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STUDENT VERSION

How Good is Your Water Bottle?

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Abstract: This modeling scenario involves Newton's Law of Cooling and should be appropriate in the first week or even the first day of an introductory differential equations course. The scenario uses an Inquiry Based Learning approach to walk the students through the creation and understanding of a differential equation describing how fluid in a water bottle will change its temperature to approach the ambient temperature in a room. The goal is for the students to understand how the right hand side of the differential equation and the initial condition relate to the physical phenomenon being described. This scenario also hints at the notion of an equilibrium and the speed at which an equilibrium is approached.

SCENARIO DESCRIPTION

We will explore how the temperature of cold water (or hot tea) in a water bottle changes when left in a room at constant temperature. Our goal is to build a differential equation describing this rate of change.

1. The temperature of our classroom is set to always be exactly $70°F$. The water inside my high-quality water bottle is currently $40°F$. Let us pretend that I accidentally leave my water bottle in this room for a long time.
   (a) Sketch a plot of water temperature by time (first quadrant only). Be sure to label your graph.
   (b) Imagine that instead of my high-quality insulated water bottle, I have a cheap thin plastic water bottle. Sketch the plot for the water temperature in this bottle.
   (c) Compare the two sketches.
2. What if instead of the $40°F$ cold water, I had filled my high quality bottle with hot tea at a temperature of $130°F$. Sketch the plot for the temperature of the hot tea.

3. Let $x(t)$ be the temperature of the liquid in the water bottle at time t.

   (a) Find $x(0)$ in each of the three previous examples (this is called the initial condition).

   (b) Find $\lim_{t\to\infty} x(t)$ in each of the three previous examples, i.e. what temperature do we expect the water to approach if I leave it the bottle out forever?

4. Give a reason why the following derivatives do NOT describe the rate of change of the water temperature in an arbitrary water bottle, $\frac{dx}{dt}$. Let T_a be the temperature of the room and $k > 0$ be a constant that depends on the quality of the water bottle.

   (a) $\frac{dx}{dt} = -kx$

   (b) $\frac{dx}{dt} = x - t$

   (c) $\frac{dx}{dt} = \sin t$

   (d) $\frac{dx}{dt} = \frac{x-T_a}{t}$

5. Let's try to determine what quantities effect the rate of change of the water temperature, i.e. the derivative $\frac{dx}{dt}$.

   (a) Does the rate at which the water changes temperature depend on the current temperature of the water? Would you expect $50°F$ water in this room to warm up faster, slower, or at the same rate as $68°F$ water?

   (b) Does the rate at which the water changes temperature depend on how much time has elapsed? Consider the scenario where the water in a bottle started at $40°F$ and a few hours later has warmed to $50°F$. At that moment we bring in an IDENTICAL water bottle with water at $50°F$. Would we expect the water in these two bottles to change temperature at the same or different rates?

   (c) What has to be true for the water in the bottle not to be changing temperature, i.e. when $\frac{dx}{dt} = 0$?

   (d) Using (a) and (b), should we see an x on the right side of the derivative? Should we see a t? What other quantities are important? How are the derivatives for my bottle and the cheap plastic bottle similar, yet different? Are there differences between the example with water and the example with hot tea?

6. Put everything together, assume the rate of change in temperature is proportional to the difference between the fluid temperature and the room temperature and find $\frac{dx}{dt}$ ($\frac{dx}{dt}$ is a 'differential equation' for the temperature of the water or tea and when combined with an initial condition is called an 'Initial Value Problem').

7. The differential equation you found above is known as 'Newton's Law of Cooling'. Newton's Law of Cooling assumes that convection is the only method of heat transfer. This problem examines the heat gain from thermal radiation. While the heat gained by convection is linearly proportional to the difference between the water and room temperatures, heat gained by radiation is proportional to the difference between the fourth power of the water temperature and the fourth power of the room temperature.

   (a) Write down the differential equation describing the change in water temperature in my water bottle coming from both convection and radiation.

   (b) Using Newton's Law of Cooling we expect that $40°F$ in a $70°F$ room would heat up at the same rate as $50°F$ water in an $80°F$ room. Would we expect the same to hold true if we include the heat gain from radiation?

8. Let's add the effect of a phase change (ice melting) to Newton's Law of Cooling. Let the water inside my water bottle be $60°F$ and the room be $70°F$. Now we put several ice cubes into my water bottle. Sketch a graph of the plot of water temperature by time.