis in reality but is an example of a differential equation worth our time for what it will teach us.

$$y''(t) = y(t)\,, \quad y(0) = 2\,, \quad y'(0) = 1\,. \tag{9}$$

This is asking us to find a function $y(t)$ such that when we take its second derivative we get the same function. Hmmmmm!!! Know any function like that? Sure you do, $y(t) = e^t$, indeed, its buddy $y(t) = e^{-t}$ does the same thing. Sure $y(t) = 0$ is a candidate, but we are really interested in substance, in stuff, and $y(t) = 0$ is not stuff!! Now remember that if we have two solutions, say $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ we can form a linear combination $c_1 y_1(t) + c_2 y_2(t) = c_1 e^t + c_2 e^{-t}$ and that combination will also be a solution. Moreover, if we can get two such linearly independent solutions and add them we know we will have found the most general solution to the differential equation (9) - in the case we have a second-order differential equation, namely

$$y_{\text{general}}(t) = c_1 e^t + c_2 e^{-t}\,.$$

How do we find these two constants c_1 and c_2 out of all the possible set of constants? Let us not forget that every differential equation we shall study usually comes from a physical situation (well, this one does not!) and in addition to some physical or natural law in the formulation of the equation itself we also have initial conditions, in this case, $y(0) = 2$ and $y'(0) = 1$.

c) Use the initial conditions $y(0) = 2$ and $y'(0) = 1$ to determine the constants c_1 and c_2 in our general solution $y_{\text{general}}(t) = c_1 e^t + c_2 e^{-t}$ of differential equation (9) and thereby recover THE solution to this differential equation.

> In class when we are working with students we often tell them to hum a mantra of the form "Hmmmmm," when they are to the point in their solution where they are going to apply the initial conditions to exact THE unique solution to their differential equation by solving for the constants c_1 and c_2, for that way we know they are on the right track during exams! The room is alive with various pitches of "Hmmmmm!" Try it. It works and it will remind you that you are almost done with solving the differential equation. While we do not even play a medical doctor on television we are doctors and highly recommend this therapy.

So we are developing a pretty good approach to solving these types of differential equations except for one thing: we are apparently expected to guess our solution and then build from our guesses. In a sense, that is true and we will make one more guess approach, but WHAT a guess it will be. We have seen enough, we believe, that you are ready to get in the serious guessing game. Ready, here we go.

Consider the differential equation (10) (repeated here for ease of reference) for our spring mass dashpot:

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{10}$$

Suppose the solution to differential equation (10) is of the form $y(t) = e^{\lambda t}$ (where λ is the lower case Greek letter "lambda.") Such a strategy could be called guessing, but we prefer (tut! tut!)) to call this conjecturing. Indeed, if we were to try $y(t) = e^{\lambda t}$ in equation (10) and it was to produce two linearly independent solutions ALL THE TIME that would be great, but let us see what this might entail. We will need $y'(t)$ and $y''(t)$ - not that hard to produce for $y(t) = e^{\lambda t}$.

d) Put your conjecture $y(t) = e^{\lambda t}$ into differential equation (10) and see what happens. Go ahead. Do not be shy. Just do it! Explain what happens. Is there anything familiar somewhere along the process? What do you do with that familiar object to continue? Explain and be sure to acknowledge that you learned this from your algebra teacher early in your mathematical travels, let us call her Ms. Baumgartner, and we shall be thanking her for all she did for us more and more as our study continues. Believe me you will come to think she was the best person in your life as we proceed. We will encourage you to send a, "Thank you," note to her from time to time.

e) There are lots of things that can happen at the end game of (d) above, but let us consider a specific model of the spring mass dashpot with numbers (for we all like numbers!) Consider differential equation (10) with $m = 1$ kg, $c = 7$ N/(m/s), and $k = 12$N/m, with initial conditions

$y(0) = 1$ m and $y'(0) = 0$. Describe this situation in words as completely as you can and draw a diagram to illustrate what we are about to study.

Let us make the conjecture, $y(t) = e^{\lambda t}$, and substitute it into the differential equation you just wrote down. Follow the trail, do the natural thing, produce two linearly independent solutions, take a linear combination of them to form a general solution, then (Hmmmmm!) use your initial conditions to determine your two constants c_1 and c_2, and sit back and look at your unique final solution to your differential equation. Plot the solution and describe what happens as time goes on for this spring mass dashpot.

f) In (e) change the resistance due to the media constant, c, from 7 N/(m/s) to 13 N/(m/s). (Ask yourself, "What did we just do physically?") Again, use your initial conditions to determine your two constants c_1 and c_2, and sit back and look at your unique final solution to your differential equation. Plot the solution and describe what happens as time goes on for this spring mass dashpot. Now compare your complete answers to those from (e). Actually plot BOTH your solution from (e) and from (f) on the same axes for $t \in [0, 5]$ s. Describe as completely as you can what is happening and what these graphics tell you. Here is a command that might help you see things better if you are using *Mathematica* :

```
Plot[{ys1[t], ys2[t]}, {t, 0, 5},
      PlotStyle -> {Thickness[.001], Thickness[.005]}]
```

Let us get back to the most general situation in differential equation (10) and use an obviously powerful command `DSolve` in *Mathematica* to point the way for future study. First we get a solution, we grab the solution, we name it, and we expand it out.

```
ysol[t_] = y[t] /. DSolve[{m y''[t] + c y'[t] + k y[t] == 0, y[0] == y0,
      y'[0] == v0}, y[t], t][[1]] // Expand
```

Here is the output *Mathematica* gives.

$$\begin{aligned} \text{ysol}(t) \quad = \quad & -\frac{m\,\text{v0}\,e^{\frac{t\left(-\sqrt{c^2-4km}-c\right)}{2m}}}{\sqrt{c^2-4km}} + \frac{m\,\text{v0}\,e^{\frac{t\left(\sqrt{c^2-4km}-c\right)}{2m}}}{\sqrt{c^2-4km}} + \frac{1}{2}\text{y0}\,e^{\frac{t\left(-\sqrt{c^2-4km}-c\right)}{2m}} \\ & + \frac{1}{2}\text{y0}\,e^{\frac{t\left(\sqrt{c^2-4km}-c\right)}{2m}} - \frac{c\,\text{y0}\,e^{\frac{t\left(-\sqrt{c^2-4km}-c\right)}{2m}}}{2\sqrt{c^2-4km}} + \frac{c\,\text{y0}\,e^{\frac{t\left(\sqrt{c^2-4km}-c\right)}{2m}}}{2\sqrt{c^2-4km}} \end{aligned} \tag{11}$$

Awesome!!! Is that beautiful or what? Let us look carefully (with a surgical scalpel) at parts of this solution. We see $\frac{-c-\sqrt{c^2-4km}}{2m}$ and $\frac{-c+\sqrt{c^2-4km}}{2m}$ popping up every where. What are these? Where did they come from? Think about what Ms. Baumgartner taught you, then answer. Try to answer this question before continuing, but think of several things (1) our original differential equation $m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,$, (2) our conjecture $y(t) = e^{\lambda t}$, (3) what you get when you substitute this conjecture into the differential equation (10), and (4) the

resulting algebra that Ms. Baumgartner taught you how to handle. Do it!

Where do these expressions, $\frac{-c-\sqrt{c^2-4km}}{2m}$ and $\frac{-c+\sqrt{c^2-4km}}{2m}$, occur in our solution? Why, in the exponent of e from our conjecture of the form $y(t) = e^{\lambda t}$.

f) Write a short essay in which you describe what you think is going on with regard to our solution of the differential equation $m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, y(0) = y_0\,, y'(0) = v_0\,.$ Be as complete as you can and do take leaps and speculate. This is not science fiction, but observational or natural science, based on what you have seen thus far. We will get back to you on this.

g) Report what happens in (11) if we set only $v0 = v_0 = 0$. Explain in detail. What if we set only $y0 = y_0 = 0$? Explain in detail. What if we set both $v0 = v_0 = 0$ and $y0 = y_0 = 0$. Explain in detail. What physically have we done in the latter case? Much like your little brother or sister says, we can say of the latter case, "This is so boring." Don't you agree? Do you agree much with your little brother or sister?

It appears that exponential functions are popping up all over the place and they might prove a fertile area for hunting solutions to our differential equation (10. Let us just do that.

h) For the differential equation (10) put a conjecture of the form $y(t) = e^{\lambda t}$ into the equation and see what it leads to. What conditions on λ exist as a direct result of your conjecture and substitution? What would Ms. Baumgartner say about your situation? Write down what you would have to do to get two (Got it? TWO) solutions $y_1(t)$ and $y_2(t)$? We shall now try it out with some numbers - for some a comfort zone.

i) Consider the differential equation (12):

$$1 \cdot y''(t) + 6 \cdot y'(t) + 5 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{12}$$

Place your conjecture, $y(t) = e^{\lambda t}$, into differential equation (12). Then, again as in (d) - it never hurts to repeat to get something down solid) follow the trail, do the natural thing, produce two linearly independent solutions, take a linear combination of them to form a general solution, then (Hmmmmm!) use your initial conditions to determine your two constants c_1 and c_2, and sit back and look at your unique final solution to your differential equation. Plot the solution and describe what happens as time goes on for this spring mass dashpot.

j) Let us try another one. Something scary might happen in this one, but do not fret. We are here to help.

$$1 \cdot y''(t) + 6 \cdot y'(t) + 25 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{13}$$

Place your conjecture, $y(t) = e^{\lambda t}$, into differential equation (13). Then, again as in (d) (it never hurts to repeat to get something down solid) follow the trail, do the natural thing, produce two linearly independent solutions, take a linear combination of them to form a general solution, then (Hmmmmm!) use your initial conditions to determine your two constants c_1 and c_2, and sit back and look at your unique final solution to your differential equation. Be sure to plot your solution.

What is the ONLY difference between the differential equation model of a spring mass dashpot in (j) from that in (i)? Hint: recall the physical meaning of each of the constant coefficients. Argue that this difference might be enough to explain the different behaviors as displayed by the plots of the position of the mass at the end of the spring.

k) Here is yet another differential equation (14)

$$1 \cdot y''(t) + 6 \cdot y'(t) + 130 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{14}$$

Again, place your conjecture, $y(t) = e^{\lambda t}$, into differential equation (14). Then follow the trail, do the natural thing, produce two linearly independent solutions, take a linear combination of them to form a general solution, then (Hmmmmm!) use your initial conditions to determine your two constants c_1 and c_2, and sit back and look at your unique final solution to your differential equation. Be sure to plot your solution.

What is the ONLY difference between the differential equation model of a spring mass dashpot in (k) from that in (i) and (j)? Argue that this difference might be enough to explain the different behaviors as displayed by the plots of the position of the mass at the end of the spring.

This complex arithmetic IS scary and we would like to eliminate it, for it may make you uncomfortable. Frankly, it does not make some people uncomfortable, for example electrical engineers thrive on the imaginary $i = \sqrt{-1}$, only they call it j! Well, Leonhard Euler comes to the rescue for us with a beautiful mathematical truth, now called Euler's Identity:

$$e^{i\omega} = \cos(\omega) + i\sin(\omega)\,. \tag{15}$$

Here i is, of course our imaginary square root of -1 while ω stands for the lower case Greek letter omega. Incidentally, *Mathematica* knows Euler's Identity (and more!). You can see this by applying the command `ComplexExpand` to, say e^{3i} or even e^{3it}.

`ComplexExpand[Exp[ 3 I ]]` = _____ .

`ComplexExpand[Exp[ 3 I t]]` = _____ .

What Euler's Identity allows us to do is to change expressions with i in the exponent of e and thus reduce messy formulae with i in an exponential function to one which has i as a coefficient of a sine term. Incidentally, for just about the most beautiful formula in mathematics use Euler's Identity to compute

$$e^{i\pi} = \_\_\_\_\_\,.$$

This set of symbols has it all for mathematics types. We have a colleague who believes this formula is so beautiful and captures the essential major numbers of mathematics that he had a coffee table made with the result tiled in to the surface. Some conversation piece that is!

Let us get back to another differential equation (16) and let us "work" it together and learn even more.

$$1 \cdot y''(t) + 6 \cdot y'(t) + 10 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{16}$$

Our conjecture $y(t) = e^{\lambda t}$ leads us to (17)

$$\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 10 e^{\lambda t} = 0\,, \tag{17}$$

from which we have been factoring out $e^{\lambda t}$ from both sides. Then, because Ms. Baumgartner told us that if the product of two terms, $e^{\lambda t}$ and $\lambda^2 + 6\lambda + 10$ is 0 then one of them must be 0 and since $e^{\lambda t}$ can never be 0 we must have

$$\lambda^2 + 6\lambda + 10 = 0\,. \tag{18}$$

We will call (18) the *characteristic equation* of differential equation (16) for its roots will characterize the solutions of (16). Often the roots of the characteristic equation are called *eigenvalues*. If you have been doing your work here all along you have seen many characteristic equations.

Now we seek the roots of (18). It would be nice and is has been so in several instances thus far if this quadratic factored the way it did early in Ms. Baumgartner's course, but this one does not so we have to use the quadratic equation which Ms. Baumgartner taught us later in that same course. Do you begin to see how important she is in your life? Wait, there will be much more she did for us. Have you found out her school address so you can write to her and tell her how you appreciate all she did for you?

What does the quadratic formula give us for the solution of (17)? It give us the two roots to (18):

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i\,. \tag{19}$$

In our method you have developed to solve the differential equation (17) we now know there are two solutions from which we can build a general solution:

$$y_1(t) = e^{(-3+i)t} \quad \text{and} \quad y_2(t) = e^{(-3-i)t}\,. \tag{20}$$

We know that we need to take a linear combination of these two solutions to form our general solution:

$$y_{\text{general}}(t) = c_1 e^{(-3+i)t} + c_2 e^{(-3-i)t}\,. \tag{21}$$

Using our initial conditions, $y(0) = 2$ and $y'(0) = 0$ yields two equations in the unknowns c_1 and c_2:

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= (-3+i)c_1 + (-3-i)c_2 \end{aligned}$$

from which we find $c_1 = 1 - 3i$ and $c_2 = 1 + 3i$. Finally, then we have our complete solution in (22):

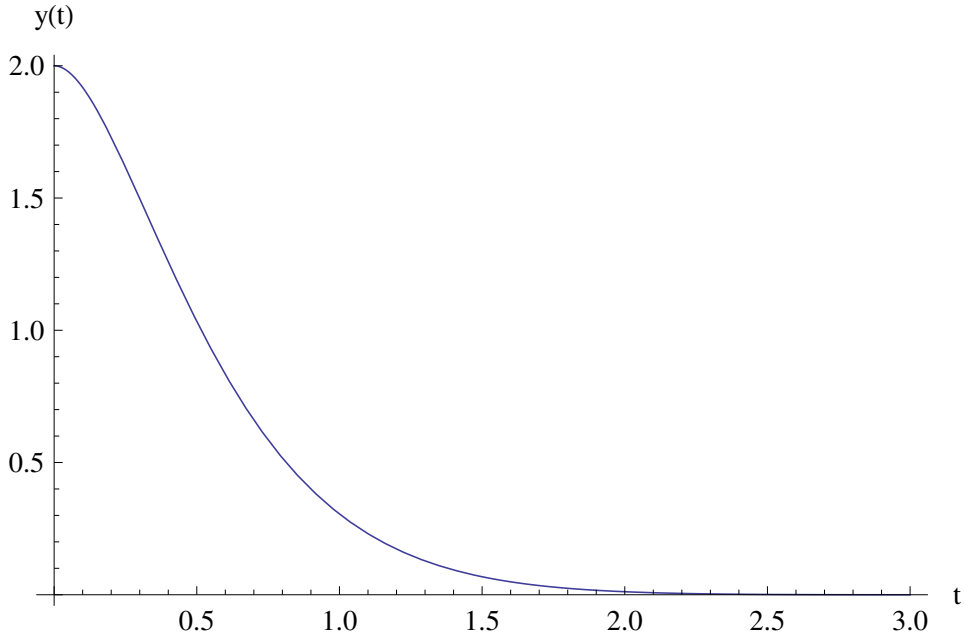

Figure 2. Plot of solution of differential equation (16).

$$y(t) = (1-3i)e^{(-3+i)t} + (1+3i)e^{(-3-i)t}\,. \tag{22}$$

We can live with this, indeed, in *Mathematica* we can plot this (see Figure 2).

Now, if we were to apply Euler's Identity or *Mathematica*'s `ComplexExpand` to (22) we would realize a real looking solution whose plot would be exactly the same as the plot in Figure 2:

$$y(t) = 6e^{-3t}\sin(t) + 2e^{-3t}\cos(t)\,. \tag{23}$$

However, if we applied Euler's Identity to JUST ONE of our solutions from $y_1(t)$ and $y_2(t)$, we could actually get the two linearly independent solutions - real looking, not complex - we need to build a general solution; all just from one complex solution. Let us see how this is possible.

$$\begin{aligned}(1-3i)e^{(3+i)t} &= (1-3i)e^{-3t}\cdot e^{it}\\ &= (1-3i)e^{-3t}(\cos(t)+i\sin(t)\\ &= 3e^{-3t}\sin(t)+e^{-3t}\cos(t)+i\left(e^{-3t}\sin(t)-3e^{-3t}\cos(t)\right)\end{aligned} \tag{24}$$

From (24) we could extract two candidates for solution:

$$yr_1(t) = 3e^{-3t}\sin(t) + e^{-3t}\cos(t) \quad \text{and} \quad yr_2(t) = e^{-3t}\sin(t) - 3e^{-3t}\cos(t)\,. \tag{25}$$

By simply substituting each of these two solutions, $yr_1(t)$ and $yr_2(t)$, into the differential equation (16) we could show that each of these two solutions, $yr_1(t)$ and $yr_2(t)$, actually solves the differential equation (16). Thus we can take the real part and the imaginary part (without the i) of one of our complex solution and form two real solutions. Remember we just need to find two linearly independent solutions to build a general solution and these will always do the trick.

To compete our analysis we need to build our general solution and find the coefficients c_1 and c_2 in our general solution (26)

$$\begin{aligned} yr_{\text{general}} &= c_1 \cdot yr_1(t) + c_2 \cdot yr_2(t) \\ &= c_1 \left(3e^{-3t}\sin(t) + e^{-3t}\cos(t)\right) + c_2 \left(e^{-3t}\sin(t) - 3e^{-3t}\cos(t)\right) \end{aligned} \tag{26}$$

Now it is a bit of an algebraic mess, but we can now use our initial conditions (Hmmmmm!) to solve for c_1 and c_2, although notice that the sine and cosine functions evaluate to 0 and 1, respectively, for an argument of 0. Somebody is looking out for our algebraic sanity. So let us go on, but we shall need our derivative of $yr_{\text{general}}(t)$ first:

$$yr'_{\text{general}}(t) = 10c_2e^{-3t}\cos(t) - 10c_1e^{-3t}\sin(t)$$

Now, finally(!), we use our initial conditions (Hmmmmmm!):

$$\begin{aligned} 2 &= yr_{\text{general}}(0) = c_1 \left(3e^{-3\cdot 0}\sin(0) + e^{-3\cdot 0}\cos(0)\right) + c2 \left(e^{-3\cdot 0}\sin(0) - 3e^{-3\cdot 0}\cos(0)\right) \\ &= c_1 - 3c_2 \end{aligned} \tag{27}$$

and

$$\begin{aligned} 0 &= yr'_{\text{general}}(0) = 10c_2e^{-3\cdot 0}\cos(0) - 10c_1e^{-3\cdot 0}\sin(0) \\ &= 10c_2 \end{aligned} \tag{28}$$

So from our initial conditions equations we see that $c_1 = 2$ and $c_2 = 0$. Thus our final solution is

$$\begin{aligned} y(t) &= c_1 \cdot yr_1(t) + c_2r \cdot yr_2(t) \\ &= = 0 \left(3e^{-3t}\sin(t) + e^{-3t}\cos(t)\right) + 2 \left(e^{-3t}\sin(t) - 3e^{-3t}\cos(t)\right) \\ &= 2e^{-3t}\sin(t) - 6e^{-3t}\cos(t) = 2e^{-3t}\sin(t) + 6e^{-3t}\cos(t) \end{aligned} \tag{29}$$

The latter equality because cosine is an even function and as such $-\cos(t) = \cos(t)$. Thus (29) gives the same result as Equation(23). This means that we have shown several ways to solve the differential equation (16). We review them all in the next activity.

l) Solve the following differential equation (30)

$$1 \cdot y''(t) + 6 \cdot y'(t) + 109 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{30}$$

in three different ways, (1) using a conjectured solution and dealing with all the complex arithmetic and finally applying Euler's Identity to convert the problem into a problem with no i's in it; (2) find the real part of one of the solutions, say the real part of $y_1(t)$ or the real coefficient of the imaginary part of $y_1(t)$. Now using these two parts alone, build a general solution; and (3) using *Mathematica*'s `DSolve` (this will be our preferred way, don't you think). In all cases completely determine the constants c_1 and c_2 (they will be different in each of (1) and (2)) and plot your solutions ALL on the same axes to make sure they are all the same.

One final situation we need to examine

So far here is what we have. If our differential equation (10) has conjectured solutions $y(t) = e^{\lambda t}$ then λ can be of the form:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4 \cdot m \cdot k}}{2 \cdot m}\,. \tag{31}$$

We obtain all this from solving the characteristic equation, $m\lambda^2 + c\lambda + k = 0$, associated with differential equation (10). Now three things can happen in this situation and we outline them as three different Cases:

1. $c^2 - 4 \cdot m \cdot k > 0$;
2. $c^2 - 4 \cdot m \cdot k < 0$; or
3. $c^2 - 4 \cdot m \cdot k = 0$

The term $c^2 - 4 \cdot m \cdot k$ is so discerning or discriminating that we shall call it the *discriminant* for it will discriminate among the various types of solutions to the differential equation (10).

We have already discussed in some detail Cases (1) and (2). Indeed, in Case (1) we get real roots and our solution just dampens to 0 while in Case (2) we get complex roots, causing us to get (through Euler's Identity) oscillations, in real applications, damped oscillations. What of Case (3)?

Each of these cases is referred to with a special name: Case (1) is called *overdamped*; Case (2) is called *underdamped*; and Case(3) is called *critically damped*.

m) Consider these differential equations all from Case (3). Use *Mathematica*'s `DSolve` command to solve them and then make some inferences about what the two solutions for differential equation (10) must look like in Case (3) situation. State your result as a general rule for differential equations of the type of (10). While there is theory to confirm this, we shall no doubt concur with your conclusions and march on!

   i) $y''(t) + 10y'(t) + 25y[t] = 0\,, \quad y(0) = 3 \quad \text{and} \quad y'(0) = 0\,.$
   ii) $y''(t) + 6y'(t) + 9y[t] = 0\,, \quad y(0) = 5 \quad \text{and} \quad y'(0) = 0\,.$
   iii) $y''(t) + 4y'(t) + 4y[t] = 0\,, \quad y(0) = 5 \quad \text{and} \quad y'(0) = 0\,.$
   iv) $2y''(t) + 16y'(t) + 32y[t] = 0\,, \quad y(0) = 5 \quad \text{and} \quad y'(0) = 0\,.$

Bookkeeping and Rearrangements

In our solutions for the second order differential equation (32),

$$a \cdot y''(t) + b \cdot y'(t) + c \cdot y(t) = 0\,, \quad y(0) = y_0 \text{ and } y'(0) = 0\,, \tag{32}$$

which we have found to be so useful, the case where the discriminant, $b^2 - 4 \cdot a \cdot c$, is negative gives rise to complex roots to the characteristic equation and the general solution looks like (33)

$$\begin{aligned} y(t) &= Ae^{\left(\frac{-b}{2a}t\right)} \sin\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) + Be^{\left(\frac{-b}{2a}t\right)} \cos\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) \\ &= e^{\left(\frac{-b}{2a}t\right)} \left(A \sin\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) + B \cos\left(\frac{\sqrt{4ac-b^2}}{2a}t\right)\right) \end{aligned} \tag{33}$$

If we let $\left(\frac{\sqrt{4ac-b^2}}{2a}\right) = \omega$ (33) simplifies to (34)

$$y(t) = e^{\left(\frac{-b}{2a}t\right)} \left(A \sin(\omega t) + B \cos(\omega t)\right)\,. \tag{34}$$

We wish to combine the sine and cosine terms in (34) into one sine function with a phase angle. We can do this because of the trigonometric identity:

$$\sin(x+y) = \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y)\,,$$

and the diagram in Figure 3. Identifying $x = \omega t$ and $y = \theta$ and multiplying and dividing each expression by $\sqrt{A^2 + B^2}$ we can convert the expression $A \sin(\omega t) + B \cos(\omega t)$ to

$$\begin{aligned} &\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}} \sin(\omega t) + \frac{B}{\sqrt{A^2+B^2}} \cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\left(\cos(\theta)\sin(\omega t) + \sin(\theta)\cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\left(\sin(\omega t + \theta)\right) \end{aligned} \tag{35}$$

where $\theta = \text{Arctan}\left(\frac{B}{A}\right)$ is called the *phase shift*.

From (35) we can write our solution (34) as (36)

$$y(t) = \sqrt{A^2+B^2} \cdot e^{\left(\frac{-b}{2a}t\right)} \cdot \sin(\omega t + \theta)\,. \tag{36}$$

The phase angle, θ permits us to see the solution as a $\sin(\omega t)$ type function, but out of phase by θ radians. More precisely, we can write

$$\sin(\omega t + \theta) = \sin(\omega\left(t + \frac{\theta}{\omega}\right))\,, \tag{37}$$

and refer to $\frac{\theta}{\omega}$ as the *phase angle*. Phase angles will be important to engineers and scientists.

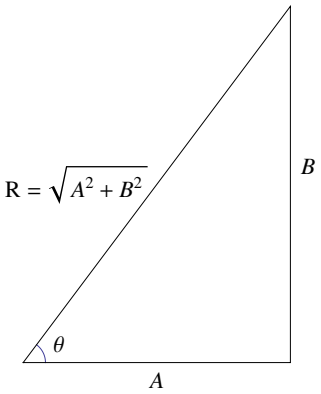

Figure 3. Useful triangle diagram in rearranging an oscillating solution to a second-order differential equation which results in a single phase shifted solution from the sum of sine and cosine terms in solution.

Consider the differential equation (38)

$$1 \cdot y''(t) + 6 \cdot y'(t) + 25 \cdot y(t) = 0 \quad y(0) = 1 \text{ and } y'(0) = 0 \tag{38}$$

whose solution is

$$\text{ysol}(t) = \frac{1}{4} e^{-3t} (3 \sin(4t) + 4 \cos(4t)) .$$

while the phase shifted form of the solution is

$$\begin{aligned} \text{ysol}(t) &= \frac{5}{4} e^{-3t} \sin\left(4t + \tan^{-1}\left(\frac{4}{3}\right)\right) \\ &= \frac{5}{4} e^{-3t} \sin\left(4\left(t + \frac{\tan^{-1}\left(\frac{4}{3}\right)}{4}\right)\right) \\ &= \frac{5}{4} e^{-3t} \sin\left(4\left(t + 0.231824\right)\right) \end{aligned} \tag{39}$$

In (39) we note that $\omega = 4$ and $\theta = \arctan\left(\frac{4}{3}\right) = 0.231824$ radians.

We see in Figure 4 the meaning of phase angle geometrically.

i) Solve the differential equation (40),

$$1 \cdot y''(t) + 10 \cdot y'(t) + 19 \cdot y(t) = 0 , \quad y(0) = 1 \text{ and } y'(0) = 0 . \tag{40}$$

ii) Convert the solution to phase angle format and compute the phase angle θ in radians.

iii) Plot both solutions in (i) and (ii) on the same axis to confirm your analyses. What should you see?

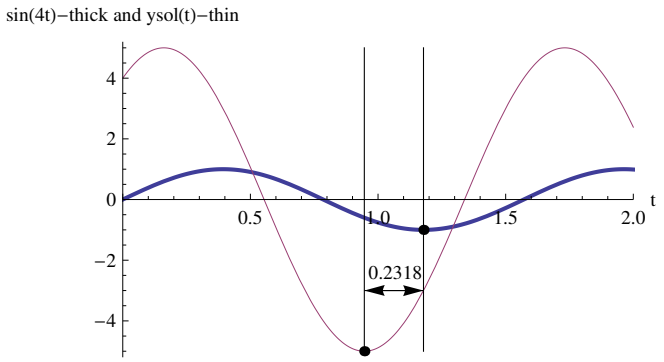

Figure 4. A plot of the oscillating portion (not damped), $3\sin(4t) + 4\cos(4t)$, of the solution (thin) to a second-order differential equation (36) with its simple frequency curve $\sin(\omega t)$ (thick). Notice the phase angle here of 0.2318 radians from bottom to bottom illustrating what we mean by out of phase by a phase angle of 0.2318 radians.

Activity 1 - Three Types of Damped

It is known that for the liquid and shape of the mass in a certain spring mass dashpot configuration the differential equation (41) models this situation quite well:

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = 0\,. \tag{41}$$

where the coefficient of resistance is $c = 7$ N/(m/s) and the mass is $m = 4$ kg.

For what range of values of the spring constant k N/m will the spring system be

i) underdamped;

ii) critically damped; or

iii) overdamped?

iv) Pick a value from each of the three regions you determine, solve the corresponding differential equation, and plot its solution to confirm your claim.

Activity 2 - Design with First Passage Times

Consider the spring mass dashpot system described by the differential equation (42)

$$2 \cdot y''(t) + 4 \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{42}$$

i) For what values of k will the system described in differential equation (42) be underdamped?

ii) For several values of k in (i) find the time the mass first has a displacement of 0. For each value of k call this *first passage time*, $FP(k)$.

iii) Plot $FP(k)$ vs. k for $k \in [0, 20]$. What is happening for the value of $k = 0$ to this function?

iv) For design purposes it may be required to offer up a spring with the lowest k value but still have a first passage time of 0.5 s. Find the smallest k for which $FP(k) = 0.5$.

Now consider the spring mass dashpot system described by the differential equation (43)

$$2 \cdot y''(t) + c \cdot y'(t) + 9.4 \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{43}$$

v) For what values of c will the system described in differential equation (43) be underdamped?

vi) For several values of c in this range of values in (i) find the time the mass first has a displacement of 0. For each value of c call this *first passage time*, $FP(c)$. What is happening for the value of $c = 0$ to this function?

vii) Plot $FP(c)$ vs. c for $c \in [0, 20]$.

viii) For design purposes it may be required to offer up a spring with the lowest c value but still have a first passage time of 0.5 s. Find the smallest c for which $FP(c) = 0.5$.

Here is a rather interesting result that seems to imply initial position has nothing to do with first passage time when all other values m, c, k, and v_0 are fixed!

ix) For the motion described by differential equation (43) show that for a given value of c no matter what initial position y_0 given the first passage time $FP(c)$ is ALWAYS the same.

x) Also show the same is true in the case of the motion described by differential equation (43), i.e. that for a given value of k no matter what initial position y_0 given the first passage time $FP(k)$ is ALWAYS the same.

xi) Incidentally, this last result does *not* apply to initial velocity, i.e. the first passage time varies as we change $y'(0) = v_0$ if every other parameter stays fixed. Prove this result.

xii) Neither does this last result hold true for mass m, i.e. the first passage time varies as we change m if every other parameter stays fixed. Prove this result.

Activity 3 - Modeling Parachuting

In modeling the descent of a parachutist there are two stages to the descent: (1) with the chute unopened, commonly referred to as skydiving and (2) with the chute open, hopefully referred to as the gentle descent.

In setting up a Free Body Diagram of the External Forces on the sky diver there is laboratory and theoretical support for a resistance term NOT of the form $c \cdot y'(t)$, but rather of the form $c \cdot (y'(t))^2$. Of course, there is still the force $(m \cdot g)$ due to the weight of the sky diver and gear, where m is the mass in kg and g is the acceleration due to gravity, usually 9.8 m/s^2.

i) Verify that differential equation (44) is a reasonable model for a sky diver by using a Free Body Diagram. Assume we measure distance and velocity positive downward from the altitude of the plane upon the start of the jump. Identify each of the terms and units in (44).

$$m \cdot y''(t) + c \cdot y'(t)^2 = m \cdot g\,, \quad y(0) = 0\,, \quad y'(0) = 0\,. \tag{44}$$

ii) Indeed, the units for the resistance coefficient, c, must be $\frac{kg}{m}$ so that the units on $c \cdot (y'(t))^2$ can be in Newtons or $kg \cdot \frac{m}{s^2}$.

iii) Suppose we make our first jump (our mass is 100 kg) out of a plane at an altitude of 3,500 m for a vertical descent first without a parachute and then with a parachute. We are quite conservative and are a bit frightened of just how long we want to free fall.

Ascertain the size of an appropriate resistance coefficient, c $\frac{kg}{m}$, of the human body in flat or layout position, perhaps with added loose material in a jump suit. You might want to look on the web for such data and then document it in your report. What we want so that before the parachute opens we reach terminal velocity of 65 m/s (not too fast, not too slow). Let the sky diver fall some 2,500 m to an altitude of 1,000 m and at that altitude pull the parachute open and glide to earth. Now determine the size of the resistance coefficient, c, for an opened chute so that we land with a slow to medium hit-the-ground velocity of 3.0 m/s.

iv) Implement the second stage (using your parachute) model and plot both altitude and velocity from when we pull the chute open to landing.

iv) How long was our total jump from plane to ground? How much time did we spend in each of the two stages, free fall and parachute fall?

v) Can we expect to design for these two values of c, the one in free fall stage and the one in chute fall stage? If so, then we are good to go? If not, then what might we advise our parachutist to do so as to have a good flight and a safe landing?

Here is some information that you should know. Reasonable terminal velocity for a parachutist hitting the ground would be slow 2.1 m/s^2, medium 3.3 m/s^2, and fast 4.6 m/s^2 [5]. Terminal velocity for a sky diver is about 53 m/s to 76 m/s [2].

Activity 4 - Frequency of Spring Mass Dashpot

Consider a spring mass dashpot system whose vertical displacement of the mass attached to the spring is given by the differential equation (45) with mass displacement $y(t)$ in m, m in kg, and spring constant k in N/m:

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = 0\,. \tag{45}$$

First, let us suppose that there is no resistance, i.e. $c = 0$. This means we are examining the following differential equation (46):

$$m \cdot y''(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = 0\,. \tag{46}$$

i) For the differential equation (46) determine its oscillation frequency in Hz (i.e. cycles per second). From the characteristic equation and its roots in this case you should know that it is underdamped and has oscillation. Explain why.

iii) For the differential equation (45) find criteria on the coefficients m, c, and k so that this system will oscillate and be damped. In that case determine its "oscillation" frequency in Hz (i.e. cycles per second). We write "oscillation" because while the system is damped the mass will go from a high to a low and back to high again in a constant length interval of time and that will help you determine the frequency.

Activity 5 - Logarithmic Decrement

Consider the underdamped oscillator (discriminant, $c^2 - 4mk < 0$) in differential equation (47):

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,, \tag{47}$$

i) Determine the frequency of the oscillator described in (47). This is called the *quasi frequency* (μ) in light of the fact that the motion is not truly periodic due to damping.

ii) Determine the frequency of the oscillator described in (47), only this time let $c = 0$, i.e. we have a true pure oscillator. This is called the *natural frequency* (ω_0) of this system.

iii) Now the ratio of μ over ω_0 will permit us to see that the result of damping $c > 0$ is to reduce the natural frequency of the system. Explain this statement.

iv) The period of the undamped system is $T = \frac{2\pi}{\omega_0}$ while the *quasi period* of the damped system can be defined as $T_d = \frac{2\pi}{\mu}$ and hence the ratio of T_d to T is yet another way to compare the two systems, damped with undamped. Find this ratio and verify that damping increases the quasi period when compared to the period.

The natural logarithm of the decrement defined in (v) below is called the *logarithmic decrement* (Δ) and can be shown to equal $\frac{\pi c}{m\mu}$. Since all the quantities k, m, μ, and T_d can be measured and since c is embedded in the expression for $\Delta = \frac{\pi c}{m\mu}$ then we can use Δ to determine c from observations.

v) We can show the ratio of the displacements of two successive maxima of the system's motion (call this ratio the system's *decrement*) in the case of the damped system is $e^{\frac{cT_d}{2m}}$. If you can determine this fact in general then do so. Otherwise, use this set of parameters, $m = 4$, $c = 0.1$, $k = 2.5$, with $y_0 = 1$ and $y'(0) = 0$, and show that the this system's decrement is 0.905418.

vi) In Figure 5 we have the graphical output of a underdamped oscillator whose governing equation is differential equation (47) with the following values of parameters and initial conditions: $m = 1$ kg, $k = 20$ N/m, $y(0) = 3$ m, and $y'(0) = 0$ m/s. Use your approach from (v) and information from (iv) to determine the parameter c N/(m/s). Once you have ascertained c then solve your differential equation (47) model using that value and compare your solution to the data through a plot of your solution. How well are you able to estimate c? Note: In Activity 9 - Inverse Problems we shall attempt to determine c in another manner.

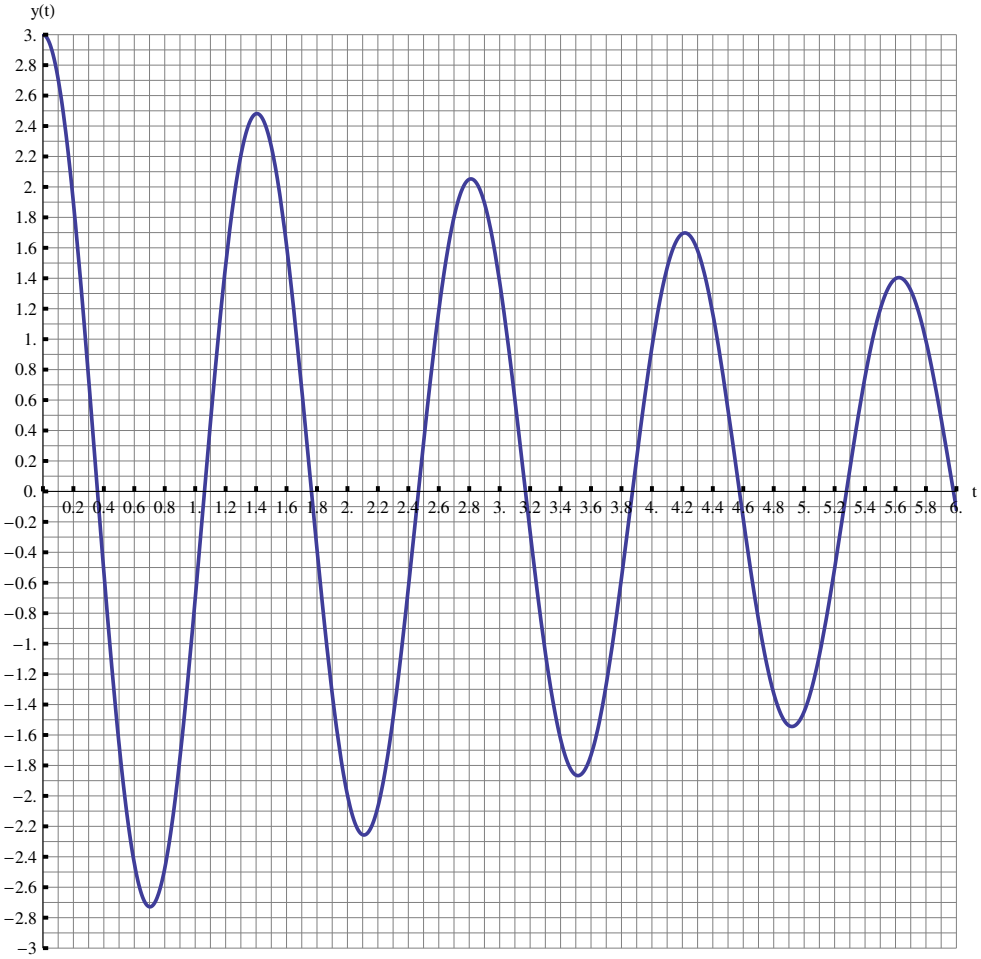

Figure 5. Plot of solution to differential Equation (47) with the following values of parameters and initial conditions: $m = 1$ kg, $k = 20$ N/m, $y(0) = 3$ m, and $y'(0) = 0$ m/s in which we seek a value for c.

Activity 6 - Buoyancy Force

Archimedes' Principle on buoyancy says that if you place an object or portion of an object in a liquid that there is an upward force on the object equal to the weight of the displaced liquid. It is known that the mass density of water at 20° C is 992 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2. Consider a right circular cylinder of radius $r = 2$ m and length $L = 5$ m which is submerged in water with its flat circular base some 4 m below the surface. Most of us know this intuitively if we have ever tried to put a beach ball under water in the summer pool, it is easy to put it in a ways, but hard to really submerge it totally.

a) Compute the vertical force on the cylinder due to Archimedes' Principle in the above case.

b) Compute the vertical force on the cylinder due to Archimedes' Principle if the cylinder is

submerged in water with its flat circular base some s m below the surface.

We would like to model the motion of a buoy made of American redwood in the shape right circular cylinder. The mass density of the redwood is 450 kg/m^3 [4] and the cylinder has dimensions of radius $r = 2$ m and length $L = 5$ m. We shall presume the cylinder stays in the configuration of its flat circular base downward and parallel to the surface of the water, i.e. no tipping.

The cylinder will float at an equilibrium depth, dEq m, measured from the bottom of the cylinder, when the upward force due to Archimedes' Principle and the force due to the weigh of the object are equal (and opposite) to each other.

c) Show that one can find the equilibrium depth, dEq, when the upward force due to Archimedes' Principle and the force due to the weigh of the object are equal (and opposite) to each other. Do this by justifying and then solving (48), where h is the depth of the cylinder in the water. Show how the equation is built first and then solve it for depth $h = dEq$, the equilibrium depth. Call the height on the cylinder of the equilibrium depth the *equilibrium point* or *level*.

$$\pi \cdot r^2 \cdot h \cdot \rho_{\text{Mass Density Water}} \cdot g = \pi \cdot r^2 \cdot L \cdot \rho_{\text{Mass Density Redwood}} \cdot g \tag{48}$$

d) Let $x(t)$ be the height at which the equilibrium point (recall the equilibrium point is dEq m, measured from the bottom of the cylinder) is above the water level. We suggest you draw a diagram at this point to keep track of "Who's on first" and "What's on second!" Incidentally if you need a study break then we suggest you go to a web site which features the great comedians Abbot and Costello routine [1], "Who's on first, what's on second!"

Back on track now? Then let us build a differential equation for the height of the equilibrium point above and below the water as the buoy bobs vertically. (We will presume, again, that the buoy stays vertical and does not tilt - that is a harder problem!)

Recall Newton's Second Law which says

$$m \cdot x''(t) \quad = \quad \sum_{i=1}^{n} (\text{External Force})_i \; = \; \underline{\qquad\qquad\qquad\qquad}$$

where m is the mass in kg of the buoy and $x(t)$ is the height in m of the equilibrium point above the water level. As a convention let us measure $x(t) > 0$ when the equilibrium is higher than the water level and $x(t) < 0$ when the equilibrium is below the water level. Then $x'(t) > 0$ means the buoy is moving upward while $x'(t) < 0$ means the buoy is moving downward.

We need to find these External Forces. At first glance these will be (1) the buoyant force due to Archimedes' Principle in which the cylinder is $dEq - x(t)$ m deep in the water, the equilibrium point dEq having risen $x(t)$ out of the water and (2) the weight of the cylinder. Use (1) and (2) to build a more fleshed out differential equation than offered in (49).

e) Solve your differential equation built in (d) and plot your solution over the time interval $t \in [0, 50]$ s. Describe what happens physically and mathematically as completely as you can.

f) Now there is something unrealistic about your plot in (e). If you have not already commented on it in your response to (e), then do so here. We shall need to make a better model to rid ourselves of this problem. To do this we will have to introduce some resistance term(s) so that the system loses energy and does not bob forever to the same height.

There will be two kinds of resistance we want to consider (1) resistance due to the bottom of the cylinder plunging into the water as the cylinder goes deeper into the liquid (it does not "plunge" into the liquid as the cylinder rises) and (2) resistance due to the rough edges of the vertical surface of the cylinder "rubbing" in the water.

Attempt to model both these forces and you put constants of proportionality in place (as we have no data on the motion of a buoy here) for each form of resistance (1) and (2). Hint: We need to keep track of which direction the resistance will apply and for that we may need to know where we are, i.e. $x(t)$, and how fast we are moving and in which direction, i.e, $x'(t)$. Since we shall have changes in sign you might want to use *Mathematica*'s `Sign` function, e.g., `Sign[x'[t]` will tell you the sign, either +1, 0, or -1, of the velocity and so the quantity `(1-Sign[x'[t]])/2` will be 1 if $x'(t) < 0$ and 0 if $x'(t) > 0$. This might help you in modeling (2).

g) Plot your solution in (f) over the time interval $t \in [0, 50]$ s and compare it with your no resistance modeling of (e). Explain what you see and what the differences are. Does this make sense? You might have to tweak your constants of proportionality for each term due to (1) and (2) above. Be sure to push your model to extremes, e.g., consider balsa wood, teak wood, or ranges of your parameters due to (1) and (2). Write up your conclusions.

Here is some *Mathematica* code which will permit you to animate your cylindrical tank in all situations, Here `xsR[t]` is the solution of your differential equation model.

```
plane = Plot3D[0, {x, -3, 3}, {y, -3, 3}, Mesh -> False]

cy[t_] :=   Graphics3D[
  Cylinder[{{0, 0, xsR[t] - dEq}, {0, 0, xsR[t] + L - dEq}}, 2]]

Animate[Show[{plane, cy[t]},
  PlotRange -> {{-3, 3}, {-3, 3}, {-10, 10}},
  Axes -> True, AxesLabel -> {"", "", "height"}, Boxed -> False,
  ViewPoint -> {0, 1, 0}, ImageSize -> 800], {t, 0, 100, .01}]
```

Here `plane` is the plane of the water surface and `cy[t_]` is the graphic object of a cylinder of height L whose base is at height `xsR[t] - dEq` and whose top is at height `xsR[t] - dEq+L`.

Activity 7 - Bad News If Positive Real Number in Exponential Function

Consider the general case for the spring mass dashpot device with mass m (> 0) kg, resistance coefficient c (> 0) N/(m/s), spring constant k (> 0) N/m, initial position $y(0) = y_0$ m, and initial velocity $y'(0) = v_0$ m/s.

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{49}$$

Explain why, in our solution for $y(t)$, there will never be exponential functions with positive real part. Why is this good and what would you say about such a model if it did have exponential functions with positive real part in the solution.

Activity 8 - Rocket Thrust Modeling

Suppose we have a small rocket with a full 100 liter fuel tank. Fuel (with a mass density of .98 kg/l) is burned at a steady rate of 3 liters per second and can provide a constant thrust force of 5900 Newtons as it burns. The rocket and fuel tank has a mass of 400 kg with no fuel in it. The shape design of the rocket causes a resistance proportional to its velocity during flight of 2 $v(t)$ where $v(t)$ is in m/s and the constant 2 is that of resistance and is really 2 N/(m/s). We aim the rocket straight up and launch it . . . 5 . . . 4 . . . 3 . . . 2 . . . 1 . . . Launch . . . We have lift off!!!

i) How long will the rocket burn and hence provide thrust? We call this time interval the *burn period*.

ii) Build a differential equation model based on all the External Forces and Newton's Second Laws.
Here are some issues to consider. Construct a Free Body Diagram - very important. What are reasonable initial conditions? The mass $m(t)$ is now a function of time; what is that function. There is an external constant force due to thrust during the burn and the weight of the rocket.

iii) Solve the differential equation from (ii) (probably best to use *Mathematica*'s `NDSolve` command) and plot the rocket's altitude over the burn period. Also plot the velocity over the burn period.

iv) Determine maximum height and velocity of this rocket?

v) Model what happens to the rocket after the burn period? Plot the rocket's altitude over the free falling period. Also plot the velocity over the free falling period.

vi) How long before it strikes the earth? That is, how long is the total flight? We may assume this is a vertical flight only.

vii) With what velocity does it strike the ground? Ouch!

In the above model we have moved away from constant coefficient differential equations. Moreover, we have introduced additional, important external forces (thrust and weight) to the model of the rocket. These forces will make our differential equation into what is called *nonhomogeneous* differential equation. We shall see more of these. Furthermore, there is no spring constant term here as the rocket is not on a tether of any sort!

Activity 9 - Inverse Problem

This is an example of an *inverse problem* in which we are given data and we seek to estimate a parameter in the model so as to fit this data with a model.

Suppose we have observations (Table 1) on position, $y(t)$, in cm, vs. time t in s of a small spring in a dashpot. We know the mass is $m = 1$ g and the spring constant is $k = 20$ dyne/cm. We can measure those.

t s	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y(t)$ cm	2.16	1.77	1.40	1.08	0.71	0.39	0.18	0.03	-0.06	- 0.01	-0.10

Table 1. Data on the displacement of a mass on the end of a spring.

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{50}$$

We seek to find the resistance coefficient c in dyne/(m/s) where these parameters are in a differential equation (50)

Discuss how you would determine c. Offer up several ideas. Carry out one approach at least. Confirm your estimate for c in some manner. If you have offered up several approaches compare your results. How does your model with your c value compare to the data?

Activity 10 - Keeping Costs Down

Stiffness in a spring (i.e. the restoration constant's k N/m) value costs money. The more the stiffness the more expensive the spring. Let us assume a cost, in dollars, for a 0.2 m long spring of $\text{Cost}(k) = 13.70 + .01k^2$.

Now suppose we are designing the following spring mass dashpot system:

$$1.2 \cdot y''(t) + 46.4 \cdot y'(t) + k \cdot y(t) = 0\,, \quad y(0) = 2\,, \quad y'(0) = 0\,. \tag{51}$$

i) Find the value(s) of k which make the system overdamped. Do the same for critically damped.

ii) Suppose in the case of the overdamped configuration we wish to bring the mass to within 0.1 m of the static equilibrium in the first 0.5 s of motion. What k value would we use to do this? How much would such a spring cost us?

Incidentally, the engineers who work for luxury car manufacturers are VERY interested in reducing the oscillations in the vehicles they design as quickly as possible for luxury comfort. The devices in cars which do this are called *shock absorbers* and they are essentially a spring dashpot. Think about the ride when you ride in your friend's 13 year old beat-up VW compared to that in your Uncle Jack's new Cadillac.

iii) What kind of spring performance can we get if we are willing to spend \$600 on the spring?

iv) It is conjectured that for an x% reduction in the tolerance, i.e. x% less than the 0.1 m close to the static equilibrium, there is an x% increase in price. This means that when $x = 80$ we are comparing costs of a spring which will get us to within 0.1 m of the static equilibrium in 0.5 s

to a spring that will get us to within $.8 * .1 = .08$ m of the static equilibrium in 0.5 s. Prove or disprove this conjecture.

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