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STUDENT VERSION

Barging Ahead: Optimizing a Trip Upriver

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Abstract: As captain of a barge, you need to determine how fast to transport your barge up river against the current in order to minimize the expended energy. Since expended energy is proportional to the force, and since the force is proportional to the speed, traveling too fast is inefficient. However, if traveling too slow, it will almost take an infinite amount of time to arrive, which is also inefficient. How fast then should you go? We find an answer to this question, and to the related question of minimizing cost of travel, where the cost is a linear combination of the energy and the time.

SCENARIO DESCRIPTION

Barges are an integral component of the freight transportation system. The capacity of a barge is sixty times that of a semi truck, and in tonnage per mile, barges are nine times as fuel efficient, saving shippers and consumers over seven billion dollars annually. Here's your opportunity, as the CEO of a freight company, to help optimize the efficiency of barges.

I) How much energy is needed for a barge to cross Lake Macatawa (with no current)?

The energy needed is equal to the work accomplished which is equal to the force (in the direction of the movement) times the distance. The primary (we assume the *only*) force to counteract in moving the barge is due to friction with the water, denoted F_{friction}. As is noticed by anyone with their hand out the window of a moving car, the force required to counteract friction is in the direction of travel and increases with speed; in fact, for slow speeds the force is proportional to the speed. Thus letting v represent the speed of the barge through the water, we have

$$F_{\text{friction}} = kv$$

where k is a proportionality constant.

Then, the energy, E, needed to move a distance from point *0* to point p (distance of p) is:

$$\begin{aligned} E &= (\text{Force})(\text{Distance}) \\ &= (F_{\text{friction}})(p) \\ &= kvp. \end{aligned}$$

We see that as v decreases to 0, so does E. This implies that there is no minimum amount of energy needed to cross the lake. The required energy will become arbitrarily small as v approaches 0.

1) Why is this not a realistic model for optimizing the efficiency of barges?

II) Suppose, more realistically, that your company wants a barge to cross Lake Macatawa (distance p) with a minimum cost, where T is the time of travel.

1) Why might the following be a good model for the cost, C, of the trip?

$$C = \alpha E + \beta T$$

2) What factors are important in determining the values of α and β?

3) Find the value of v that minimizes C. (Hint: It will be in terms of α, β, and k.) Explain why your answer is physically reasonable.

III) Now let's suppose that your company wants to minimize the energy needed to transport coal a distance p upstream along a straight portion of the Grand River from Lake Michigan to Grand Rapids, where the current of the river has a constant speed, v_c.

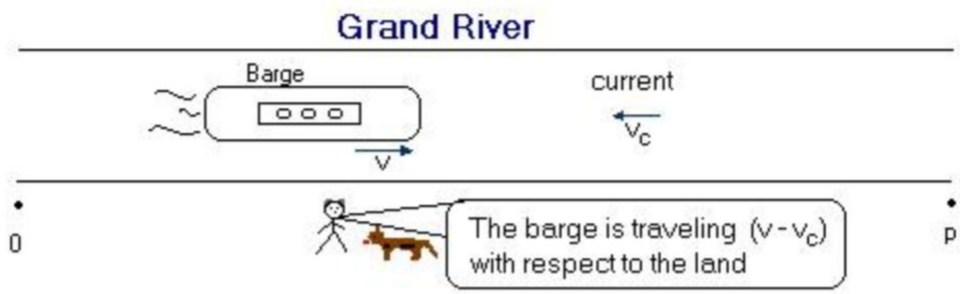

Figure 1. Depiction of river-barge configuration.

Since the energy is required to push the barge through the water, the "distance" we need for $E =$ (Force)(Distance) is not how far the barge moves relative to the land, but how far it moves relative to the water, i.e., the linear distance of water that passes the front of the barge. We will call this the *Effective Distance* and denote it, D_E. Then,

$$E = (F_{\text{friction}})(D_E)\,.$$

1) Express D_E in terms of v, v_c, and p. (Hint: Find T, the travel time from 0 to p.)

2) Express the energy, E, needed to travel from 0 to p as a function of v.

3) Find the value of v that minimizes the total energy. If the water in the Grand River flows at 3 mi/hr, at what rate should the barge move through the water to minimize the energy of transport?

4) Since we do not want the barge going backwards, we are concerned only with $v > v_c$.

Graph E vs v for $v > v_c$. (Choose, e.g., $k = 1$, $p = 2$ and $v_c = 3$.)

5) Explain why the shape of the graph is physically reasonable. In particular, explain why the graph is shaped as it is for values of v slightly larger than v_c.

IV) Suppose now that your company is in danger of going bankrupt due to your sub-par management skills. Thus we need to minimize the cost of transporting freight upriver rather than just the energy. Remember that the cost function is of the form

$$C = \alpha E + \beta T.$$

1) Express C as a function of v.

2) Find the velocity that minimizes the cost. Simplify.

3) How does the optimal velocity compare with the optimal velocity from Part III? Explain why your answer is physically reasonable.

V) The previous model explored the case of traveling upstream against a current of constant speed, v_c. More realistically, the speed of the current may change along the river due to widening or narrowing of the river or the addition of water from tributaries. We now model traveling upstream against a variable current, expressing the speed of the current as $v_c(x)$, where x represents the position along the bank of the river. (We assume $v > v_c(x)$ for all values of x.) As before, the required energy, E, is the product of the force and the Effective Distance, D_E:

$$\begin{aligned} E &= (F_{\text{friction}})(D_E) \\ &= (kv)(vT). \end{aligned}$$

The time, T, it takes to go distance p can no longer be found through a simple algebraic method because $v_c(x)$ is no longer constant. Instead, the variable rate must now be expressed using a differential equation.

1) Express $\frac{dx}{dt}$ in terms of v and $v_c(x)$, then use the equation to find, T, the time of travel. Use this to determine an expression for E (which will include an integral).

2) Consider a river whose current is increasing as you travel upstream so that $v_c(x) = \alpha x$ for $\alpha > 0$.

Simplify the expression for E by evaluating the integral.

3) Find the value of v (in terms of α and p) that minimizes the energy $E(v)$. (Hint: Show $E'(v) = (kv/\alpha)(1 - \beta + 2\ln(\beta))$, where $\beta = \frac{v}{v-\alpha p}$. Solve numerically for a value of β between 3 and 4 and then use that value to solve for v. $\beta = 1$ is the other solution, but notice that it requires $p = 0$.)

4) For $\alpha = 1/5$ and a 10 mile trip up the river, what value of v will minimize the expended energy of the trip?

During this 10 mile trip, the speed of the river increases uniformly from 0 to 2 mi/hr, so (as a function of position) the current has an average speed of 1 mi/hr. Recall from (**III**) that for constant v_c, the energy is minimized when v is twice the speed of the current. Thus one might naively expect the optimal speed for this variable current model to be $v = 2$ mi/hr which is twice the average speed of the current.

5) Explain why the optimal speed is, in fact, significantly greater than 2 mi/hr. (Hint: Refer to the graph from **III** and consider the rate of energy increase on both sides of the minimum at $v = 2v_c$.)

VI) Recap: What simplifying assumptions were used in these models? As the CEO of the freight company, how might you make a better - more realistic - model?

Acknowlegements

The author was assisted with this project by Ryan TerLouw and Ryan Weaver, two students in an NSF-REU project during the summer of 2004.