www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Mirror, Mirror

Kurt Bryan
Rose-Hulman Institute of Technology
Terre Haute IN 47840 USA
kurt.bryan@rose-hulman.edu

Abstract:

This project models the "Foucault Knife Edge Test," an optical test commonly used by amateur astronomers who make their own mirrors for reflecting telescopes. The goal of the test is to estimate the shape of the surface of a mirror from optical reflection data. The model results in a very simple first order nonlinear ordinary differential equation (ODE) that has no closed-form solution. Moreover, one of the components of the ODE is a function that must be estimated from the collected data; some real data from a mirror the author made is included. A computer algebra system can be helpful for performing computations, and sample Maple and Mathematica worksheets are provided.

SCENARIO DESCRIPTION – TELESCOPES AND MIRROR MAKING

It is a time-honored tradition among amateur astronomers to build a telescope from scratch, right down to grinding the optical elements. Building a telescope is an interesting exercise in "multiple precision" construction. The telescope's tube and base are often made of wood, which is typically cut to an accuracy of a few hundredths of an inch. Machined metal parts might be accurate within a few ten-thousandths of an inch. But in order to obtain good images, the optical elements themselves must be made to tolerances on the order of a *millionth of an inch*. How can such precision be attained at home (or anywhere else)? Remarkably, this can be done by using only bare human hands, aided by a bit of clever optics and geometry.

In this project we'll look at one facet of constructing an accurate parabolic mirror for use in a reflecting telescope. A schematic for a typical Newtonian reflecting telescope is shown in Figure 1. Incoming parallel light rays enter the telescope and reflect off of the *primary* mirror at the back of the scope (shown in red). They then reflect off of the *secondary* mirror or *diagonal* mirror (also in red) and exit the side of the telescope, where they are focused by an eyepiece to form an image for visual inspection or photography. (Despite being in the optical path, the secondary mirror doesn't

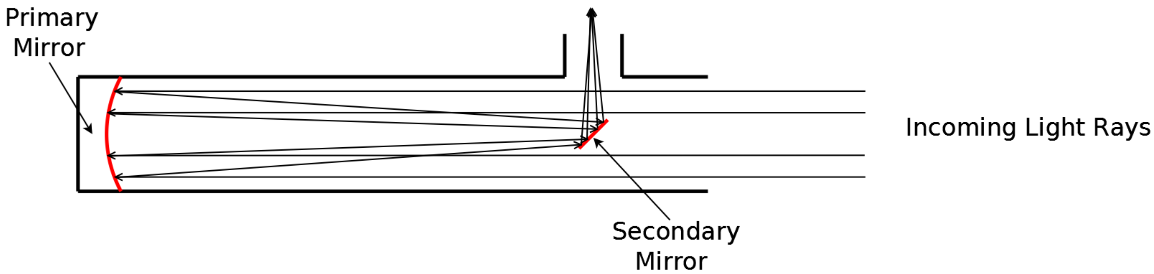

Figure 1. A typical reflecting telescope.

really appear in or otherwise ruin the image, as long as the secondary is not too big.) In order to focus these incoming parallel rays to a point and form a good image, the primary mirror must be parabolic in shape, to extremely high precision. The secondary mirror doesn't alter this fact, but merely serves to bring the light rays to a focus outside of the telescope.

One of the standard hallmarks of a good mirror is that the surface is parabolic to an accuracy of at least 1/8 that of the wavelength of light in the visual spectrum, though more stringent figures of 1/10, 1/20, or finer are often cited. Green light has a wavelength of about 5.5×10^{-7} meters (this is the "middle" of the visual spectrum), so 1/8 wave corresponds to about 6.9×10^{-8} meters, comparable to the size of a virus! We obviously need a very accurate method to test the shape of the mirror. It should also be noted that a statistic like "1/10 wave accuracy" is only one figure of merit for the quality of a mirror. Moreover, the mirror is only one component in the optical system that makes up a telescope. Still, the knife edge test for the shape of the primary mirror is a traditional staple for amateur mirror makers.

How does one make a parabolic mirror? Very briefly, the traditional process of grinding a mirror consists of starting with two glass disks. One is placed flat-side down on an immobile waist-high stand or barrel; this disk is called the "tool." The other disk is placed on top of the tool, flat face to flat face, with water and abrasive in between; this disk will become the mirror. The mirror-maker then grinds the top disk back-and-forth while walking around the barrel for 20 to 100 hours (or more!), periodically refreshing the abrasive and moving to finer and finer abrasives. Consistent grinding combined with the natural randomness in the process results in the top face of the tool becoming a convex portion of a sphere and the bottom of the top disk becoming the corresponding concave spherical piece, to very high precision, perhaps a millionth of an inch. Unfortunately, a spherical mirror (unless quite small) does not adequately focus light rays. We have to reshape the spherical mirror into a parabola! This is called "figuring" the mirror. For an introduction to building telescopes see [2] (a good introduction) or [3] (a classic, slightly more technical reference).

Taking the tiny amounts of glass off of a spherical mirror to turn it into a parabola isn't the difficulty, since glass is very hard. With a fine abrasive and gentle pressure the amount of glass removed with each stroke may be only a few atoms thick. The problem is that we need to know where to remove glass, and this requires that we can measure and monitor the shape of the emerging mirror to very high precision as we form it into a paraboloid. This is where the Foucault Knife Edge Test comes into play.

THE KNIFE EDGE TEST

To understand the test, we're going to restrict our attention to a two-dimensional setting. The process of mirror-making, if done with any care, results in a mirror that is rotationally symmetric about its central point, even if the mirror is not precisely spherical or parabolic in shape. This means that to determine the shape of the mirror we really only need to consider a cross-section of the mirror through the center, or even just a radial portion.

Refer to Figure 2, in which the mirror to be tested is the curve on the left. The Foucault Knife Edge Test consists of placing a light source (the green dot) on the central axis of the mirror. If the light source is at just the right distance from the mirror, light rays that strike the mirror at certain points will reflect orthogonally and return to the light source; this is illustrated by the solid blue lines. Note that due to the symmetry assumption, rays that hit the mirror at the same distance above or below the central axis both return to the source. Contrast this to rays that strike the mirror elsewhere, shown as dashed lines. These rays may not reflect back to the light source (but will if the light source is moved closer to or further from the mirror).

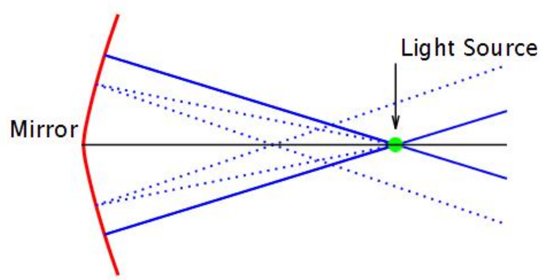

Figure 2. Mirror with incident and reflected rays.

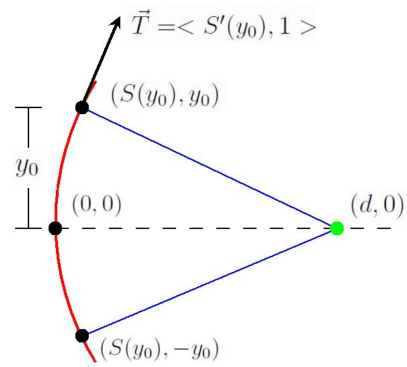

Figure 3. Geometry of orthogonal reflection.

To quantify this, see Figure 3 in which we suppose the mirror is described by a curve $x = S(y)$ for some function S, where we'll assume $S(0) = 0$ for convenience. We'll also assume the vertical extent of the mirror is given by $-R < y < R$ for some R. The symmetry assumption means that $S(y) = S(-y)$. The goal is to find the function S. The knife edge test consists of limiting our attention to a specific point $(x, y) = (S(y_0), y_0)$ on the mirror, then adjusting the light source so that rays reflecting off the mirror at this point (or its symmetric partner $(S(y_0), -y_0)$) return to the light source. We then measure the horizontal distance "d" from the light source to the center of the mirror (at $x = 0$). This is repeated for each value of y_0 in the range $0 < y_0 \leq R$. For the reader who is interested in how one can determine the value of d for any fixed y_0 using a simple homemade apparatus (which explains where a "knife edge" comes into play) see the Appendix.

In Figure 3 the light source is at the point $(d, 0)$. Consider the displacement vector from $(d, 0)$ to the point $(S(y_0), y_0)$ on the mirror, given by $\vec{v} =< S(y_0) - d, y_0 >$. This vector is orthogonal to the surface of the mirror. Equivalently, $\vec{v}$ is orthogonal to the vector $\vec{T} =< S'(y_0), 1 >$, which is tangent to the mirror.

Problem 1: Show that $\vec{T} =< S'(y_0), 1 >$ is in fact tangent to the mirror at the point $(S(y_0), y_0)$.

Since $\vec{v}$ and $\vec{T}$ are orthogonal we conclude that $\vec{v} \cdot \vec{T} = 0$. When written out explicitly this yields $(S(y_0) - d)S'(y_0) + y_0 = 0$. In this Knife Edge Test we perform the above measurement of d for "every" y_0 in the range $0 < y_0 \leq R$. Since d depends on y_0, we'll write $d(y_0)$, so we have

$$(S(y_0) - d(y_0))S'(y_0) + y_0 = 0$$

for $0 < y_0 \leq R$. Let's drop the subscript on y and just write $(S(y) - d(y))S'(y) + y = 0$. A bit of rearrangement gives

$$S'(y) = \frac{y}{d(y) - S(y)}. \tag{1}$$

If we know (measure) $d(y)$ for all $y \in (0, R)$ we can solve the differential equation (1) with initial condition $S(0) = 0$ to find $S(y)$, the shape of the mirror!

There are some additional practical issues that are addressed below (we can't take data for every y, for example), but the above is the basic idea. Before going on, let's stop to try some problems:

Problem 2: Suppose we measure $d(y) = 1.5$ meters for every y (that is, when the light source is at a distance of 1.5 meters from the mirror, all incident rays reflect orthogonally off of the mirror, at every point on the mirror, and return to the source). Solve (1) with $S(0) = 0$. What shape is the mirror, and how does the light source position relate to this shape? (The answer should seem "obvious" after the fact.)

Problem 3: For a general (unspecified) function $d(y)$, is (1) a linear ODE in the unknown S? Is it separable? Is it autonomous? Is it exact? Does it fit into any standard category that you know of? (Try feeding it to a computer algebra system with $d(y)$ left undefined.)

Problem 4: Look up the Existence-Uniqueness Theorem for ODEs in a DE textbook. If we assume that $d(y)$ is a "reasonable" function, (say d is continuously-differentiable, and that $d(y) \geq M > 0$ for some constant M), is (1) guaranteed to have a unique solution with $S(0) = 0$? That would be good—it means the data $d(y)$ we collect is sufficient for the purpose of determining S.

Problem 5: Suppose $d(y) = \frac{1}{2} + y^2$. Solve (1) with $S(0) = 0$ to find the shape of the mirror.

y (meters)	$d(y)$ (meters)
7.62×10^{-3}	2.4384
2.29×10^{-2}	2.4385
3.81×10^{-2}	2.4387
5.33×10^{-2}	2.4390
6.68×10^{-2}	2.4394

Table 1. Knife edge data at $y = 0.1R, 0.3R, 0.5R, 0.7R, 0.9R$, with $R = 0.076$ meters.

(You may need a computer algebra system).

Problem 6: Suppose $d(y) = \frac{1}{4} + y^2$. Solve (1) with $S(0) = 0$ to find the shape of the mirror. (You WILL need a computer algebra system). Compare to Problem 5.

SOME PRACTICAL ISSUES AND EXAMPLES

The first obvious difficulty with the knife edge test as outlined above is that we cannot measure $d(y)$ for every y. In practice one typically measures $d(y)$ at 3 to 7 values of y between $y = 0$ and $y = R$. One approach is to then interpolate this data to form a function $d_p(y)$, then solve the ODE (1) using $d_p(y)$ in place of the true $d(y)$. This solution would be computed numerically using any standard numerical ODE solver and provides an estimate of S.

Example 1 As an example, let's suppose the mirror shape is given by $S(y) = 0.205052y^2$ for $-R < y < R$ and $R = 0.076$ meters. This corresponds to a classic six-inch diameter parabolic mirror with a focal length of 48 inches. In this case the "true" function $d(y)$ is given by $d(y) = 2.438405 + 0.205052y^2$ with y and $d(y)$ in meters (to see this just plug the given $S(y)$ into (1) and solve for $d(y)$.) Suppose we measure $d(y)$ at y values $0.1R, 0.3R, 0.5R, 0.7R, 0.9R$. This yields the data in Table 1. These points are shown (black dots) in Figure 4. Note that $d(y)$ is measured to an accuracy of 10^{-4} meters, just 1/10 of a millimeter, with the light source over 2 meters from the mirror. This is not typically possible, but not necessary either! We'll address this a bit later in Problem 7.

We can fit a curve, say a cubic spline, to these data points. The computation is made easy by using a computer algebra system; see the allied notebook `knifeedge.mw` (Maple) or `knifeedge.m` (Mathematica). The graph of the spline $d_p(y)$ that interpolates this data is shown as the red curve in Figure 4. We use the interpolating function $d_p(y)$ in (1) (instead of $d(y)$, which we don't really know). In this example the actual formula for $d_p(y)$ is a bit messy and not shown. The numerical solution to (1) turns out to be quite close to the correct parabola $S(y) = 0.205052y^2$, but it's more illuminating to plot the difference of this estimate from the true $S(y)$ as in the left panel of Figure 5. Even more informative is this same difference as shown in the right panel, but with the vertical units as wavelengths of green light, 5.5×10^{-7} meter. The error introduced by using only 5 measurements

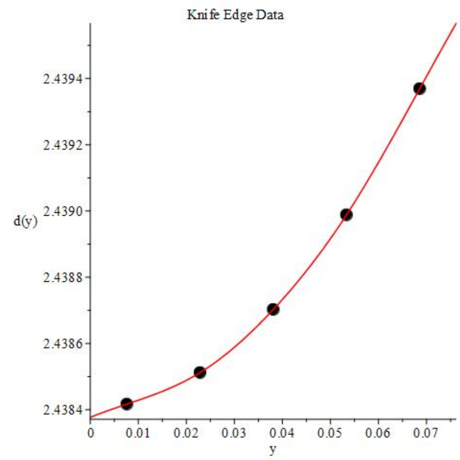

Figure 4. Measured data and interpolating curve (in this case, a cubic spline).

of $d(y)$, interpolating, and then solving (1) for $S(y)$ is negligible (anything under $1/20 = 0.05$ of a wavelength is generally considered negligible in this setting.)

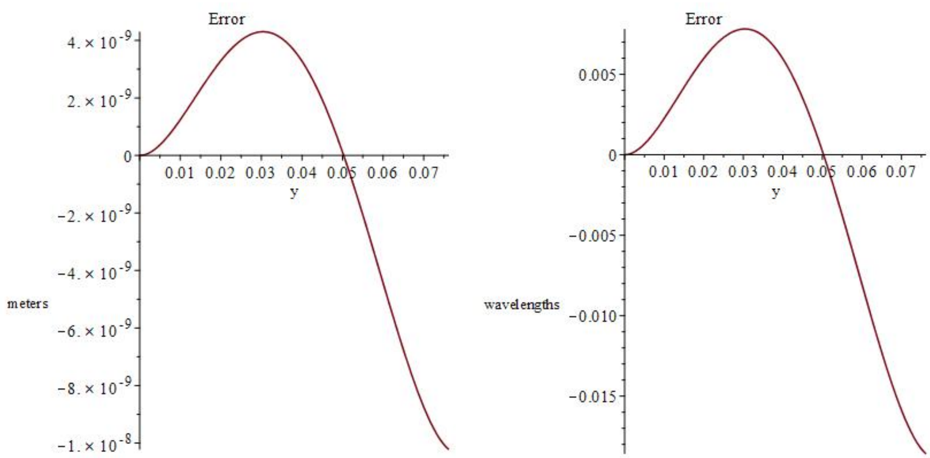

Figure 5. Error in estimated mirror shape (in meters on the left, wavelengths on the right).

Example 2 Most mirror makers are not trying to obtain a very specific parabola, e.g., $S(y) = 0.205052y^2$ (corresponding to about a 48 inch focal length). Any parabola that is "close" will do; the mirror maker would be just about as happy if the mirror under construction had a 48.2 inch focal length (corresponding to $S(y) = 0.204201y^2$), since this is easily accommodated into the telescope's construction. However, a 6 inch mirror with shape $S(y) = 0.204201y^2$ deviates from the mirror with shape $S(y) = 0.205052y^2$ by almost 10 wavelengths at the mirror edge. Hence in determining the quality of a parabolic mirror's shape we shouldn't compare it to a pre-chosen hypothetical parabola,

y (meters)	$d(y)$ (meters)
5.44×10^{-3}	2.4857
1.63×10^{-2}	2.4859
2.72×10^{-2}	2.4863
3.81×10^{-2}	2.4869
4.90×10^{-2}	2.4877
5.99×10^{-2}	2.4884
7.08×10^{-2}	2.4892

Table 2. Knife edge data at points $y = (k + 0.5)R/7$ for $k = 0, 1, \ldots, 6$.

but rather to that parabola that it best fits in some quantifiable sense. The quality of the mirror will be judged by how far it deviates from this profile.

To illustrate, consider the data in Table 2. Here we have used 7 points for the knife edge measurements, at locations $y = (k+0.5)R/7$ for $k = 0, 1, \ldots, 6$. We reconstruct an estimate of $S(y)$ as in Example 1, by fitting a cubic spline to the data in Table 2, then solving (1) numerically with $S(0) = 0$. This gives us an estimate $\tilde{S}(y)$ for the shape of the mirror.

To determine its quality we can then fit a parabola $S(y) = Ay^2$ to the resulting estimate $\tilde{S}(y)$ by minimizing the least squares function

$$Q(a) = \sum_{j=0}^{N} (\tilde{S}(y_j) - Ay_j^2)^2$$

with respect to A, where the y_j span the interval $[0, R]$, say $y_j = Rj/N$ with $N \approx 10$. In the present example this yields $A \approx 0.201049$. The resulting parabola $S(y) = 0.201049$ (corresponding to a perfect parabola with focal length 48.96 inches) deviates from $\tilde{S}(y)$ by at most 0.28 wavelengths. That is, the mirror that gives rise to the data in Table 2 deviates from even the best fitting parabola by 0.28 wavelengths (which means this mirror needs additional work). A plot of the difference $\tilde{S}(y) - Ay^2$ is shown in Figure 6.

Problem 7: How can the data for $d(y)$, as in Table 1 or Table 2, be measured to such accuracy (0.1 mm) over a 2 or 3 meter span? We don't need to! It's not the absolute value for the various $d(y)$ that's critical, it's their relative values. To illustrate, redo Example 1 but first add an "error" of 5 mm (0.005 meter) to each value of $d(y)$, then reconstruct an estimate $\tilde{S}(y)$ of the mirror shape, and compare the error to the "best fit" parabola. How is this error manifest in the estimated $S(y)$? Try adding 1 cm or even 5 cm to each entry in $d(y)$ and repeat. What conclusion do you draw?

Problem 8: In Table 3 there is some data from a 4.25" mirror the author made (so $R = 0.054$ meters, roughly).

The relative accuracy between the $d(y)$ offsets is probably around 2×10^{-4} meters. Estimate the mirror's profile $S(y)$, and in particular, how far it deviates from a best-fit parabola. What is

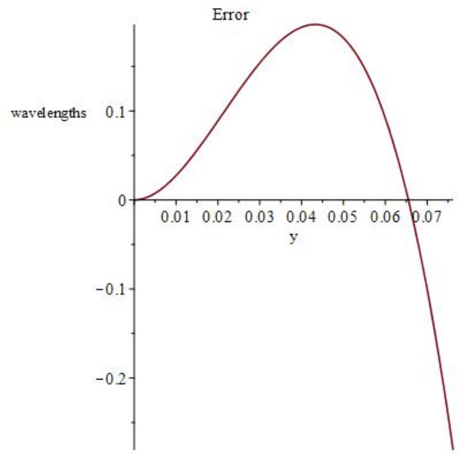

Figure 6. Error in estimated mirror shape from best fit parabola.

y (meters)	$d(y)$ (meters)
1.44×10^{-2}	0.9208
2.50×10^{-2}	0.9211
3.23×10^{-2}	0.9214
3.82×10^{-2}	0.9215
4.33×10^{-2}	0.9216
4.78×10^{-2}	0.9217
5.20×10^{-2}	0.9221

Table 3. Knife edge data for 4.25 inch mirror.

the maximum error, in wavelengths of green light?

CONCLUSION

The Knife Edge Test was developed by the French physicist Foucault in the mid 19th century. It is one of a large number of optical tests that can be made on a mirror. It's advantage is that it is relatively simple to build a tester and analyze the data, and the test is quite sensitive. Most treatments of the test do not formulate the ODE (1), since this would be beyond what many amateur telescope builders are familiar with. The references [2] and [3] give alternative quantitative recipes for analyzing knife edge data, but for those familiar with differential equations, the ODE (1) provides a simple conceptual framework for determining the shape of the mirror.

APPENDIX: PERFORMING THE KNIFE EDGE TEST

Spherical Mirrors

How can we determine if light rays that reflect off the surface of a mirror do so orthogonally? Let's first consider the case of a "spherical mirror" (in the two-dimensional setting, a mirror that is a portion of a circle, as illustrated in Figure in the left panel). Let C on the x axis denote the center of the circle of which the mirror is a part, and imagine a "point" source of light located precisely at C (shown in green). Intuitively, all light rays (shown in blue) that emanate from the source and strike the mirror do so at a right angle, and so are reflected back to C. The rays will continue past C; if an observer is properly positioned behind C these rays will enter the observers "eye" (or eyepiece) and the mirror will appear to be brightly and uniformly illuminated.

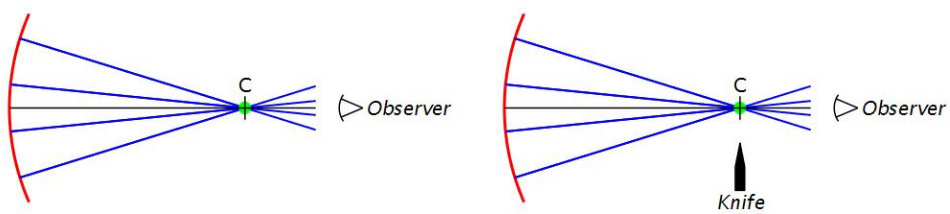

Figure 7. Light source (green dot), incident and reflected rays (blue) and spherical mirror (left panel), and with knife edge (right panel).

Now imagine a "knife" positioned as illustrated in the right panel of Figure , with the knife tip slightly below C so that it doesn't block any of the reflected rays. The situation is essentially the same as in the left panel, and the observer sees a brightly and uniformly illuminated mirror. However, if the knife edge is advanced vertically, it will eventually block the reflected rays and the observer will see the mirror go dark. This will happen "instantaneously" when the knife tip passes the central (horizontal) axis of the mirror at C. (It's not really instantaneous due to the small imperfections in the apparatus and the wave nature of light.)

Consider now what happens if we do the same experiment, but with the light source and knife edge positioned horizontally somewhere other than C, say closer to the mirror, as illustrated in the left panel of Figure . In this case the reflected rays do not return to C; in fact, they don't converge to any single point, but we don't need to prove this right now. As the knife edge is advanced upward it cuts off light rays from the bottom of the mirror first, then rays from the middle, finally rays from the top. The observer will see the mirror go dark from bottom to top (assuming he/she accounts for the reversed image behind C). Contrast this to the case in which the light source/knife edge are

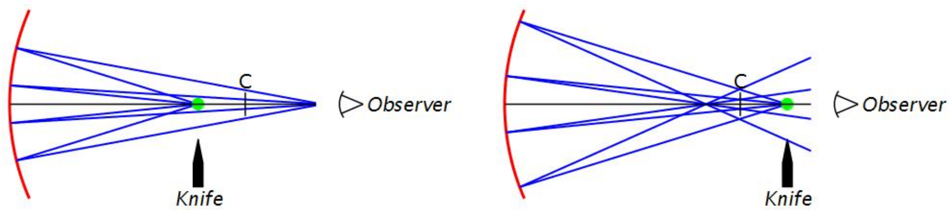

Figure 8. Light source (green dot), incident and reflected rays (blue) and spherical mirror (left panel), and with knife edge (right panel).

moved farther from the mirror, illustrated in the right panel of Figure . In this case as the knife edge advances it cuts off light from the top of the mirror first; the observed will see the mirror go dark from top to bottom, the opposite of the case in the left panel. In the Knife Edge Test the light source and knife edge are always moved (horizontally) together.

The moral of the above examples is this: If the mirror is circular/spherical and the light source/knife is closer to the mirror than the center C, then as the knife edge is advanced upward the mirror will appear to go dark in the same direction that the knife edge advances (bottom to top). If the light source/knife is farther from the mirror than C, the mirror will appear to go dark from top to bottom (opposite the motion of the knife edge). When the source/knife are exactly at C the mirror appears to go dark "instantly."

If one wanted to determine the center of curvature of a spherical mirror, a test such as this could be used. Simply place the source/knife somewhere on the central axis and advance the knife edge upward while observing from behind. If the mirror goes dark in the direction of the knife edge travels, you're closer to the mirror than C; move farther out. If the mirror goes dark the other way, move closer. An iterative process will lead you to the point C, when the entire mirror will go dark simultaneously. If no such point can be found, the mirror is not spherical!

Mirrors of General Shape

Refer to Figure 9. Consider what the observer will see as the knife edge advances upward. The first ray to be cut off is a dashed one, and the observer will see the region of the mirror off of which that ray reflected go dark. However, the other dashed ray will not be cut off, and that diametrically opposite portion of the mirror remains light. As the knife edge continues upward it will eventually cut off both solid rays simultaneously, since both reflect off the mirror at a right angle and return to the source. The observer will see the corresponding portions of the mirror (diametrically opposite

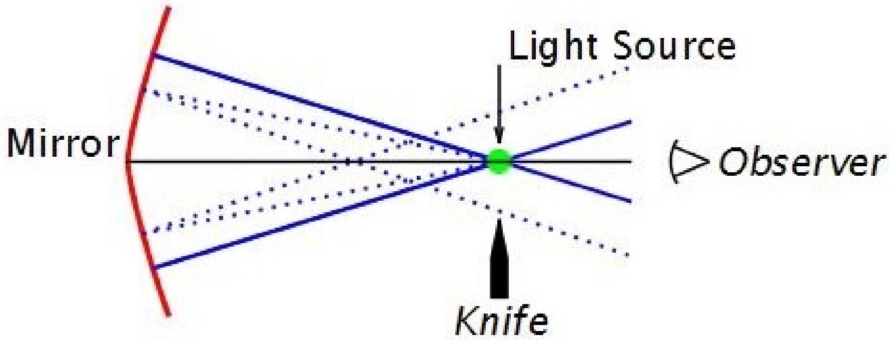

Figure 9. Knife edge test on a non-spherical mirror.

each other) go dark simultaneously. Only after the knife edge has advanced further will the region of the mirror off of which the second dashed line go dark.

The grand conclusion:

If the light source/knife edge is placed at a given distance from the mirror on the central axis and the knife edge is advanced, and two diametrically opposite regions on the mirror go dark simultaneously, then the light rays that reflected from that region did so at a right angle.

We can use the above observation to measure $d(y_1)$ for any specific y_1. A Foucault tester consists of a point light source and knife edge on an adjustable carriage that is placed in front of the mirror. One places a non-reflective "mask" over the mirror that allows only light from zones near $y = \pm y_1$ to pass, and then adjusts the distance from the test to the mirror so that as the knife edge advances the two zones near $y = \pm y_1$ dim simultaneously. This provides a measurement of $d(y_1)$. See [1] for some photographs of a Foucault test and what a mirror looks like through the instrument as measurements are taken.

REFERENCES

[1] Astrobananas. http://astrobananasenglish.blogspot.com/2011/07/foucault-test.html. Accessed 31 July 2019.

[2] Berry, R. 2001. *Build Your Own Telescope, 3rd Edition.* Richmond VA: Willmann-Bell.

[3] Texereau, J. 1984. *How to Make a Telescope, 2nd English Edition.* Richmond VA: Willmann-Bell.