MODELY SVĚTOVÝCH POPULACÍ

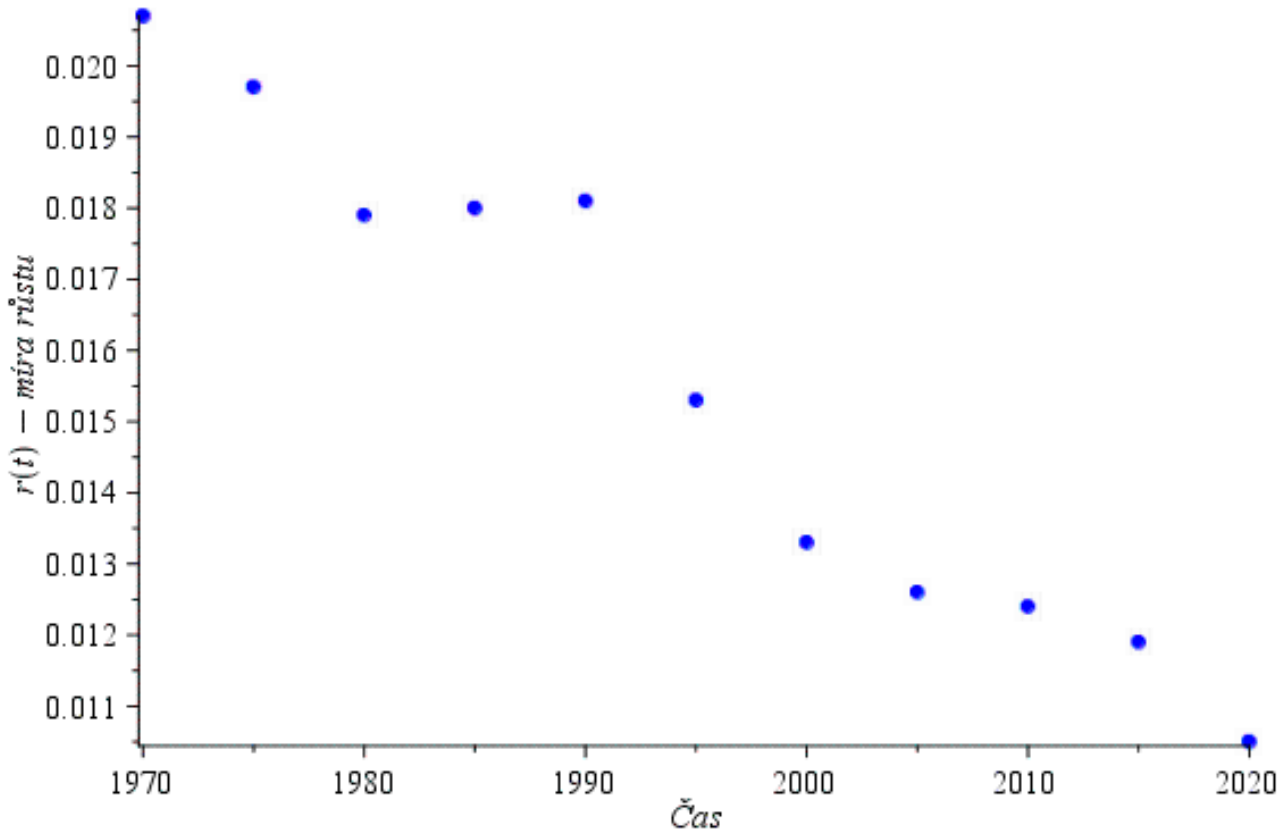
– maple worksheet

autoři : *Lenka Přibylová, Jan Ševčík,*
Pavel Morcinek, Brian Winkel

https://www.math.muni.cz/~pribylova/planet/

Data míry růstu světové populace v letech 1970-2020

```
> restart :
> with(LinearAlgebra) :
> rr := Vector(11, [0.02070, 0.01970, 0.01790, 0.01800, 0.01810,
      0.01530, 0.0133 , 0.01260, 0.0124 , 0.01190, 0.0105]) :
> yr := Vector(11, [1970, 1975, 1980, 1985, 1990, 1995, 2000, 2005,
      2010, 2015, 2020]) :
> plot(yr, rr, style = point, symbol = solidcircle, color = "Blue", labels
      = [Čas, r(t) - míra růstu], labeldirections = [horizontal,
     vertical] )
```



Malthusův model (exponenciální růst)

Míru růstu aproximujeme konstantou - průměrnou hodnotou.

```
> with(ArrayTools) :
> r_average := add(rr) / Size(rr, 1)
```

$$r_average := 0.01549090909$$

Řešením Malthusova modelu je exponenciální funkce

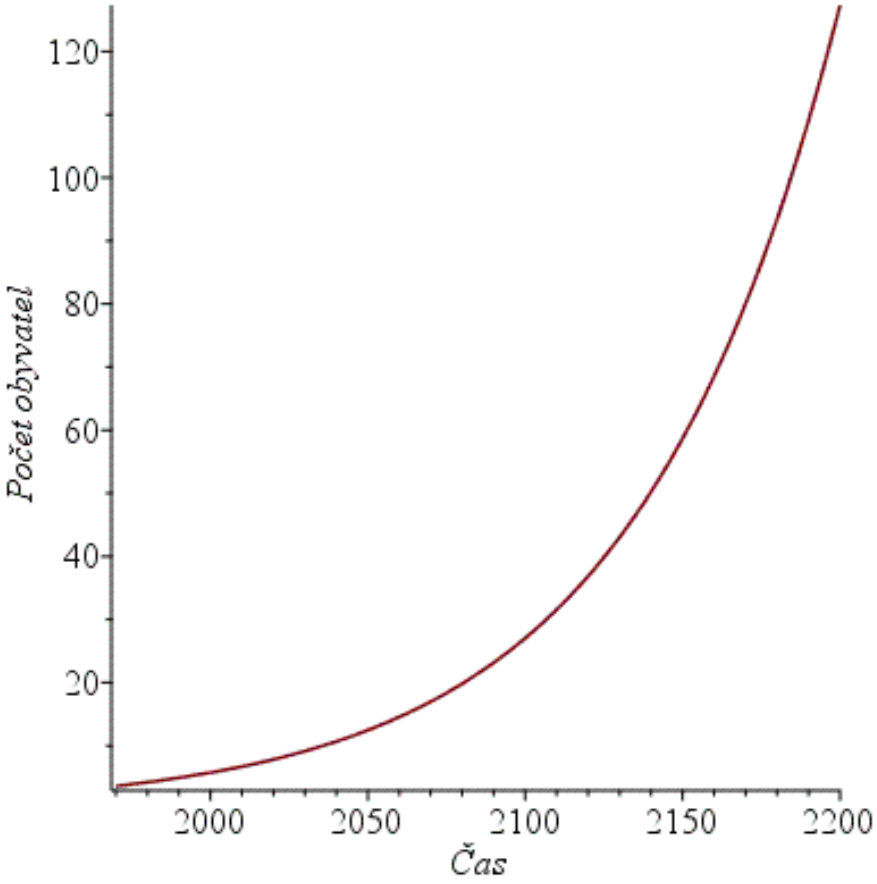
```
> y_general := dsolve({y'(t) = r·y(t), y(0) = y0})
```

$$y_general := y(t) = y0\, e^{r\,t}$$

```
> y_0 := evalf(dsolve({y'(t) = r_average·y(t), y(2019) = 7.714}))
```

$$y_0 := y(t) = 2.014755038\ 10^{-13}\ e^{0.01549090909\,t}$$

```
> plot(rhs(y_0), t = 1970 ..2200, labels = [Čas, Počet obyvatel],
      labeldirections = [horizontal, vertical])
```



MODEL (a)

Míru růstu aproximujeme lineární funkcí času.

```
> with(Statistics) : with(plots) :
> r(t) := a0 + a1·t :
```

Řešení půjde i v tomto případě najít obecně:

```
> y_general := simplify(evalf(dsolve({y'(t) = r(t)·y(t), y(2019)
      = 7.714}))) )
```

$$y_general := y(t) = 7.714000000\, e^{0.5\,(t-2019.)\,(a1\,t+2.\,a0+2019.\,a1)}$$

> $LinearFit(r(t), yr, rr, t)$

$$0.430088181818182 - 0.000207818181818182\, t$$

> $a := LinearFit(r(t), yr, rr, t, output = parametervalues)$

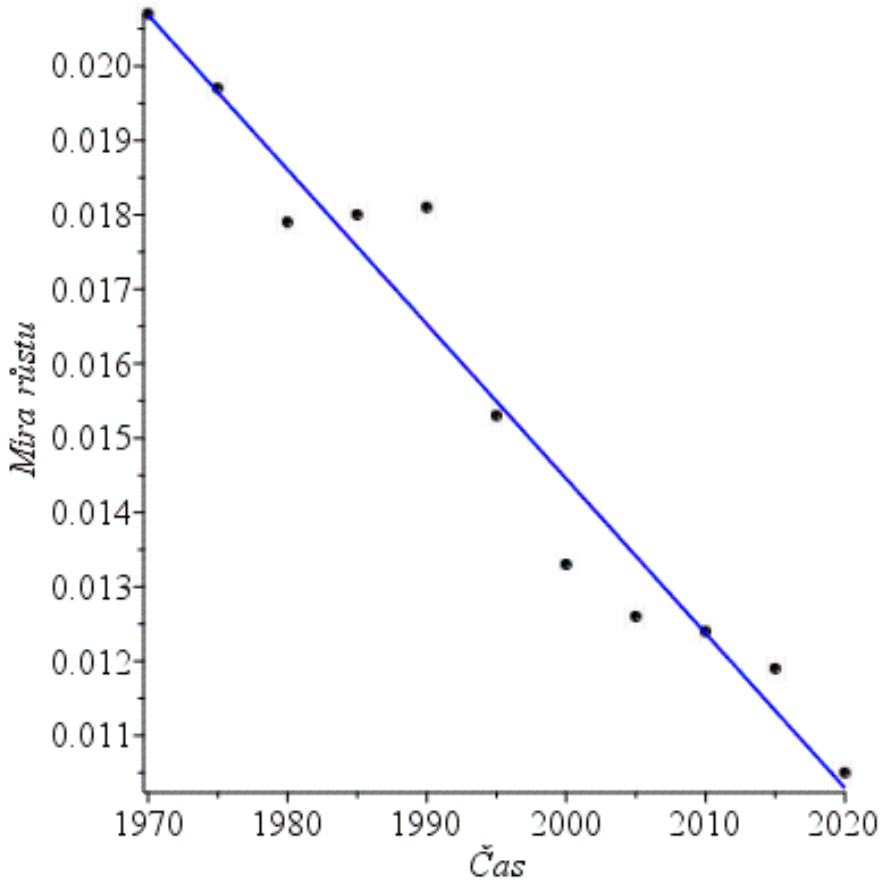
$$a := \begin{bmatrix} 0.430088181818182 \\ -0.000207818181818182 \end{bmatrix}$$

> $a0 := a[1] : a1 := a[2] :$

> $P1 := plot(r(t), t = 1970..2020, color = "blue") :$

> $P2 := plot(yr, rr, style = point, symbol = solidcircle, color = "black", labels = [Čas, Míra růstu], labeldirections = [horizontal, vertical]) :$

> $display(P2, P1)$

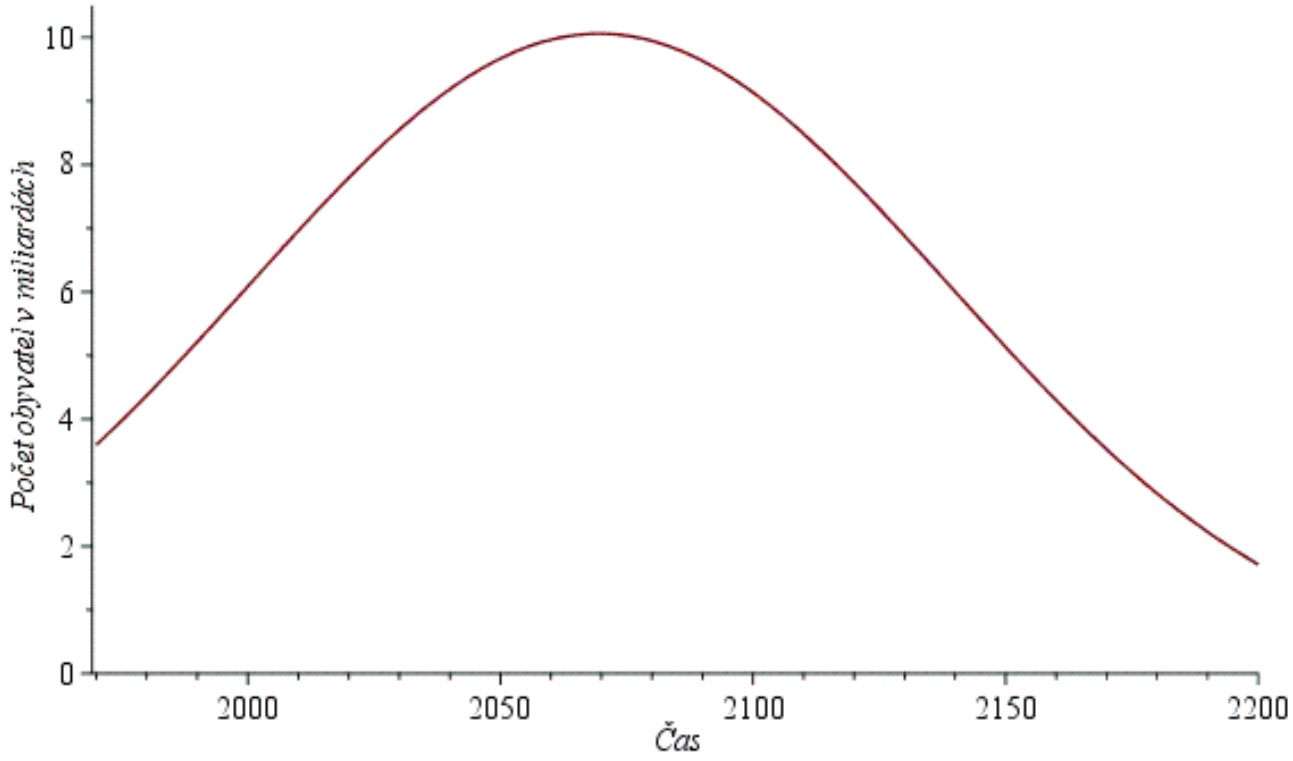


Řešením modelu s touto mírou růstu je

> $y_1 := simplify(evalf(dsolve(\{y'(t) = r(t)\cdot y(t), y(2019) = 7.714\})))$

$$y_1 := y(t) = 5.285072871\, 10^{-193}\, e^{-0.0001039090909\,(t-4139.081365)\,t}$$

> $N1 := plot(rhs(y_1), t = 1970..2200, labels = [Čas, Počet obyvatel v miliardách], labeldirections = [horizontal, vertical], view = [1970..2200, 0..10.5])$



> $T[\max] := -\frac{a0}{a1};$

$$T_{\max} := 2069.54068241470$$

> $evalf(subs(t = T[\max], rhs(y_1)));$

$$10.05890294$$

Maximální velikost populace bude asi 10 miliard v polovině roku 2069.

MODEL (b)

Míru růstu aproximujeme funkcí s periodickou složkou.

> $unassign('a0'); unassign('a1');$

> $with(Statistics) : with(plots) :$

> $L := 30 :$

> $r2(t) := a0 + a1 \cdot t + a2 \cdot \sin\left(\frac{2 \cdot \mathrm{Pi} \cdot t}{L} + a3\right);$

$$r2 := t \mapsto a0 + a1\, t + a2 \sin\left(\frac{2\,\pi\, t}{L} + a3\right)$$

> $simplify(rhs(dsolve(\{y'(t) = r2(t) \cdot y(t), y(2019) = N[0]\})));$

$$N_0\, e^{\frac{1}{2\,\pi}\left(-30\, a2 \cos\left(\frac{\pi\, t}{15} + a3\right) - 30\, a2 \sin\left(\frac{\pi}{10} + a3\right) + \pi\,(t - 2019)\,(a1\, t + 2\, a0 + 2019\, a1)\right)}$$

> $y_general := simplify(evalf(dsolve(\{y'(t) = r2(t)\cdot y(t), y(2019) = 7.714\})))$:

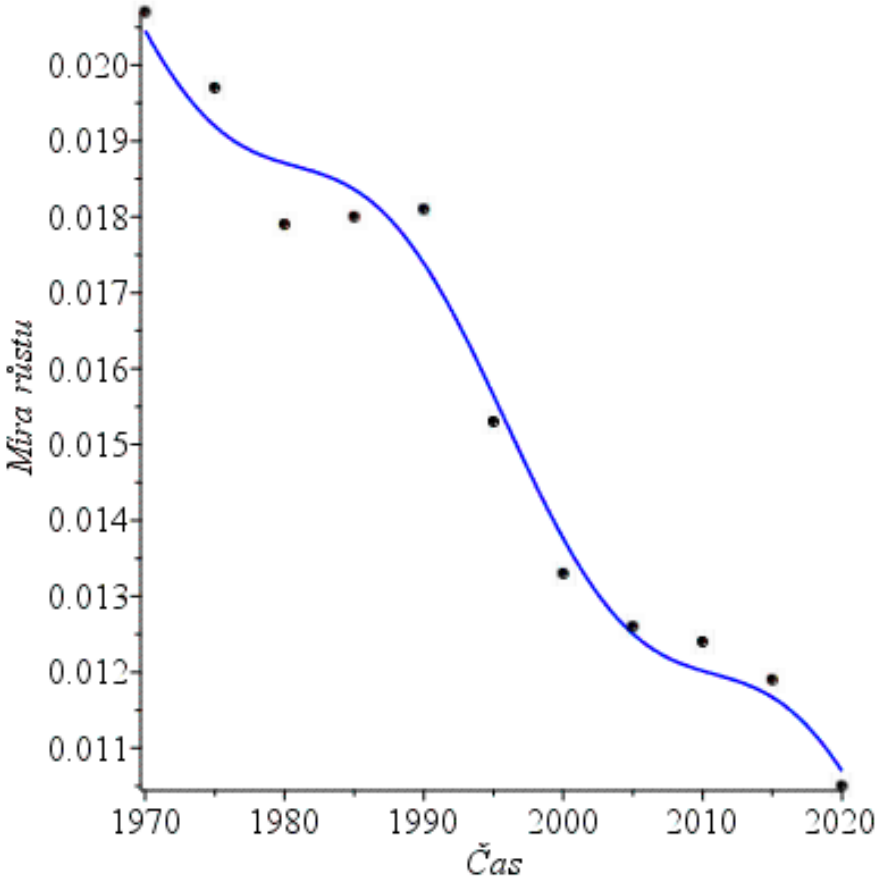
> $Fit(r2(t), yr, rr, t)$

$$0.460475443158841 - 0.000223043478425951\, t + 0.000817978984624507 \sin\left(\frac{\pi t}{15} - 0.172919515467782\right)$$

> $r22(t) := Fit(r2(t), yr, rr, t)$:

> $P1 := plot(r22(t), t = 1970..2020, color = \text{"blue"}, labels = [Čas, Míra růstu], labeldirections = [horizontal, vertical])$:

> $display(P1, P2)$

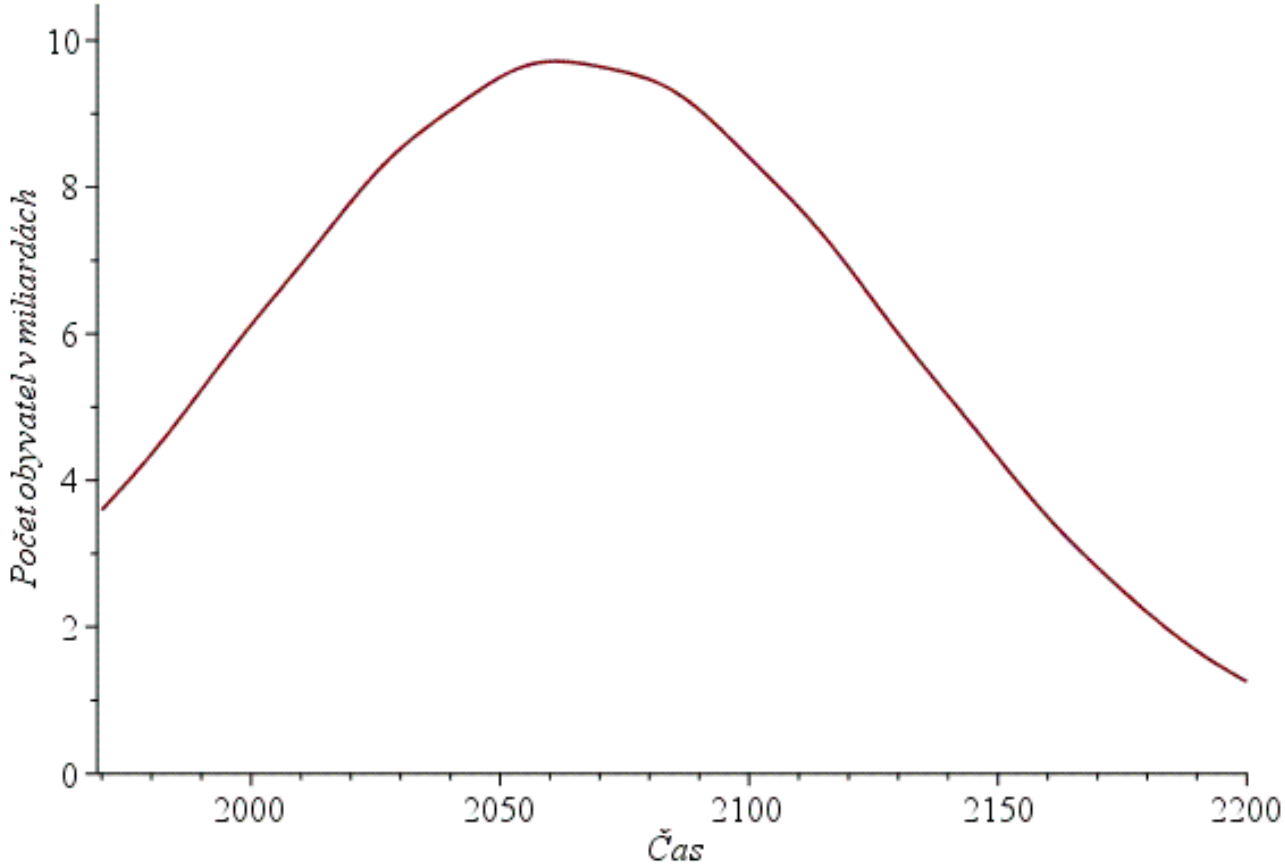


Řešením modelu s touto mírou růstu je

> $y_2 := evalf(dsolve(\{y'(t) = r22(t)\cdot y(t), y(2019) = 7.714\}))$;

$$y_2 := y(t) = 3.589327706\, 10^{-206}\, e^{-0.0001115217392\, t^2 + 0.4604754430\, t - 0.003905561962 \cos(0.2094395103\, t - 0.1729195155)}$$

```
> N2 := plot(rhs(y_2), t = 1970 ..2200, labels = [Čas,
      Počet obyvatel v miliardách], labeldirections = [horizontal,
     vertical], view = [1970 ..2200, 0 ..10.5])
```




```
> data := seq([evalf(subs(t = i, rhs(y_2)))], i = 2019 ..2200) :
> 2018 + max_index([data])[1]; max([data])
```

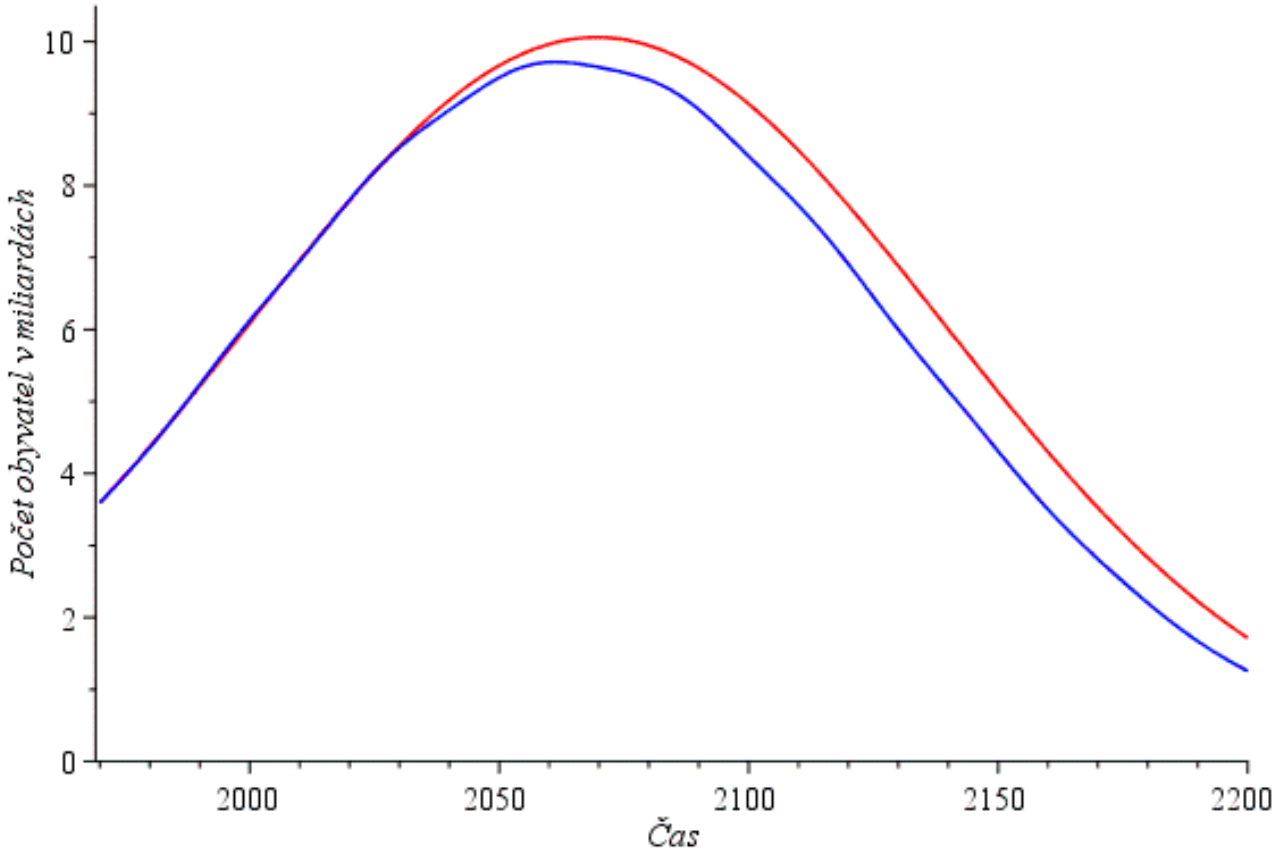
2061

9.717400649

Maximální velikost populace bude asi 9,7 miliard v polovině roku 2061.

Porovnání modelů s mírou růstu klesající v čase (a) - (b)

```
> display(N1, N2, color = ["red", "blue"])
```



MODEL (c)

Logistický model růstu populace, míra růstu je lineární funkcí velikosti populace.

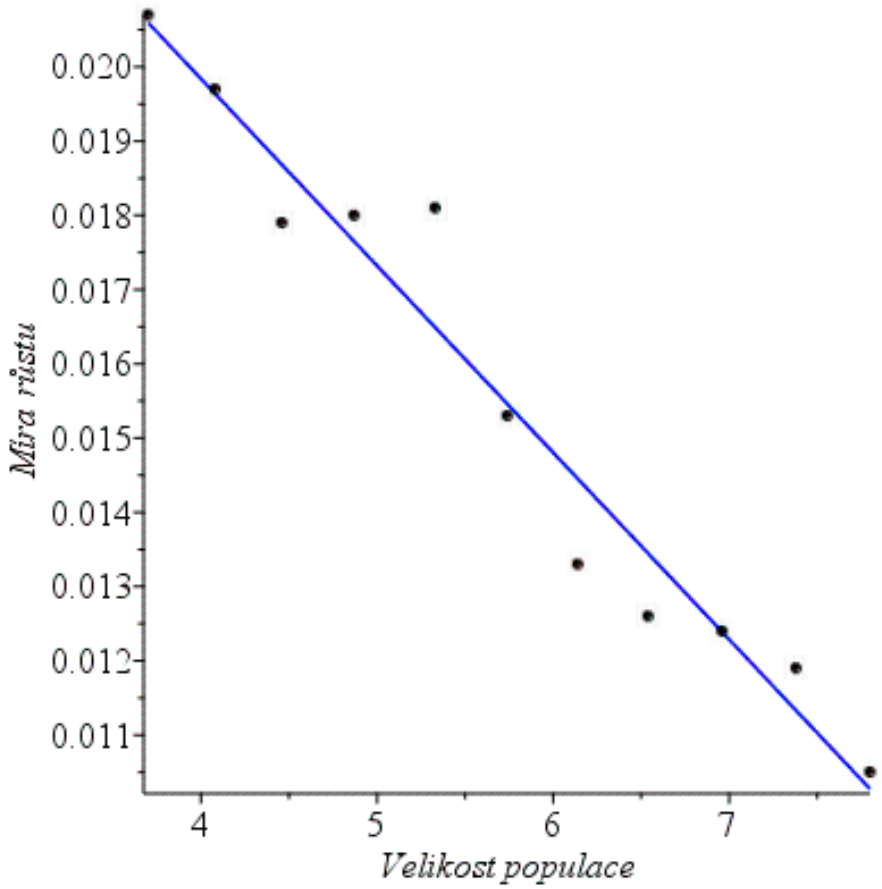
```
> unassign('a0'); unassign('a1'); unassign('a2'); unassign('a3');
> with(Statistics) : with(plots) :
> Nr := Vector(11, [3.7, 4.08, 4.46, 4.87, 5.33, 5.74, 6.14, 6.54, 6.96,
      7.38, 7.8]) :
> r3(N) := a0 + a1·N :
> y_general := simplify(evalf(dsolve({y'(t) = r3(t)·y(t), y(2019)
      = 7.714})))
```

$$y_general := y(t) = 7.714000000\, e^{0.5\,(t - 2019.)\,(a1\,t + 2.\,a0 + 2019.\,a1)}$$

```
> Fit(r3(N), Nr, rr, N)
```

$$0.0299099215088044 - 0.00251760534280711\, N$$

```
> r33(N) := Fit(r3(N), Nr, rr, N) :
> P1 := plot(r33(N), N = 3.7 ..7.8, color = "blue") :
> P2 := plot(Nr, rr, style = point, symbol = solidcircle, color
      = "black" ) :
> display({P1, P2}, labels = [Velikost populace, Míra růstu],
      labeldirections = [horizontal, vertical])
```



> $y_3 := dsolve(\{y'(t) = subs(N = y(t), r33(N)) \cdot y(t), y(2019) = 7.714\})$:

> $evalf(y_3)$

$$y(t) = \frac{0.00002284080275}{1.748185649\ 10^{20}\ e^{-0.02990992151\ t} + 1.922576995\ 10^{-6}}$$

> $data := seq([i, evalf(subs(t = i, rhs(y_3)))], i = 1970..2200)$:

> $N3 := plot(evalf(rhs(y_3)), t = 1970..2200, labels = [Čas, Počet obyvatel v miliardách], labeldirections = [horizontal, vertical])$:

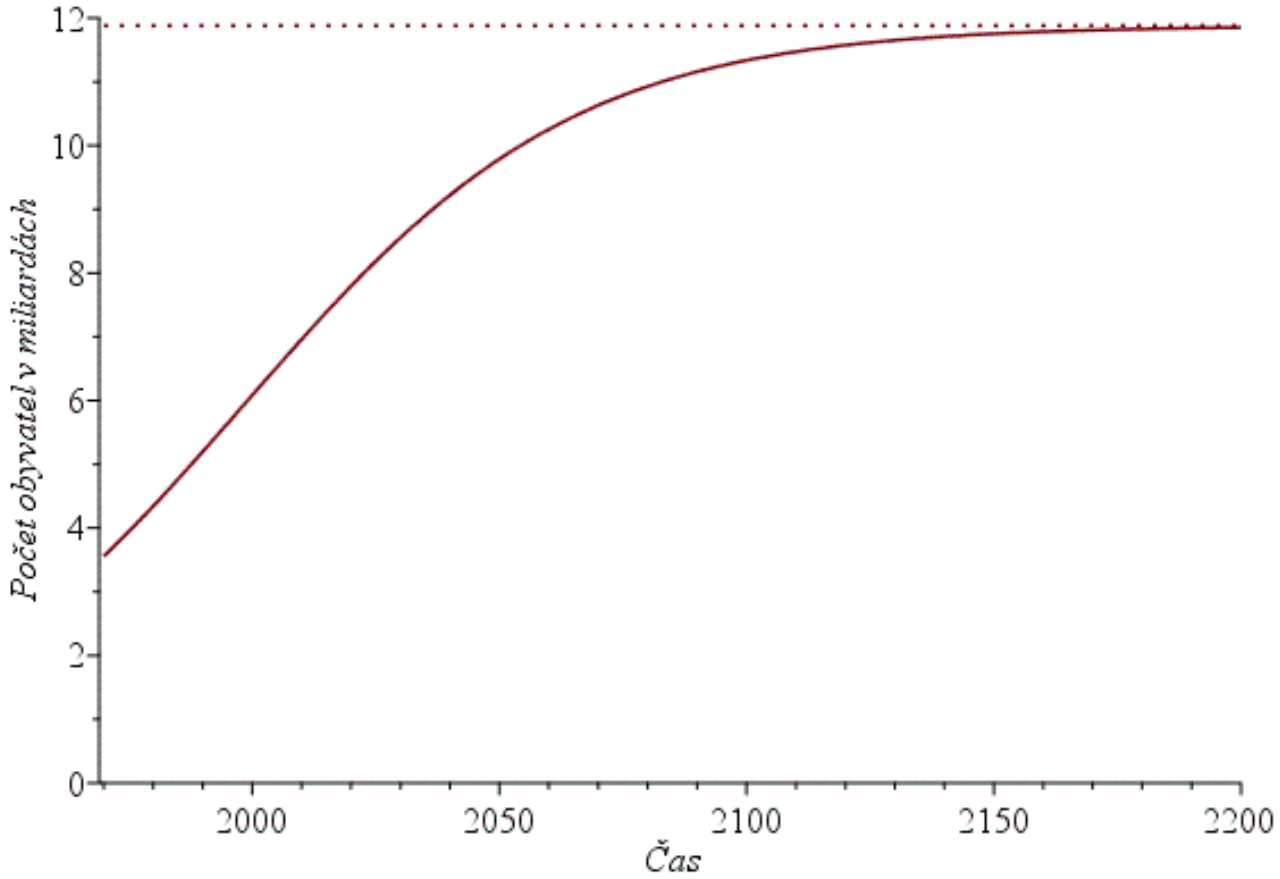
> $capacity := evalf(limit(rhs(y_3), t = \text{infinity}))$;

$$capacity := 11.88030586$$

Maximální velikost populace nemůže převýšit 11,88 miliard.

> $N4 := plot(capacity, t = 1970..2200, linestyle = dot)$:

> $N5 := display(\{N3, N4\}, view = [1970..2200, 0..12])$



MODEL (d)

Model s Alleeho efektem.

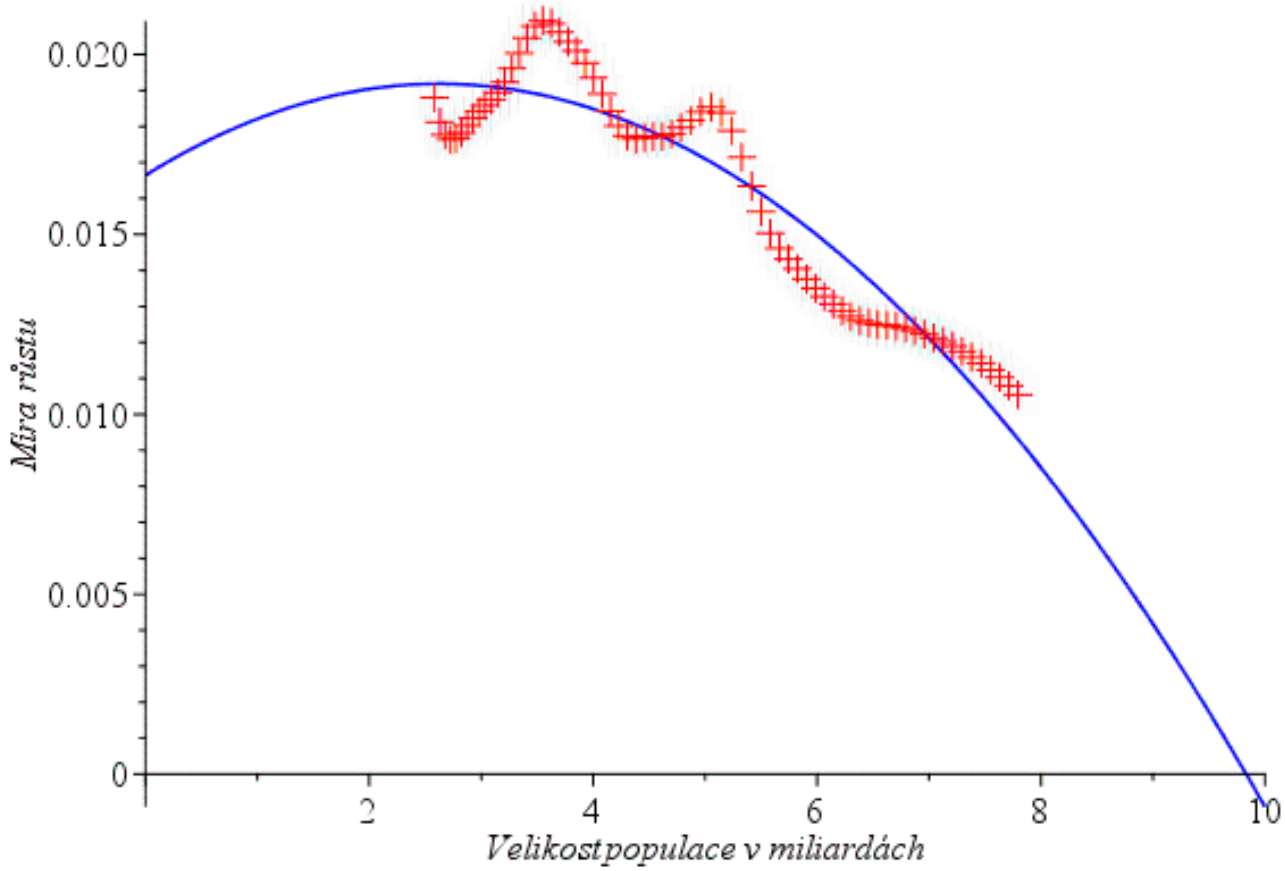
```
> unassign('a0'); unassign('a1'); unassign('a2'); unassign('a3');
> with(Statistics) : with(plots) :
> A := ImportMatrix("populace.xlsx") :
> Yr := A[ .., 1 ] : Nr := A[ .., 2 ] / 10^9 : Rr := A[ .., 3 ] :
> r4(N) := a0 + a1·N + a2·N^2 :
> Fit(r4(N), Nr, Rr, N);
```

$$0.0166446504391002 + 0.00193824270656437\,N - 0.000369320922627608\,N^2$$

```
> r44(N) := 0.0166446504391002 + 0.00193824270656437 N
      - 0.000369320922627608 N^2;
```

$$r44 := N \mapsto 0.0166446504391002 + 0.00193824270656437\,N - 0.000369320922627608\,N^2$$

```
> P1 := plot(r44(N), N = 0 ..10, color = "blue", labels
      = [Velikost populace v miliardách, Míra růstu], labeldirections
      = [horizontal, vertical]) :
> P2 := plot(Nr, Rr, style = point, symbol = cross, symbolsize = 20,
     color = "red") :
> display(P1, P2)
```



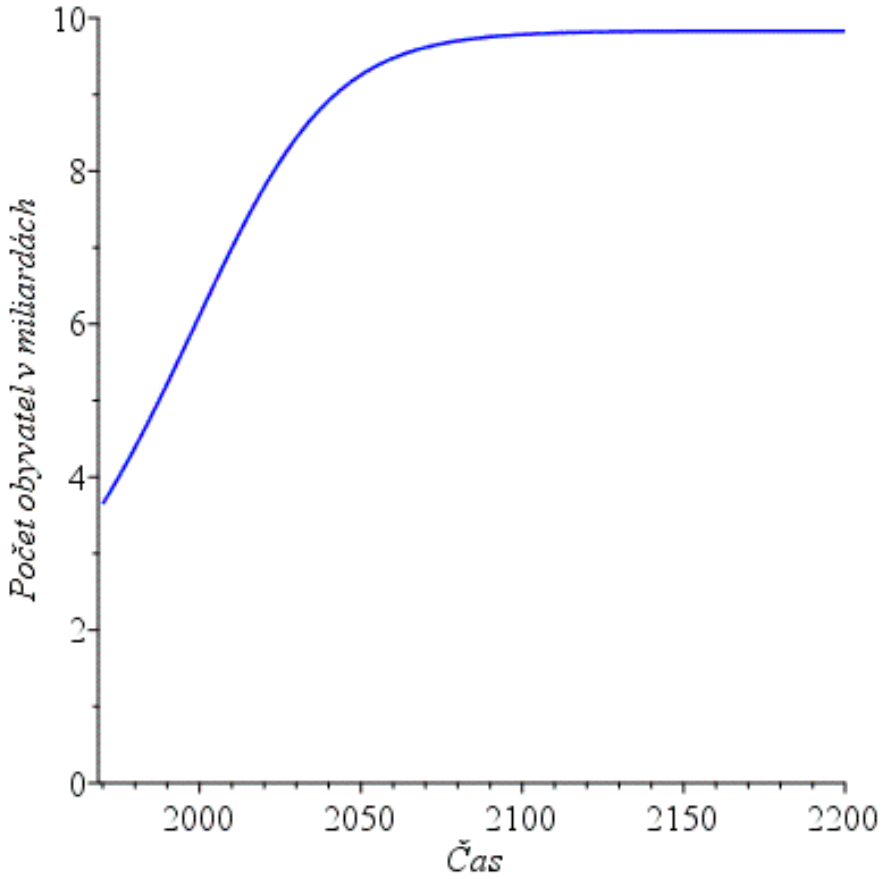

```
> dsys := {N'(t) = r44(N(t))·N(t), N(2019) = 7.714}
```

$$dsys := \{N(2019) = 7.714,\ \mathrm{D}(N)(t) = (0.0166446504391002 + 0.00193824270656437\,N(t) - 0.000369320922627608\,N(t)^2)\,N(t)\}$$

```
> dsn1 := dsolve(dsys, numeric,'range'= 1970 ..2200);
```

$$dsn1 := \mathbf{proc}(x_rkf45)\ \ldots\ \mathbf{end\ proc}$$

```
> data := seq([i, rhs(dsn1(i)[2])], i = 1970 ..2200) :
> N6 := plot([data[ .., 1]], [data[ .., 2]], labels = [Čas,
     Počet obyvatel v miliardách], labeldirections = [horizontal,
     vertical], color = blue, view = [1970 ..2200, 0 ..10]);
```




```
> unassign('a0'); unassign('a1'); unassign('a2'); unassign('a')
```

```
> collect(a·(N/Al − 1)·(1 − N/K) − (a0 + a1·N + a2·N^2), N)
```

$$\left(-\frac{a}{Al\,K} - a2\right) N^2 + \left(\frac{a}{K} + \frac{a}{Al} - a1\right) N - a - a0$$

```
> sol := solve({{(-a/(Al K) − a2), (a/K + a/Al − a1), -a − a0}, {Al, K, a}})
```

$$sol := \left\{Al = \frac{a0}{RootOf\left(a2_Z^2 + a1_Z + a0\right) a2},\ K = RootOf\left(a2_Z^2 + a1_Z + a0\right),\ a = -a0\right\}$$

```
> a0 := 0.0166446504391002; a1 := 0.00193824270656437; a2 := -0.000369320922627608;
```

$$a0 := 0.0166446504391002$$

$$a1 := 0.00193824270656437$$

$$a2 := -0.000369320922627608$$

```
> evalf(sol);
```

$$\{Al = -4.583846857,\ K = 9.831972777,\ a = -0.01664465044\}$$

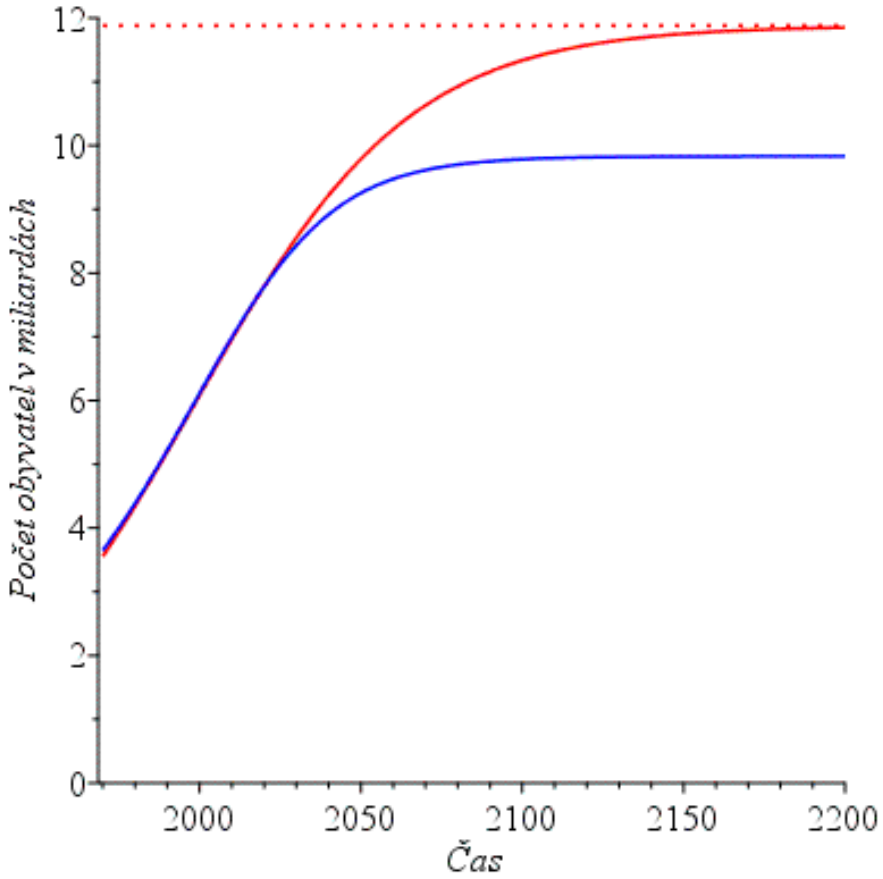
```
> dsn1(2200);
```

$$[t = 2200.,\ N(t) = 9.83172611336770]$$

Kapacitu prostředí K a Alleho práh Al dostáváme z odhadu kvadratické r(N). K odpovídá velikosti populace v roce 2200. Práh je záporný, jde proto o slabý Alleeho efekt, což lze očekávat.

Porovnání modelů s mírou růstu závislou na velikosti populace (c) - (d)

```
> display(N5, N6, color = ["red", "blue"])
```



Červená křivka odpovídá velikosti populace v případě použití logistického modelu (lineární závislost míry růstu na velikosti populace), modrá křivka odpovídá velikosti populace v případě použití logistického modelu se slabým Alleeho efektem (kvadratická závislost míry růstu na velikosti populace).