1-058-T-Mma-WaterClocks-TeacherVersion

Sania Qureshi, Basic Sciences and Related Studies,
Mehran University of Engineering and Technology, Jamshoro, Sindh PAKISTAN
and
Brian Winkel, Director SIMIODE, Chardon OH USA

Given right circular CYLINDER with fixed Spigot Hole size. What shape container would we have to offer so that the water falls at a constant rate, meaning for a right circular cylinder what cross-sectional area A(h) at height h would we have to offer so that the water falls at a constant rate?

We begin with Torricelli' s which computes the rate at which water leaves a cylindrical tank at height h(t) and cross sectional area A(h(t)).

```
TorricelliLaw = A[h[t]]×h'[t] == - α a Sqrt[2 g h[t]]
```

$A[h[t]]\, h'[t] == -8.76231 \sqrt{h[t]}$

We seek to determine A(h(t)), the cross sectional area at height h(t) so that the water falls at a constant rate all the time, i.e. h’(t) = - c, (c > 0). Here,
a is the cross sectional area of the tiny hole near the bottom of our clock;
α is the discharge or contraction coefficient, i.e. the effective percent of the tiny hole’s area which actually permits water to flow out;
g is the acceleration due to gravity - different clocks for different altitudes(!); and
c is the actual rate in cm/(unit time) we want our water level to fall.

We solve for h'[t] in Torricelli' s Law differential equation:

```
hsolTor = h'[t] /. Solve[TorricelliLaw, h'[t]][[1]]
```

$-\frac{8.76231 \sqrt{h[t]}}{A[h[t]]}$

```
ASol = A[h[t]] /. Solve[-c == hsolTor, A[h[t]]][[1]]
```

$525.738 \sqrt{h[t]}$

We see from the above equation that if we are to have our column of water fall at a constant rate, say h’[t] = - c (c > 0), then our Surface area at the top of the column, A(h(t)) has to be proportional to $\sqrt{h(t)}$. Indeed, should we desire to make the cross sectional area circular, i.e. A(h(t)) = $\pi\,\text{rad}^2$ then our radius at the surface level of our column has to be the positive root of the solution of A(h(t)) = πr^2 .

```
r = rad /. Solve[π rad^2 == ASol, rad][[2]]
```

```
12.9363 h[t]^(1/4)
```

Combining terms and rewriting we see that r = $\sqrt{\frac{\alpha a \sqrt{2 g h(t)}}{\pi c}}$ where r is the radius of a circular cross sectional clock at height h; a is the cross sectional area of the tiny hole near the bottom of our clock; α is the discharge or contraction coefficient, i.e. the effective percent of the tiny hole's area which actually permits water to flow out; g is the acceleration due to gravity - different clocks for different altitudes(!); and c is the actual rate in cm/(unit time) we want our water level to fall.

We now use this radius in Torricelli' s Law to get confirmation that the column of water does fall at a constant rate.

```
TorrLaw = π r^2 h'[t] == -α a Sqrt[2 g h[t]]
```

```
525.738 √h[t] h′[t] == -8.76231 √h[t]
```

```
ht[t_] = h[t] /. DSolve[{TorrLaw, h[0] == h0}, h[t], t][[1]] /. {h0 → 60}
```

DSolve: For some branches of the general solution, the given boundary conditions lead to an empty solution.

```
60 - 0.0166667 t
```

So, how long does it take to empty this container?

```
tempty[c_] = t /. Solve[ht[t] == 0, t][[1]]
```

```
3600.
```

Thus if we set c = 1 cm/min = 1 cm/ (60 sec) we can compute the height at any time, t, and confirm our clock does have its water level fall at a rate of 1/60 cm/sec.

We construct the hole at the base to be circular of radius r = 0.3 cm and hence area a = π $(0.3)^2$ = 0.282743.

```
a = π (.3)^2;
```

Here we will presume (and we can determine this experimentally) that $\alpha = 0.70$.

```
α = 0.70;
```

We shall use g = 980 cm/s^2

```
g = 980;
```

And since we wish the water to fall 1 cm every 1 minute we shall use c = 1 cm/(1 min) = 1 cm/ (1*60 s) = 1/60 cm/s.

```
c = 1/60
```

$\frac{1}{60}$

Our height of the column of water as a function of time is given by hh (t) in cm at time t seconds. We confirm the height at the start, t = 0 s, and when the tank should be empty, t = 60*60 s.

```
hh[t_] = 60 - 1 / 60 t;
```

```
hh[0]
```

60

```
hh[60 * 60]
```

0

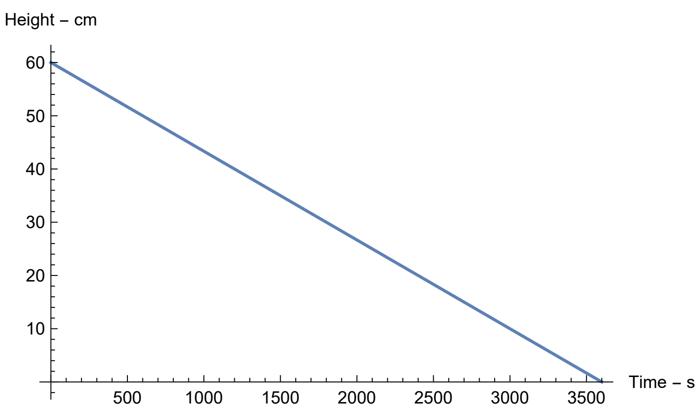
Thus the radius of our cylinder at height h is

$$\text{rc[t\_]=}\sqrt{\frac{\alpha\, a\, \sqrt{2\, g\, \text{hh}[t]}}{\pi\, c}}$$

$$12.9363 \left(60 - \frac{t}{60}\right)^{1/4}$$

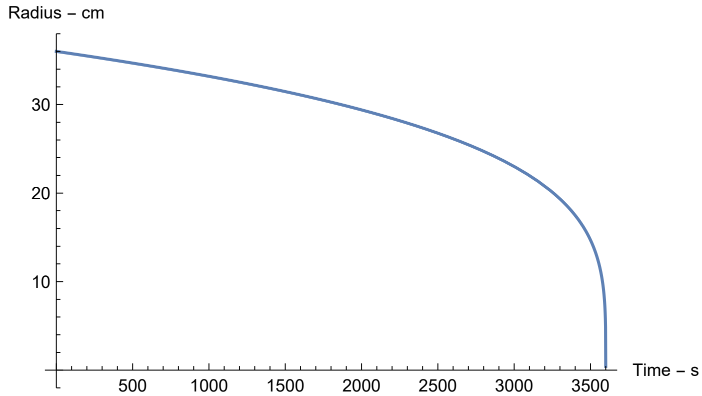
We plot the relationship between height of the water in the tank and radius of the surface of the water at that height, .

```
Plot[hh[t], {t, 0, 60*60}, AxesLabel → {"Time - s","Height - cm"}]
```



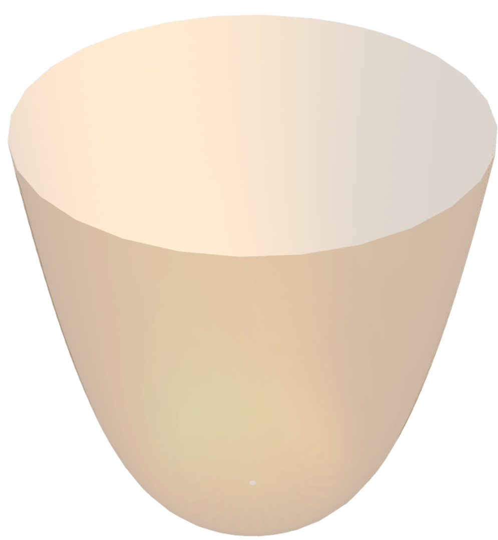
We plot the relationship between the radius of the surface of the water as a function of time.

```
Plot[rc[t], {t,0,60*60}, AxesLabel → {"Time - s","Radius - cm" }]
```



We create the container.

```
container = ParametricPlot3D[{Sin[θ] rc[t], Cos[θ] rc[t], ht[t]}, {t, 0, 60*60}, {θ, 0, 2 π}, Bo
AxesLabel → {"Radius - cm", "Radius - cm", "Height - cm"}, Axes → False, Boxed → False, Mesh → F
```



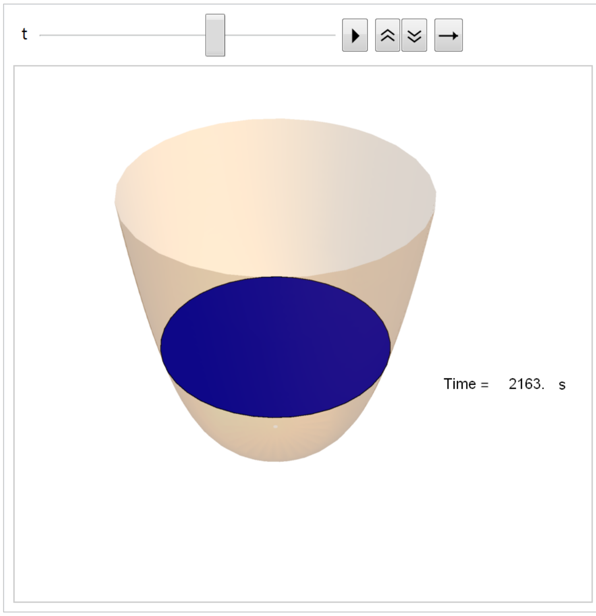
We create graphic for the surface of the water.

```
Surf[t_] := Graphics3D[{Blue, Cylinder[{{0, 0, ht[t]}, {0, 0, ht[t] + .001}}, rc[t]]}]
```

We attempt to do a reasonable job on the text for the graphics.

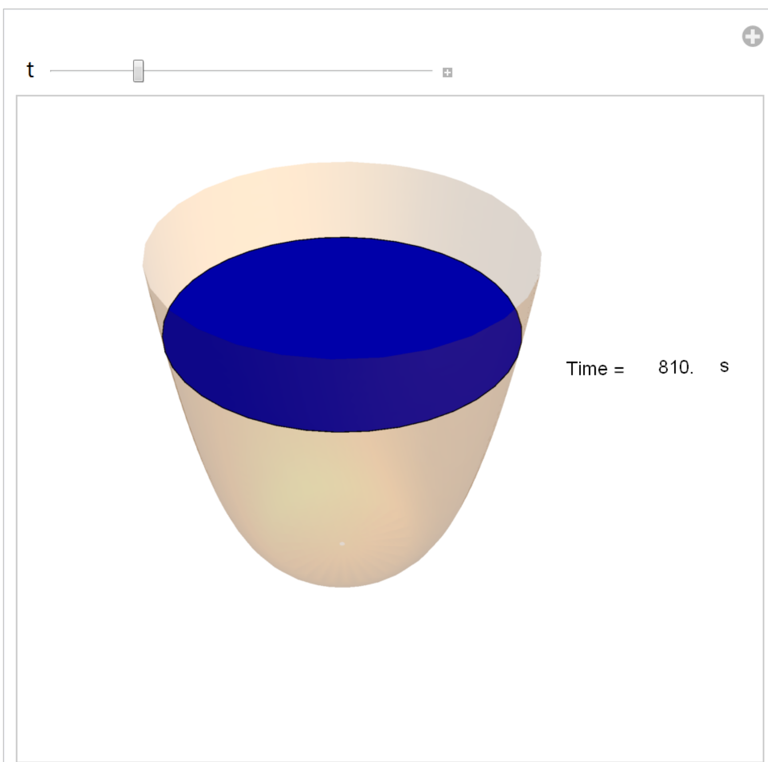
```
txt[t_] := Graphics3D[{Text["Time = ", {50, 0, ht[t]+5}], Text[t, {62, 5, ht[t]+6}],  Text["s ",
```

First, we offer an animation of the water falling out of the tank with time stamp.




```
Animate[Show[container, Surf[t], txt[t]], {t, 0, 60 * 60}]
```

Next, we offer a manipulation of the water falling out of the tank with time stamp.




```
Manipulate[Show[container, Surf[t], txt[t]], {t, 0, 60 * 60}]
```