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STUDENT VERSION

Leaky Bucket

Brian Winkel
Director SIMIODE
Chardon OH 44024 USA
BrianWinkel@SIMIODE.org

Abstract: We seek to model the height of water in a cylindrical tank (bucket) in which water flows out the bottom of the tank through a small bore hole while we are pouring water into the tank at the top of the tank at a constant (or varying) rate.

SCENARIO DESCRIPTION

The tank is a right circular cylinder of radius 4 cm and height 13 cm and the initial height of the water in the tank is 10 cm. See Figure 1. Water is flowing out of the tank through a small bore hole of radius r_C cm. Water is coming into the tank at a steady rate of $F\text{cm}^3/\text{s}$.

a) Form a differential model for the height of water in the tank at time t s. The material from [1, pp. 1-4] will help you do this.

First we need to determine the governing equations or model for the height of a column of liquid in a cone with a hole in the bottom from which the liquid leaks out. We refer to Torricelli' s Law development as outlined in [1, pp. 1-4].

Per this analysis, we calculate the flow out of the hole at the bottom of the cone in two different ways and set them equal.

The left hand side of our differential equation (1) is given by a small element of height change, $h'(t)$, over a cross sectional area $A(h(t))$ at height $h(t)$. Hence.

$$A(h(t)) \cdot h'(t)\,, \tag{1}$$

where $A(h(t))$ is the cross sectional area in our cone at height $h(t)$ in the cone.

The right hand side of our differential equation (2), according to the Conservation of Energy Principle derivation developed for Torricelli's Law in the above reference, is

$$\alpha\,\pi\,(r_C)^2\sqrt{2\,g\,h(t)}\,, \tag{2}$$

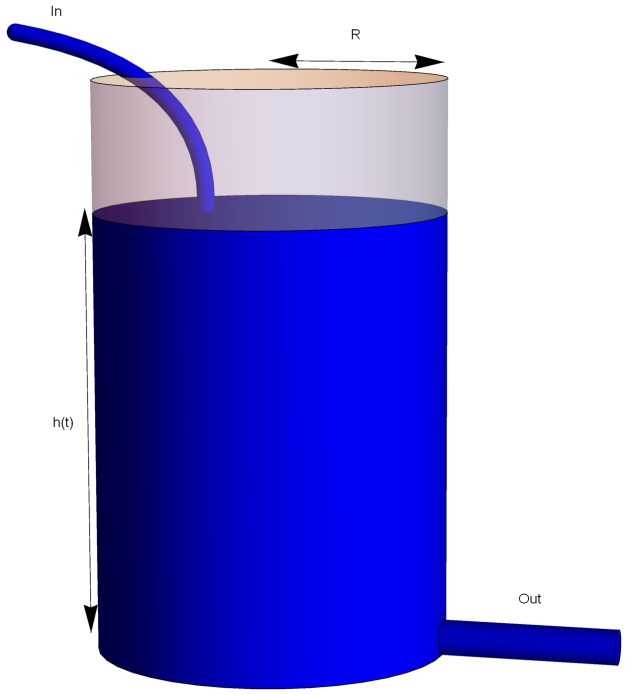

Figure 1. Diagram of the Leaky Bucket with water flowing in and out.

where α is the effective area (percentage due to friction at the opening called the discharge or contraction coefficient) for fluid flow through the small opening at the bottom of the cone; r_C is the radius of the hole on the bottom of our cone; and g is the acceleration due to gravity.

Now you need to account for the additional input flow at constant rate of F cubic units of water per unit time.

(b) Use the following parameters to solve the differential equation model for the amount of water in the tank at time t s. $\alpha = 0.70$, $r_C = 0.5$ cm, $g = 980\,\text{cm/s}^2$, $R = 4$ cm, $H = 10$ cm, and $F = 3\,\text{cm}^3/\text{s}$.

Describe what happens to the water level in the tank, in the short term and in the long term.

Is there a time at which the rate at which the water enters is equal to the rate at which the water leaves? If so determine this value of time. If there were we would call it an *equilibrium value* or height.

(c) Plot the water level over the first 20 s of the situation and determine when the tank is half full.

REFERENCES

[1] Winkel, Brian. 2015. 1-015-T-Torricelli.