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STUDENT VERSION
WATER FALLING IN A CONE

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Abstract: We offer an opportunity to model the height of a falling body of water in a right circular cone (funnel) and to estimate an appropriate parameter based on data collected from a video of the experiment found on YouTube. This is an application of Torricelli's Law.

SCENARIO DESCRIPTION

We have an inverted (point down) right circular conical tank (top of funnel) with a top radius R centimeters and a total height of H centimeters, filled with water to a height of h centimeters. There is a circular hole of radius r_C centimeters at the very bottom of the cone from where water leaks. See Figure 1 of the apparatus used to in the video [2] from which we collected data which we offer below.

We begin our analysis using data sampled from the movie of our experimental set up as found in SlowMoCapture-2 found at [2]. See Appendix for a sample set of data taken by the author if the reader cannot collect the data from the videos [1, 2, 3]. These videos are also available for download on YouTube.

First we need to determine the governing equations for the height of a column of liquid in a cone with a hole in the bottom from which the liquid leaks out. We refer to Torricelli' s Law development as outlined in [5, pp. 1-4].

Per this analysis, we calculate the flow out of the hole at the bottom of the inverted right circular cone in two different ways and set them equal. Here $h(t)$ is the height of the water from the base of the inverted cone and $A(h(t))$ is the cross sectional area of the surface of the column of water at height $h(t)$.

The left hand side (1) of our differential equation is given by a small element of height change, $h'(t)$, over a cross sectional area $A(h(t))$ at height $h(t)$:

$$A(h(t)) * h'(t)\,. \tag{1}$$

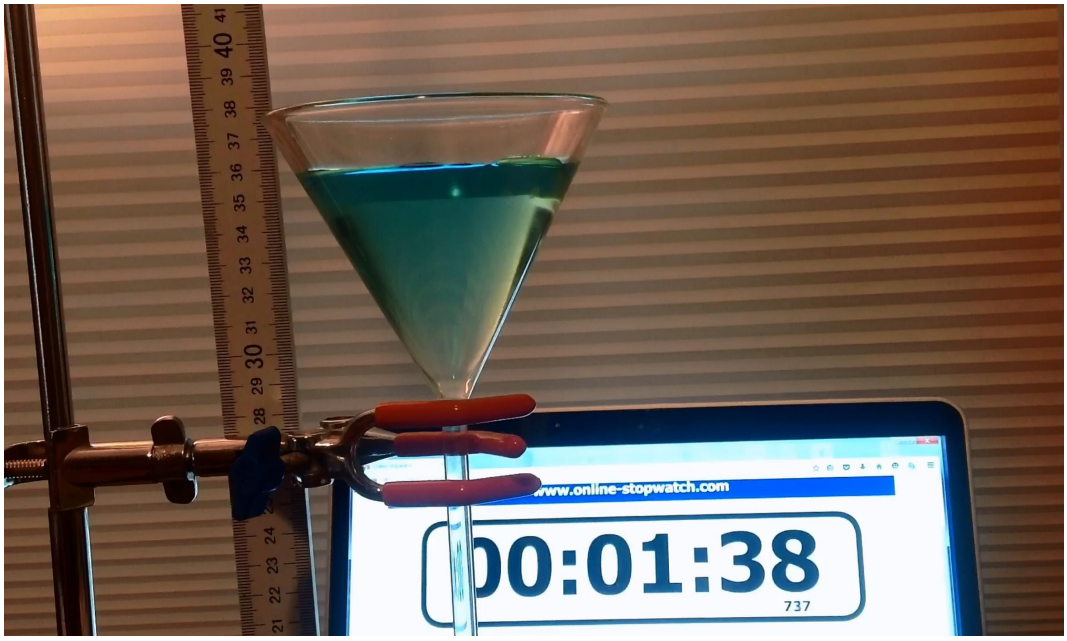

Figure 1. Set up used to collect data on falling column of water from cone. Distance on ruler in cm and time on digital clock in minutes, seconds, and milliseconds.

The right hand side (2) of our differential equation, according to the application of the Conservation of Energy Principle derivation developed for Torricelli's Law in [5], is

$$\alpha\pi(r_C)^2\sqrt{2gh(t)}\,, \tag{2}$$

where α is the effective area or percentage of total available area for fluid flow due to friction at the opening called the *discharge* or *contraction coefficient*) for fluid flow through the small opening at the bottom of the cone; r_C is the radius of the hole on the bottom of our cone; and g is the acceleration due to gravity.

Activity 1

(1) Justify the differential equation (3) for the height of the water, $h(t)$, in an inverted (point down) right circular cone with total height H and top radius R is given by:

$$\pi\left(h(t)\frac{R}{H}\right)^2 h'(t) = \alpha\pi(r_C)^2\sqrt{2gh(t)}\,. \tag{3}$$

Activities 2-5

Here are the main questions we need to address:

(2) Solve the differential equation model (3) for $h(t)$.

(3) Explain what you have to do to find out how long it takes the conical container to empty in terms of parameters? What might you do with this information to estimate the parameters α? Explain why this would be very difficult.

(4) Suppose we had data and measurements. Could we estimate the parameters from collected data? In the Appendix we offer data collected from [2] for use if the reader cannot collect data.

(5) Further, what if the circular holes in the containers offer friction or resistance?

Activities 6 - 8

(6) Create a mathematical model for a column of falling water based on Torricelli's Law as offered in [5, pp. 2-4] when the container holding the water is an inverted right circular cone of height H and base radius R. Here is the set of known working parameters and an unknown constant for our configuration as depicted in the videos on YouTube [1, 2, 3]. These videos are also available for download on YouTube.

- top radius of the right circular cone, $R = 4.$ cm;
- height of the right circular cone, $H = 9$ cm;
- radius of hole at bottom tip of the cone, $r_C = 0.4$ cm;
- acceleration due to gravity, $g = 980\text{cm}^2/\text{s}$; and
- the empirically determined resistance percentage, α, i.e. the percent of the flow rate of the liquid which actually goes through the hole in the *contraction coefficient*.

(7) Using the data from one of the videos on YouTube [1, 2, 3] estimate the parameter α.

(8) Validate your model by comparing your final model using the estimated parameter with your collected data.

VIDEO RESOURCES

We have placed three videos on YouTube [1, 2, 3] which can be used to collect data on the height of a column of water (with blue food dye for visual ease) in a right circular cone (glass funnel) as it falls due to a hole in the bottom of the cone. The dimensions of the inner cone of this glass funnel used in this experiment is 4.95 cm for the radius of the top circular cross section of the cone and a height of 9 cm (roughly from visual inspection) from the top cross section to the "point" at the base of the inverted cone. The radius of the hole at the bottom of the cone is 0.4 cm. One can stop and start the video to collect data on the height of liquid as a function of time and build a mathematical model for this process. In this model (using Torricelli's Law as the guiding principle) one can then estimate the main parameter, namely the effective area (as a percent of the cross sectional area of the hole through which the water exits at the bottom of the cone) for fluid flow. That percent or fraction is called the *discharge* or *contraction coefficien*.

Videos [1, 2] were taken on 24 February 2016 in Cornwall NY USA using the camera in an iPad

5 and then converted using an app called SloMo so that the user could more readily collect data. Video [3] was not converted.

APPENDIX

In Table 1 we offer data (transformed for ease of use) from an experiment videoed using equipment as depicted in configuration Figure 1 and shown in the YouTube video [2]. Data such as this can be collected directly from the video and cleaned up for use in the modeling. However, in the event the user cannot gain access to the video we over a set of data which can serve.

Time (s)	Height (cm)
0.0	8.60
0.46	8.40
1.30	8.10
2.66	7.40
3.66	7.00
4.56	6.50
5.45	5.90
6.40	5.30
6.96	4.60
7.40	4.00
7.90	3.20

Table 1. Data from [2].

REFERENCES

[1] SlowMoCapture 1. 2016. Video. `https://youtu.be/RV0reeXxfpw`. YouTube. Accessed 26 February 2016.

[2] SlowMoCapture 2. 2016. Video. `https://youtu.be/s9CZ-yR-j2Q`. YouTube. Accessed 26 February 2016.

[3] Capture 3. 2016. Video. `https://youtu.be/9CpNqiMpHjU`. YouTube. Accessed 26 February 2016.

[4] Neutrium. 2016. Discharge Coefficient for Nozzles and Orifices. `https://neutrium.net/fluid_flow/discharge-coefficient-for-nozzles-and-orifices/`. Accessed 27 February 2016.

[5] Winkel, Brian. 2015. 1-015-S-Torricelli.